Abstract

The main topics of this dissertation are the sequences of measures on Boolean algebras and their connections with ideals. This lies at the intersection of set theory and measure theory.

We study the *Nikodym property* of Boolean algebras, which asserts that every pointwise bounded sequence of measures on $\mathcal{A}$ is norm bounded. This definition is motivated by the classical result due to Nikodym and Andô, asserting that all the σ-complete algebras have this property. It is well-known that if the Stone space $St(\mathcal{A})$ contains a non-trivial convergent sequence, then $\mathcal{A}$ cannot have the Nikodym property. This example is a motivation for considering the class of simple topological spaces $N_{\mathcal{F}}$ associated with filters $\mathcal{F}$ on ω. We study the class of such filters that the space $N_{\mathcal{F}}$ supports a sequence of finitely supported measures with particular properties (connected with the Nikodym property), and prove that for any such a filter $\mathcal{F}$, if $N_{\mathcal{F}}$ is embeddable into the Stone space $St(\mathcal{A})$ of a given algebra then $\mathcal{A}$ does not have the Nikodym property. We characterize the dual class $\mathcal{AN}$ of ideals on ω in terms of density ideals, and using this we conclude that $\mathcal{I} \in \mathcal{AN}$ if and only if $\mathcal{I}$ is contained in some summable ideal. We also investigate the cofinal structure of $\mathcal{AN}$ ordered by the Katětov order $\leq_K$, which is an important tool for comparing the structural complexity of ideals and filters on ω.

We consider also the *Grothendieck property* of Boolean algebras, which has its origin in Banach space theory and is closely related to the Nikodym property—it is difficult to find an algebra having exactly one of those properties. For an ideal $\mathcal{I}$ in a σ-complete Boolean algebra $\mathcal{A}$, we show that if the Boolean subalgebra $\mathcal{A}\langle\mathcal{I}\rangle$ of $\mathcal{A}$ generated by $\mathcal{I}$ does not have the Nikodym property, then $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Grothendieck property either. We construct a family of $\mathfrak{c}$ many pairwise non-isomorphic Boolean subalgebras of the power set $\mathcal{P}(\omega)$ of the form $\mathcal{P}(\omega)\langle\mathcal{I}\rangle$ which, considered as subsets of the Cantor space 2^ω, belong to the Borel class $\mathbb{F}_{\sigma\delta}$ and have the Nikodym property but not the Grothendieck property, and a family of $2^{\mathfrak{c}}$ many pairwise non-isomorphic non-Borel Boolean algebras of the form $\mathcal{P}(\omega)\langle\mathcal{I}\rangle$ with the Nikodym property but without the Grothendieck property. The main tool used for proving these results is a new characterization of the Nikodym property of Boolean algebras $\mathcal{P}(\omega)\langle\mathcal{I}\rangle$ in terms of sequences of non-negative measures. We also study relations between various notions of boundedness of ideals on ω and of the Nikodym property. In connection with this, we investigate the so-called *hypergraph ideals*, concluding with a construction of a family of $\mathfrak{c}$ many non-isomorphic hypergraph ideals which are not *totally bounded* but have the Nikodym property.

In the last part of the thesis, we will study combinatorial problems related to Rosenthal's lemma, which is an important result concerning sequences of measures on Boolean algebras (in particular, it is a key step in proving the Grothendieck's result that all the σ-complete Boolean algebras have the Grothendieck property). The core of Rosenthal's lemma is a purely combinatorial statement: that $[\omega]^\omega$ is a *Rosenthal family.* There is a connection of Rosenthal families with free sets, where, given a function $f \in \omega^\omega$, a set $A \in [\omega]^\omega$ is *free for f* if $f[A] \cap A$ is finite. Functions from ω to ω without fixed points are also related to Rosenthal families, as they correspond to Rosenthal matrices of zeros and ones. For a class of functions $\Gamma \subseteq \omega^\omega$ without fixed points, we define $\mathfrak{ros}_\Gamma$ as the smallest size of a family $\mathcal{B} \subseteq [\omega]^\omega$ such that for every $f \in \Gamma$ there is a set $A \in \mathcal{B}$ which is free for f, and Δ_Γ as the smallest size of a family $\mathcal{F} \subseteq \Gamma$ such that for every $A \in [\omega]^\omega$ there is $f \in \mathcal{F}$ such that A is not free for f. We prove upper and lower bounds for several versions of these cardinal invariants, in terms of the classical cardinal characteristics of the continuum. In some cases, these bounds are consistent results proved by the method of forcing.