Abstract

In this dissertation we investigate convolution semigroups of probabilistic measures on the Heisenberg group $\mathbb{H}^n$, that is manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with multiplication

$$xy = (x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 \cdot y_2 - x_2 \cdot y_1)).$$

They have already been studied, also with more general setting of nilpotent Lie groups, by Wrocław mathematicians among others. Semigroups of measures on the Heisenberg group are characterized by their generating functionals which are exactly generalized laplacians. Generalized laplacians on $\mathbb{H}^n$ are the same as on the abelian group $\mathbb{R}^{2n+1}$.

In our work, we consider semigroups whose generating functionals satisfy conditions of admissibility. Such conditions are expressed in terms of their Fourier transforms, i.e. associated continuous negative definite functions. An example of admissible generalized laplacian is

$$\langle \Gamma, f \rangle = c_n \int_{\mathbb{R}^{2n+1}\setminus\{0\}} \frac{f(x) - f(0)}{\|x\|^{\frac{2n+1}{2}}} K_{\frac{2n+1}{2}}(\|x\|)dx, \qquad f \in C_c^\infty(\mathbb{R}^{2n+1}),$$

where K is the modified Bessel function of the second kind. Then we have $-\widehat{\Gamma} = \log(1 + \|\xi\|^2)$. In the setting of the abelian group Γ is the generating functional of the so-called symmetric gamma semigroup. In this work we present also a quite large class of admissible generalized laplacians.

Let P be the generating functional of a semigroup of measures μ_t on the Heisenberg group and also the generating functional of a semigroup of measures ν_t on the abelian group $\mathbb{R}^{2n+1}$. We prove estimates of the difference of the transforms $\widehat{\mu_t} - \widehat{\nu_t}$ (and their derivatives) and show that they are small with respect to the space variable $\xi \in \mathbb{R}^{2n+1}$ and the time variable $t > 0$. This leads to our main theorem which says that the difference between measures μ_t and ν_t agrees wich a function k_t which is smooth outside zero and for all $t < 1$, all $N \in \mathbb{N}$ and all $\alpha \in \mathbb{N}^{2n+1}$ satisfies

$$|\partial^\alpha k_t(x)| \leqslant \begin{cases} c_{n,\alpha} t^2 \|x\|^{-(2n+1)+2-|\alpha|} & \text{for } \|x\| \leqslant 1, \\ c_{n,\alpha,N} t^2 \|x\|^{-N} & \text{for } \|x\| \geqslant 1. \end{cases}$$

As an application we get some conditions for the measures μ_t to have densities in L^1 or in L^p. We also apply the above estimates obtaining pointwise estimates for densities of semigroups of measures on the Heisenberg group. In particular, for the generalized laplacian Γ, we can deduce the following estimates

$$v_t(x) \leqslant c_n t \|x\|^{-(2n+1)+2t}, \qquad \|x\| \leqslant 1, \ t < 1,$$

or the asymptotic behaviour for $t < 1$,

$$v_t(x) \asymp t \|x\|^{-(2n+1)+2t}, \qquad \|x\| \to 0.$$

One of the more important tools which is used in the work is a symbolic calculus for convolution operators on the Heisenberg group which is similar to the symbolic calculus of pseudodifferential operators. The idea of the calculus consists in describing the product $a\#b = (a^\vee * b^\vee)^\wedge$ for some classes of symbols. One of the obstacles in extending the Weyl calculus to general nilpotent Lie groups is the lack of a formula allowing one to calculate the derivatives $\partial^\alpha(a\#b)$. We find such a "Leibniz rule" on two step nilpotent Lie groups which includes the Heisenberg group.