University of Wrocław
Faculty of Mathematics and Computer Science
Mathematical Institute

Aleksander Antasik

Rational approximation of surface group representations

doctoral thesis
under the supervision of
Jan Dymara

Wrocław 2025

Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny

Aleksander Antasik

Wymierna aproksymacja reprezentacji grup powierzchni

praca doktorska
napisana pod opieką
Jana Dymary

Wrocław 2025

Abstract

We prove that every Hitchin representation $\pi_1(\Sigma_g) \to \mathrm{SL}_n(\mathbb{R})$ can be rationally approximated. For Teichmüller space (n=2) an approximation can be realized by lifts of Fenchel-Nielsen twists, hence with prescribed Witt class. Computer analysis shows approximation by Goldman twists is possible in an open subset of the Hitchin component for $n = 3, g = 2$.

Abstrakt

Dowodzimy, że każda Hitchinowska reprezentacja $\pi_1(\Sigma_g) \to \mathrm{SL}_n(\mathbb{R})$ może być wymiernie przybliżona. Dla przestrzeni Teichmüllera (n=2) przybliżenie można zrealizować poprzez podniesienia twistów Fenchela-Nielsena, więc także z zadaną klasą Witta. Analiza komputerowa pokazuje, że przybliżenie przez twisty Goldmana jest możliwe na otwartym podzbiorze składowej Hitchina dla $n = 3, g = 2$.

Contents

Introduction

Let Σ_g denote a surface of genus $g > 1$. From now on assume all mentioned representations are representations of $\pi_1(\Sigma_g)$. In [Tak71] Takeuchi showed that every Fuchsian representation, i.e. discrete faithful representation into $SL_2(\mathbb{R})$, can be approximated by representation into $SL_2(\mathbb{Q})$. His proof was quite direct. Since then a natural generalization of Fuchsian representations appeared in the form of Hitchin representations - these are certain well behaved representations in $SL_n(\mathbb{R})$ that form a connected component of the representation space (two components for n even). It is one of the main subjects of higher Teichmüller theory. A natural question arises - can Hitchin representations be rationally approximated? We will answer this question in this document.

Our first approach was to use dynamics. In [Gol86] Goldman introduced a notion of a twist flow - a deformation of Hitchin representations. Goldman's twist gives us a direct control over matrices of the deformed representation. They can be rationally approximated. Now take a rational Hitchin representation and deform it by twists to get a family of rational representations. A natural question arises: what is the set of points reachable by twists? If we knew that vectors tangent to twists span the tangent space at every point, this set would be the whole Hitchin component. We were not able to verify this condition in full generality, but had some partial success. We were able to prove this in the case $n = 2$ i.e. in the case of Teichmüller space. This allowed us to show a strengthened version of the result of Takeuchi. For $n = 3$ we converted a curve of representations provided by [LRT11] to a curve of $SL_3(\mathbb{R})$ Hitchin representations. Then, using computer algebra software, we showed that Goldman's twists along some explicit collection of curves allow us to locally move in every direction. We conclude that in a neighborhood of this curve every representation can be rationally approximated.

We then moved to another approach, resembling the original method of Takeuchi. Instead of starting from a rational representation and attempting to 'spread' it to a dense rational subset of the Hitchin component, we took a concrete Hitchin $SL_n(\mathbb{R})$-representation and constructed its rational approximation. A certain genericity property for eigenvectors was provided by a result of Labourie.

In final stages of writing this thesis the author learned of an independent work of Audibert and Zshornack in the preprint [AZ23]. The authors of [AZ23] first claim that rational approximation follows from a result by Benyash-Krivetz, Chernousov and Rapinchuk in [RBC96]; we were not able to verify this claim. In [AZ23] approximability is shown for any n and genus $g > 2$ using methods very similar to our own - Goldman twists by a rational matrix. The crucial step that eluded us - the fact that derivatives of twists span the tangent space (for $g > 2$) - is inferred from

[BCL20]. Note that our second approach, and also the successful cases of the first approach, do work for $g = 2$.

Acknowledgements

I would like to thank my advisor, Jan Dymara, for his guidance and support throughout this project.

Chapter 1

Preliminaries

1.1 Simple spectrum matrices

In this section we develop a toolset for working with a special type of matrices.

Definition 1.1.1. We say that a matrix $M \in \mathrm{M}_{n\times n}\,\mathbb{R}$ is a *Simple Spectrum Matrix* (SSM), or has a simple spectrum, if it has n distinct eigenvalues.

The following lemma shows that diagonalization of a simple spectrum matrix A can be smoothly extended to diagonalize matrices inside some neighborhood of A. It is commonly known, but we could not find an authoritative source, so we provide the proof.

Lemma 1.1.2. *Let A be an n-dimensional SSM and PDP^{-1} be its diagonalization with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then, there exists a neighborhood U of A in $\mathrm{M}_{n\times n}\,\mathbb{R}$ and smooth functions $\lambda_1, \ldots, \lambda_n \colon U \to \mathbb{R}$ and $v_1, \ldots, v_n \colon U \to \mathbb{R}^n$ such that for any $M \in U$ the numbers $\lambda_1(M), \ldots, \lambda_n(M)$ are distinct eigenvalues of M and $v_1(M), \ldots, v_n(M)$ are corresponding eigenvectors, with $\lambda_i(A) = \lambda_i$ and $v_i(A)$ equal to the i-th column of P, for each $i \in \{1, \ldots, n\}$.*

Proof. Consider

$$f \colon \mathbb{R} \times \mathrm{M}_{n\times n}\,\mathbb{R} \to \mathbb{R}$$

defined by

$$f(t, X) = \det(X - tI).$$

Since χ_A only has distinct roots $\lambda_1, \ldots, \lambda_n$, for any $i \in \{1, \ldots, n\}$ we have

$$\frac{\partial f}{\partial t}(\lambda_i)\bigg|_{t=0,\, X=M} \neq 0.$$

By the Implicit Function Theorem there is a neighborhood $U_i \subseteq \mathrm{M}_{n\times n}\,\mathbb{R}$ of A and a smooth function

$$\lambda_i \colon U_i \to \mathbb{R}$$

such that

$$\lambda_i(A) = \lambda_i$$

and

$$f(\lambda_i(X), X) = 0$$

for any $X \in U_i$. Hence, there exists an open $U \subseteq U_1 \cap \cdots \cap U_n$ containing A, such that for any $X \in U$ its eigenvalues $\lambda_1(X), \ldots, \lambda_n(X)$ are distinct. For $i \in \{1, \ldots, n\}$ and $X \in U$ consider

$$g_i(X) := X - \lambda_i(X).$$

Then g_i is a smooth function defined in U and for any $X \in U$ we have $\dim \ker g_i(X) = 1$. We denote the rows of $g_i(X)$ by $R_i^1(X), \ldots, R_i^n(X)$:

$$g_i(X) = \begin{bmatrix} R_i^1(X) \\ \vdots \\ R_i^n(X) \end{bmatrix}.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$. For any $v \in \mathbb{R}^n$ we have:

$$\begin{aligned} g_i(X)v = 0 &\iff \forall j \in \{1, \ldots, n\}\ R_i^j(X)v = 0 \\ &\iff \forall j \in \{1, \ldots, n\}\ \langle R_i^j(X)^{\mathrm{T}}, v \rangle = 0. \end{aligned}$$

Thus

$$\operatorname{span}\{R_i^1(X)^{\mathrm{T}}, \ldots, R_i^n(X)^{\mathrm{T}}\} = \ker g_i(X)^{\perp}.$$

Since $\ker g_i(A)$ is one-dimensional, there exist indices $1 \leq j_1 < \cdots < j_{n-1} \leq n$ such that

$$\operatorname{span}\{R_i^{j_1}(A)^{\mathrm{T}}, \ldots, R_i^{j_{n-1}}(A)^{\mathrm{T}}\} = \ker g_i(A)^{\perp}.$$

By possibly shrinking the neighborhood U of A, we can ensure that for every $X \in U$ we have

$$\operatorname{span}\{R_i^{j_1}(X)^{\mathrm{T}}, \ldots, R_i^{j_{n-1}}(X)^{\mathrm{T}}\} = \ker g_i(X)^{\perp}.$$

Let $K_i(X)$ be the generalized cross product of $(R_i^{j_1}(X)^{\mathrm{T}}, \ldots, R_i^{j_{n-1}}(X))$, hence, also a smooth function of X. Note that $K_i(X)$ is an eigenvector of X for $\lambda_i(X)$. Finally, recall that A can be diagonalized as PDP^{-1}. For each i, let P_i be the i-th column of P and hence an eigenvector of A for λ_i. Then, there exists a real number $\alpha_i \neq 0$ such that $\alpha_i K_i(A) = P_i$. As a result, setting $v_i(X) = \alpha_i K_i(X)$ fulfills the conditions stated in the lemma. □

Corollary 1.1.3. *Being an SSM is an open condition, i.e. around each SSM there is an open set of matrices which are also SSMs.*

Lemma 1.1.4. *For a simple spectrum matrix $M \in \mathrm{SL}_n(\mathbb{Q})$ the group centralizer $Z_{\mathrm{SL}_n(\mathbb{Q})}(M)$ is dense in $Z_{\mathrm{SL}_n(\mathbb{R})}(M)$.*

The following Lemma will allow us to twist a representation by a rational matrix later.

Proof. We briefly delve into the world of algebraic groups in order to cite a result from [PRR93]. Let $M = PDP^{-1}$ be the diagonalization of M. Note that D, P need not have rational entries. Consider the equations $MX = XM$ and $\det X = 1$ where X is a $n \times n$-matrix of variables. These form a system of polynomial equations with rational coefficients, and therefore define an algebraic group Z over $\mathbb{Q}$. For a field K denote the set of K-points of Z by Z_K. Let L be any field extension of $\mathbb{Q}$ such that $\chi_M(t)$ splits in L. A simple calculation shows that

$$Z_L = \left\{ P \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & (l_1 l_2)^{-1} \end{bmatrix} P^{-1} \mid l_1, l_2 \in L \right\}.$$

We have an algebraic group morphism

$$(L^\times)^2 \to Z_L$$

defined by

$$(l_1, l_2) \mapsto P \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & (l_1 l_2)^{-1} \end{bmatrix} P^{-1},$$

with inverse

$$A \mapsto ((P^{-1}AP)_{11}, (P^{-1}AP)_{22}),$$

hence it is an L-isomorphism. This shows that Z is a $\mathbb{Q}$-torus (although the fact that the isomorphism may not be defined over $\mathbb{Q}$ means it is not necessarily $\mathbb{Q}$-split). Corollary 1 to Proposition 7.8 in [PRR93] then implies that $Z_{\mathbb{Q}}$ is dense in $Z_{\mathbb{R}}$ with respect to the Euclidean topology. □

Lemma 1.1.5. *Let $M \in \mathrm{M}_{n\times n}\mathbb{R}$ be an SSM. Then for any $N \in \mathrm{M}_{n\times n}\mathbb{R}$ we have $M \sim N \iff \chi_M = \chi_N$.*

Proof. The $\implies$ is clear since χ is an invariant of conjugation. For $\impliedby$, since M has n distinct eigenvalues and N has the same characteristic polynomial, N has the same n eigenvalues as M. Hence, both M and N are diagonalizable and have the same diagonal form, so they are conjugate. □

1.2 Curves and Surfaces

We briefly explore some of the basic notions related to surfaces and curves, mostly for the sake of setting the notation and the nomenclature. For details, the reader is referred to the literature e.g. [Lab17] and [FM11]. All of the curves we consider in this document will be oriented - we might sometimes omit specifying this. A curve in a topological space X is a continuous map $[0, 1] \to X$. A curve γ is called:

- closed, if $\gamma(0) = \gamma(1)$
- simple, if it is injective except at the ends i.e. $\gamma(s) \neq \gamma(t)$ for all $(s, t) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$
- essential, if it is not freely homotopic to a constant curve
- separating, if its complement in C is not connected, where C is the component containing the curve

- non-separating, if it is not separating

The compact connected oriented surface of genus g, with n boundary components will be called a surface of type (g, n). The Euler characteristic of (g, n)-surface is equal to $2 - 2g - n$. We will only consider compact surfaces in this document. The

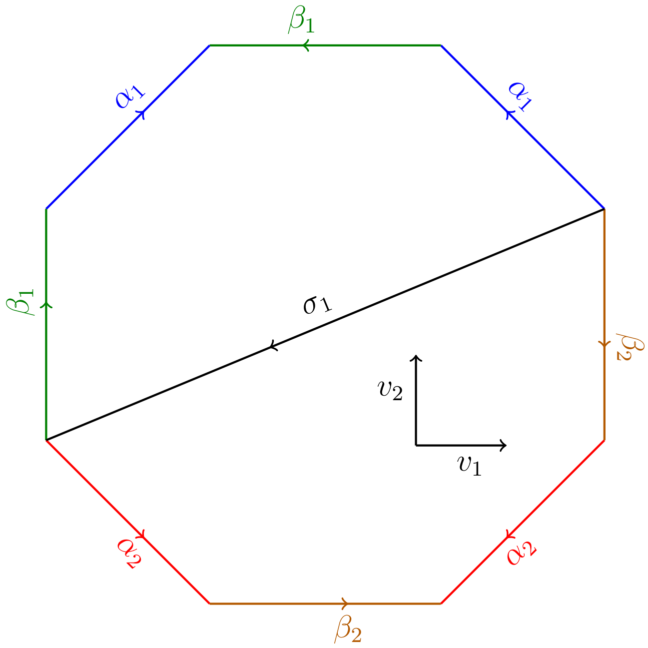


Figure 1.1: Fundamental octagon of Σ_2.

surface Σ_g of type $(g, 0)$ is obtained as follows:

- start by taking a $2g$-gon
- label its sides in order $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \alpha_2, \beta_2, \alpha_2^{-1}, \beta_2^{-1}, \ldots, \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1}$ (see Figure 1.1)
- glue each α_i to α_i^{-1} and β_i to β_i^{-1} with opposite orientations

After these identifications, all vertices of the $2g$-gon become a single vertex x_0, each of α_i, β_i becomes a simple essential closed curve, and the quotient is a surface of type $(g, 0)$. Moreover α_i, β_i generate $\pi_1(\Sigma_g)$, namely:

$$\pi_1(\Sigma_g, x_0) = \left\langle \alpha_1, \beta_1, \ldots \alpha_g, \beta_g \;\middle|\; \prod_{i=1}^{g} [\alpha_i, \beta_i] \right\rangle.$$

We will refer to α_i, β_i defined above as the standard generators of $\pi_1(\Sigma_g)$. We will abusively denote the curves and their based homotopy classes by the same symbol, and, like above, omit mentioning the basepoint of the fundamental group. We set the orientation on all Σ_g so that the basis (v_1, v_2) in Figure 1.1 (and its analogs in fundamental $2g$-gons) is positively oriented. It follows, that the algebraic intersection number

$$i(\alpha_i, \beta_i) = 1$$

for any i. For $i \neq j$ we have

$$i(\alpha_i, \alpha_j) = i(\alpha_i, \beta_j) = i(\beta_i, \beta_j) = 0.$$

The curve σ_1 in Figure 1.1 (considered in Σ_2) is an example of a simple separating closed curve. It is based (at x_0) homotopic to $[\alpha_1, \beta_1]$. In general, for Σ_g, we will denote by σ_k a curve in Σ_g, that is represented by a diagonal in the fundamental $2g$-gon, based homotopic to $\prod_{i=1}^{k} [\alpha_i, \beta_i]$. We will call σ_k the standard $(k, g-k)$-partitioning curve. Clearly, σ_k is closed, simple and separating.

Let S be of type (g, n), equipped with a smooth structure and let α be a simple closed curve disjoint from boundary of S. By a *cut surface* of α (or S *cut along* α) we mean a (possibly disconnected) surface $S \mid \alpha$ with two distinguished boundary components α^1, α^2 identified by a homeomorphism φ and a continuous map $\iota \colon S \mid \alpha \to S$ satisfying:

- $\iota \circ \alpha^1 = \iota \circ \alpha^2 = \alpha$
- there exists a homeomorphism $f \colon (S \mid \alpha)/\varphi \to S$ such that $f \circ \pi = \iota$, where π denotes the projection map $S \mid \alpha \to (S \mid \alpha)/\varphi$.

Such a surface is guaranteed to exist, given α is reasonably regular, for example piecewise smooth. For a distinguished component α^i of boundary of $S \mid \alpha$ and any $t_0 \in [0, 1]$ such that α^i is smooth at t_0 let w_{t_0} denote a vector towards the surface, transversal to $\frac{d\alpha^i}{dt}(t_0)$. Then $(\frac{d\alpha^i}{dt}(t_0), w_{t_0})$ is either a positively oriented or negatively oriented basis of $T_{\alpha^i(t_0)}(\Sigma_g \mid \alpha)$, and the same construction for α^{1-i} leads to opposite orientation. For a positively (resp. negatively) oriented basis as above we will call α^i the left (resp. right) component of α in $\Sigma_g \mid \alpha$ and denote it by α^l (resp. α^r). An analogous construction can be performed for a simple open arc α in the interior of S. For the rest of this document we will assume that Σ_g is hyperbolic i.e. $g > 1$.

Chapter 2

Teichmüller and Hitchin components

2.1 Teichmüller space

In this section we present some of the classical results of the Teichmüller theory. Interested reader can find more in [FM11]. It is a consequence of the Uniformization Theorem that for $g > 1$ any surface of type $(g, 0)$ can be equipped with a hyperbolic metric. A part of the process is equipping the universal cover $\widetilde{S}$ of S with a hyperbolic metric, that descends to S. For $S = \Sigma_g$ this means we can identify $\widetilde{\Sigma}_g$ with $\mathbb{H}^2$ via an orientation-preserving isometry. The fundamental group $\pi_1(\Sigma_g) = \pi_1(\Sigma_g, x_0)$ acts on $\widetilde{\Sigma}_g$ via orientation-preserving isometries, hence it can be identified with a subgroup of the group $\mathrm{PSL}_2(\mathbb{R})$ of orientation-preserving isometries of $\mathbb{H}^2$. This construction depends on the choice of the identification $\widetilde{\Sigma}_g \to \mathbb{H}^2$ hence is defined up to conjugation. We can therefore associate with a hyperbolic metric an element of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$, where $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))$ by conjugating the whole representation. Two isotopic metrics (i.e. such that one of them is a pullback of the other via some diffeomorphism isotopic to identity) yield the same conjugacy class. Let us define the *Teichmüller space* $\mathcal{T}(\Sigma_g)$ of Σ_g as

$$\{\text{orientation-compatible hyperbolic metrics on } \Sigma_g\ \}/\operatorname{Diff}_0(\Sigma_g),$$

where $\operatorname{Diff}_0(\Sigma_g)$ denotes the group of diffeomorphisms of Σ_g isotopic to identity. It turns out that the procedure described above defines an identification of $\mathcal{T}(\Sigma_g)$ with some subset of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$. A representation in

$$\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))$$

is called *Fuchsian* if it is discrete and faithful. The properties of the action of $\pi_1(\Sigma_g)$ on the universal cover of Σ_g imply that every representation ρ, such that $[\rho] \in \mathcal{T}(\Sigma_g)$, is Fuchsian. Conversely, the set of Fuchsian representations consists of two components of $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))$, with one projecting to $\mathcal{T}(\Sigma_g)$ (via the identification discussed above) and the other to $\mathcal{T}^-(\Sigma_g)$ - the Teichmüller space for Σ_g with reversed orientation. The set $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))$ can be identified with

$$\{a_1, b_1, \ldots, a_g, b_g \in \mathrm{PSL}_2(\mathbb{R}) \mid [a_1, b_1] \cdots [a_g, b_g] = 1\}.$$

This identification gives us a smooth structure on the non-singular part of the space $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))$, that descends to $\mathcal{T}(\Sigma_g)$. Note that there are many equivalent ways of defining the Teichmüller space and its smooth structure. It turns out that $\mathcal{T}(\Sigma_g)$ is diffeomorphic to a ball of dimension $6g-6$ - we will now define some new terms so that we can describe this diffeomorphism.

By a *pants decomposition* of Σ_g we mean a collection of essential simple closed curves on Σ_g partitioning Σ_g into surfaces of type $(0,3)$. A simple Euler characteristic calculation shows that a pants decomposition consists of $3g-3$ curves. To a pants decomposition

$$(\gamma_1, \dots, \gamma_{3g-3})$$

(and a set of auxiliary curves called seams that help track twisting) we associate the *Fenchel-Nielsen* coordinates

$$(\ell_1, \dots, \ell_{3-3g}, \tau_1, \dots, \tau_{3g-3}),$$

a coordinate system for $\mathcal{T}(\Sigma_g)$, where the first $3g-3$ coordinates are called *length coordinates* and the last $3g-3$ coordinates are called *twist coordinates*. These coordinates are related to cutting Σ_g along the curves in the pants decomposition, possibly changing lengths of boundary components and gluing them back possibly with a twist. For more details see [FM11]. A fundamental result says that for any pants decomposition this yields a diffeomorphism onto ${\mathbb{R}_+}^{3g-3} \times \mathbb{R}^{3g-3}$. A Fenchel-Nielsen twist can also be considered separately, as it does not depend on the choice of other curves in the pants decomposition. By τ_α^t we will denote a Fenchel-Nielsen twist along a simple closed curve α by $t \in \mathbb{R}$.

2.2 Gromov boundary

A geodesic ray in a metric space (X, d) is an isometric embedding $[0, \infty) \to X$. Two geodesic rays R_1, R_2 are asymptotic if $\sup d(R_1(t), R_2(t)) < \infty$. We write $R_1 \sim R_2$. For X equal to $\mathbb{H}^2$ or Cayley graph of $\pi_1(\Sigma_g)$ (or more generally a proper geodesic Gromov hyperbolic space) we define the *Gromov boundary at infinity* $\partial_\infty(X)$ as $\{\text{geodesic rays in } X\}/\sim$. There is a natural topology on $\partial_\infty(X)$. Isometries of X act on $\partial_\infty(X)$ by homeomorphisms. A Fuchsian representation ρ induces a $\pi_1(\Sigma_g)$ action on $\mathbb{H}^2$ by isometries. This yields a ρ-equivariant homeomorphism $f\colon \partial_\infty(\pi_1(\Sigma_g)) \to \partial_\infty(\mathbb{H}^2)$. Note that for any $\gamma \in \pi_1(\Sigma_g)$ the element $\rho(\gamma)$ is a hyperbolic isometry of $\mathbb{H}^2$. Hence its action on boundary has two fix points - an attracting and a repelling point. Denote them by $\rho(\gamma)^+$ and $\rho(\gamma^-)$ respectively. Then $f^{-1}(\rho(\gamma)^+) \in \partial_\infty(\pi_1(\Sigma_g))$ is an attracting point of γ and similarly for the repelling point. Note that this point does not depend on the choice of ρ. We will denote the attracting point of γ by γ^+. The repelling point will be denoted γ^-.

2.3 Hitchin components

The Higher Teichmüller Theory studies of the character variety $\mathrm{Hom}(\pi_1(\Sigma_g), G)//G$ for a simple real Lie Group G of higher rank. The goal is to find subsets that share some of the nice features of the Teichmüller space. It seems necessary as even in the definition of the character variety using a regular quotient would lead to pathological space - GIT quotient has to be used instead. There are multiple approaches including

- Hitchin components,
- space of representation where Toledo number reaches maximum possible value,
- the space of Anosov representations.

More about this topic can be read in [Wie18]. In this thesis we are mostly interested in Hitchin components. They were defined and studied using Higgs bundles techniques by Hitchin in [Hit92]. Goldman provided characterization for $G = \mathrm{SL}_3(\mathbb{R})$ in terms of convex $\mathbb{RP}^2$ structures in [Gol90]. Labourie showed that they are Anosov, discrete and faithful in [Lab04].

Before we can define Hitchin components, we need to provide some definitions. There is the unique (up to conjugation) irreducible representation

$$\iota_n\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R}),$$

defined as follows. Take any identification of $\mathbb{RP}^{n-1}$ with the set of real homogeneous polynomials of degree $n-1$ in variables X, Y. Then, for $A \in \mathrm{PSL}_2(\mathbb{R})$ we set

$$\iota_n(A^{-1}) \cdot [P(X,Y)] = [P(A_{11}X + A_{12}Y, A_{21}X + A_{22}Y)].$$

It is an easy exercise to check that this is a well-defined homomorphism $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$. A *quasi-Fuchsian* representation is a composition of ι_n with a Fuchsian representation. By a *Hitchin component* we mean a component of

$$\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_n(\mathbb{R}))/\,\mathrm{PSL}_n(\mathbb{R})$$

that contains some quasi-Fuchsian representation. There are two Hitchin components when n is even, and one when n is odd. Some authors distinguish one of the components and call it *the* Hitchin component, but we chose not to. Note a slight inconsistency, as we did distinguish a Hitchin component in dimension 2 - namely the Teichmüller space. As before, we imbue Hitchin components with smooth structure induced from $\mathrm{PSL}_n(\mathbb{R})$. Hitchin components extend many wonderful properties of the Teichmüller space to higher dimensions:

- every Hitchin $\mathrm{PSL}_n(\mathbb{R})$-representation lifts to a (Hitchin) $\mathrm{SL}_n(\mathbb{R})$-representation
- every Hitchin representation is faithful and discrete
- each Hitchin component is diffeomorphic to a ball of dimension $(n^2-1)(2g-2)$
- for a Hitchin $\mathrm{SL}_n(\mathbb{R})$-representation ρ and any $\gamma \in \pi_1(\Sigma_g)$ the matrix $\rho(\gamma)$ has n distinct eigenvalues (see Lemma 2.3.1)

A representation in either

$$\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R})) \text{ or } \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_n(\mathbb{R}))$$

is called *Hitchin* if it projects to the Hitchin component by the obvious map. We will denote the set of Hitchin representations as

$$\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))^{Hit}$$

and

$$\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_n(\mathbb{R}))^{Hit},$$

while the union of Hitchin components will be denoted by

$$\mathrm{Hit}_n(\Sigma_g).$$

The following lemma is crucial for the entire thesis.

Lemma 2.3.1 ([BD14][Lemma 9] and [Lab04, Theorem 1.5])**.** *Let $\rho\colon \pi_1(\Sigma_g) \to \mathrm{SL}_n(\mathbb{R})$ be a Hitchin representation. Then, for every non-trivial $\gamma \in \pi_1(\Sigma_g)$, the element $\rho(\gamma)$ has distinct eigenvalues. Moreover the eigenvalues are either all positive or all negative.*

Labourie defined in [Lab04] a ρ-equivariant map $\xi\colon \partial_\infty(\pi_1(\Sigma_g)) \to \mathrm{Flag}(\mathbb{R}^n)$ called the osculating flag, where $\mathrm{Flag}(\mathbb{R}^n)$ denotes the space of full flags in $\mathbb{R}^n$. We can retrieve it as follows (see [Can22, Theorem 2.2]). Take a non-trivial $\gamma \in \pi_1(\Sigma_g)$ and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\rho(\gamma)$. By possibly reordering we may require that $|\lambda_1| > \cdots > |\lambda_n|$. Let v_i denote the eigenvector for λ_i. Then, for any k we have $\xi^k(\gamma^+) = \mathrm{span}\{v_1, \dots, v_k\}$, $\xi^k(\gamma^-) = \mathrm{span}\{v_n, \dots, v_{n-k+1}\}$. We are interested in one of its properties, namely for $k_1 + \cdots + k_j = n$ and different points $x_1, \dots, x_j \in \partial_\infty(\pi_1(\Sigma_g))$ we have that

$$\xi^{k_1}(x_1) + \xi^{k_2}(x_2) + \cdots + \xi^{k_j}(x_j) \tag{2.1}$$

is a direct sum of linear subspaces.

Recall that the contragradient representation ρ^* of ρ is defined by $\rho^*(\gamma) = (\rho(\gamma^{-1}))^T$.

Lemma 2.3.2. *A representation ρ is Hitchin whenever ρ^* is.*

Proof. It suffices to show that ρ being Hitchin implies that ρ^* is Hitchin. Assume that $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_n(\mathbb{R}))$ is Hitchin. Let σ be a Fuchsian representation. Then

$$(\iota_n \circ \sigma)^*(\gamma) = (\iota_n \circ \sigma(\gamma^{-1}))^T = (\iota_n((\sigma(\gamma))^{-1}))^T = \iota^*(\sigma(\gamma)).$$

Since ι_n is irreducible, so is ι_n^*. Hence $\iota_n^* \circ \sigma$ is conjugate to a quasi-Fuchsian representation, hence a Hitchin representation. Since the map $\eta \mapsto \eta^*$ is continuous ρ must be mapped to the same component as $\iota_n \circ \sigma$, so ρ^* is Hitchin. This implies the same property for $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$. □

Lemma 2.3.3. *Let* $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))^{Hit}$ *. Let* $\alpha, \beta \in \pi_1(\Sigma_g)$ *be non-trivial, distinct and with* $\alpha^+, \alpha^-, \beta^+, \beta^-$ *- four distinct points in* $\partial_\infty \pi_1(\Sigma_g)$*. Denote the the eigenvalues of* $\rho(\alpha)$ *by* $\lambda_1, \dots, \lambda_n$*, with* $|\lambda_1| > \cdots > |\lambda_n|$ *and choose corresponding eigenvectors* $a_1, \dots, a_n$*. Finally, let* $b_1, \dots, b_n$ *denote the eigenvectors of* $\rho(\beta)$*. Then* $b_1, \dots, b_n \notin \mathrm{span}\{a_2, \dots, a_n\}$.

Proof. After reordering if necessary, we may assume that the eigenvectors $b_1, \dots, b_n$ correspond to the eigenvalues $\mu_1, \dots, \mu_n$ (where $|\mu_1| > \cdots > |\mu_n|$) of $\rho(\beta)$. Consider the contragradient representation $\rho^* \colon \pi_1\Sigma_g \to \mathrm{SL}_n(\mathbb{R})$. We will tentatively treat $\rho^*(\gamma)$ as a linear automorphism of $\mathbb{R}^{n*}$ and its eigenvectors as functionals. Let $\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_n\}$ be the dual bases of $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ respectively, that is, for all $i, j \in \{1, \dots, n\}$:

$$\varphi_i(a_j) = \delta_{ij}, \quad \psi_i(b_j) = \delta_{ij}.$$

Then $\varphi_1, \dots, \varphi_n$ are eigenvectors of $\rho^*(\alpha^{-1})$, while $\psi_1, \dots, \psi_n$ are eigenvectors of $\rho^*(\beta^{-1})$. We take the image of α^{-1}, since under $\rho^*(\alpha^{-1})$ the eigenvector φ_i corresponds to the eigenvalue λ_i. The same consideration applies to β, where one must likewise take β^{-1}. Lemma 2.3.2 implies that ρ^* is Hitchin (as a representation into $\mathrm{SL}_n(\mathbb{R})$). By the property 2.1 of osculating flag ξ of ρ^* and since

$$(\alpha^{-1})^{\pm} = \alpha^{\mp} \quad \text{and} \quad (\beta^{-1})^{\pm} = \beta^{\mp},$$

we have that

$$\xi^{k-1}(\beta^-) + \xi^1(\alpha^-) + \xi^{n-k}(\beta^+)$$

is a direct sum of linear subspaces for any $k \in \{1, 2, \dots, n\}$. Since

$$\xi^{k-1}(\beta^-) = \mathrm{span}\{\psi_1, \dots, \psi_{k-1}\} \quad \text{and} \quad \xi^{n-k}(\beta^+) = \mathrm{span}\{\psi_n, \dots, \psi_{k+1}\}$$

we have

$$\xi^1(\alpha^-) = \varphi_1 \notin \mathrm{span}\{\psi_1, \dots, \widehat{\psi_k}, \dots, \psi_n\}.$$

This implies that $\varphi_1(b_k) \neq 0$. Hence $b_k \notin \mathrm{span}\{a_2, \dots, a_n\}$, for any $k \in \{1, 2, \dots, n\}$. □

Chapter 3

Approximation of Hitchin representations

Our current goal is to approximate a Hitchin representation by a rational representation. Takeuchi showed that this is possible for 2-dimensional representations in [Tak71]. In Theorem 3.1.4, we provide a generalization of his Theorem to any dimension. The sketch of the proof is as follows. Take a Hitchin representation ρ_0. Lemma 3.1.2 implies that any representation conjugate to ρ_0 is approximable if and only if ρ_0 is approximable. Let us then take a representation ρ conjugate to ρ_0, such that $\rho(\beta_1)$ is diagonal. Then, for every $i > 1$, approximate $\rho(\alpha_i), \rho(\beta_i)$ by an $\mathrm{SL}_n(\mathbb{Q})$ matrix. The proof reduces then to finding rational approximations of $\rho(\alpha_1)$ and $\rho(\beta_1)$ so that surface group relation is satisfied. This reformulates as a matrix equation in a form $XY = ZYX$, where Z is a given matrix in $\mathrm{SL}_n(\mathbb{Q})$ and X, Y are variables. We need the find matrices X, Y in $\mathrm{SL}_n(\mathbb{Q})$ solving this equation and close to $\rho(\alpha_1), \rho(\beta_1)$ respectively. We first find, with help from the Higher Teichmüller Theory, a $Y \in \mathrm{SL}_n(\mathbb{Q})$ approximating $\rho(\beta_1)$ such that the equation $XY = ZYX$ (in variable X) has a real solution. Then we argue that there exists a rational solution approximating $\rho(\alpha_1)$ and with determinant 1.

3.1 Rational approximability

The goal of this short section is to convince the reader that considering rational approximability of a conjugacy class of representations makes sense. A representation in $Hom(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ is *rationally approximable* if for any open neighborhood U of ρ we have $U \cap Hom(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{Q})) \neq \emptyset$.

Lemma 3.1.1. *Every* $\mathrm{SL}_n(\mathbb{R})$ *matrix can be approximated by a* $\mathrm{SL}_n(\mathbb{Q})$ *matrix.*

Proof. Take any $M \in \mathrm{SL}_n(\mathbb{R})$, and let M_{ij} denote its (i, j)-th entry. Let M^{ij} denote the matrix obtained by removing the i-th row and j-th column. By the Laplace expansion along the first column, we have

$$1 = \det M = \sum_{i=1}^{n} (-1)^{i+1} M_{i1} \det M^{i1}.$$

Hence, there exists some k such that both M_{k1} and $\det M^{k1}$ are non-zero. Then M_{k1} can be expressed as

$$M_{k1} := 1 - (-1)^{k+1}(\det M^{k1})^{-1} \sum_{\substack{i=1 \\ i \neq k}}^{n} (-1)^{i+1} M_{i1} \det M^{i1}.$$

Note that the expression on the right, treated as a function of M depends smoothly on coefficients on M in some small neighborhood of M. We approximate each entry M_{ij} of M, except for M_{k1}, by some rational N_{ij}. We will build rational matrix N approximating M. Moreover it does not depend on M_{k1}. If entries of N_{ij} are close enough to corresponding entries of M, we have $\det N^{k1} \neq 0$ and we can find rational N_{k1} so that $N = (N_{ij})_{i,j=1}^{n}$ approximates M and $\det N = 1$. We do this by setting

$$N_{k1} := 1 - (-1)^{k+1}(\det N^{k1})^{-1} \sum_{\substack{i=1 \\ i \neq k}}^{n} (-1)^{i+1} N_{i1} \det N^{i1}.$$

This N approximates M. The equality of determinant is automatic by definition of N_{k1}.

□

Lemma 3.1.2. *Let $\rho, \rho' \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$ be conjugate in $\mathrm{SL}_n(\mathbb{R})$. Then ρ is rationally approximable if and only if ρ' is rationally approximable.*

Proof. Assume ρ is rationally approximable. Then there exists $\sigma \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{Q}))$ approximating ρ. There exists an $M \in \mathrm{SL}_n(\mathbb{R})$ such that $M\rho M^{-1} = \rho'$. By Lemma 3.1 we can find a $Q \in \mathrm{SL}_n(\mathbb{Q})$ approximating M. Then as $Q \to M$ and $\sigma \to \rho$ we have $Q\sigma Q^{-1} \to M\rho M^{-1} = \rho'$ and $Q\sigma Q^{-1} \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{Q}))$. □

We say that a conjugacy class of representations is *rationally approximable* if any (equivalently all) of its representatives are rationally approximable. We introduce a notation, useful in this chapter:

$$E_{\bar{x}} := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & \cdots & 0 \\ x_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \cdots & 1 \end{bmatrix},$$

for $\bar{x} = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. We first prove an auxiliary lemma.

Lemma 3.1.3. *Let ρ be such that $[\rho] \in \mathrm{Hit}_n(\Sigma_g)$. Let $\alpha, \delta \in \pi_1(\Sigma_g)$ be non-trivial, dinstinct and with distinct fixed points $\alpha^+, \alpha^-, \delta^+, \delta^-$ in $\partial_\infty \pi_1(\Sigma_g)$. Let $A := \rho(\alpha)$ and $D := \rho(\delta)$ with $A = PDP^{-1}$ and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then A is conjugate to $AE_{\bar{x}}$ in $\mathrm{GL}_n \mathbb{R}$ only if $x_2 = \cdots = x_n = 0$.*

Proof. Let $C = \{FAF^{-1} \mid F \in \mathrm{GL}_n \mathbb{R}\}$. We will first prove that C is an embedded submanifold of $\mathrm{GL}_n \mathbb{R}$ and describe its tangent space at a point. By Lemma 2.3.1

we have that A is an SSM. Hence, Lemma 1.1.2 implies that there is an open neighborhood $U \subseteq \mathrm{GL}_n \mathbb{R}$ of the set C consisting of simple spectrum matrices and a smooth map

$$\bar{\lambda} \colon U \to \mathbb{R}^n$$

assigning to each element of U its eigenvalues in decreasing order. For any $M \in \bar{\lambda}^{-1}[\bar{\lambda}(A)]$ the function $\bar{\lambda}$ is a submersion at M, because for any direction $v \in \mathbb{R}^n$ we can deform M to $(M_t)_{t\in[-1,1]}$ with $M_0 = M$ and

$$\frac{d}{dt}\bar{\lambda}(M_t)\bigg|_{t=0} = v.$$

We do this by diagonalizing M as PDP^{-1} and setting $M_t = PD_tP^{-1}$ for $D_t = D + \mathrm{diag}(tv_1, \ldots, tv_n)$. Since C is a level set of a regular value $\bar{\lambda}(A)$ of smooth $\bar{\lambda}$ it is a smoothly embedded submanifold of $\mathrm{GL}_n \mathbb{R}$. Since $g \cdot M := gMg^{-1}$ describes a smooth action of a Lie group $\mathrm{GL}_n \mathbb{R}$ on itself, it defines a smooth submersion of $\mathrm{GL}_n \mathbb{R}$ onto C. Hence each vector tangent to C at M is of the form

$$\frac{d}{dt}\exp(tX)M\exp(-tX)\bigg|_{t=0} = XM - MX,$$

for some $X \in \mathfrak{gl}_n(\mathbb{R})$.

We now move to the main part of the proof of the lemma. In order to investigate whether A and $AE_{\bar{x}}$ are conjugate, we consider equality of their characteristic polynomials:

$$\det(A - tI) = \det(AE_{\bar{x}} - tI).$$

This equality amounts to equating the corresponding coefficients of the characteristic polynomials. Since $x_2, \ldots, x_n$ are only present in the first column of $AE_{\bar{x}}$, in total degree 1, this yields a system of linear equations in $x_2, \ldots, x_n$. Note that $\bar{x} = 0$ is a solution, and hence this is a homogeneous system. Assume this system has a non-zero solution $\bar{a} \in \mathbb{R}^{n-1}$. Then for any $t \in [0,1]$, $t\bar{a}$ is also a solution. For

$$Y_{\bar{x}} := E_{\bar{x}} - I = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ x_2 & 0 & 0 & \cdots & 0 \\ x_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathfrak{gl}_n(\mathbb{R})$$

we have that $A\exp(tY_{\bar{a}}) = AE_{t\bar{a}}$ is a smooth curve that for $t \in [0,1]$ lies inside C with derivative at 0 equal to $AY_{\bar{a}}$. It remains to show that the tangent space T_AC contains no vector of the form $AY_{\bar{x}}$, with non-zero $\bar{x}$. For convenience we will first move both T_CA and $AE_{\bar{x}}$ to $\mathfrak{gl}_n(\mathbb{R})$ by applying $dL_{A^{-1}}$. We obtain

$$\begin{aligned} dL_{A^{-1}}[T_CA] &= dL_{A^{-1}}[\{XA - AX \mid X \in \mathfrak{gl}_n(\mathbb{R})\}] \\ &= \{A^{-1}XA - X \mid X \in \mathfrak{gl}_n(\mathbb{R})\} \end{aligned}$$

and

$$dL_{A^{-1}}(AY_{\bar{x}}) = Y_{\bar{x}}.$$

Recall that $A = PDP^{-1}$, with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. To simplify further, let us apply $Ad_{P^{-1}}$ to both $dL_{A^{-1}}[T_C A]$ and $dL_{A^{-1}}(AY_{\bar{x}})$ yielding:

$$\begin{aligned} T := &\mathrm{Ad}_{P^{-1}}\left[dL_{A^{-1}}\left[T_A C\right]\right] \\ &= Ad_{P^{-1}}\left[\{A^{-1}XA - X \mid X \in \mathfrak{gl}_n(\mathbb{R})\}\right] \\ &= \{(P^{-1}A^{-1}P)(P^{-1}XP)(P^{-1}AP) - P^{-1}XP \mid \mathfrak{gl}_n(\mathbb{R})\} \\ &= \{D^{-1}XD - X \mid X \in \mathfrak{gl}_n(\mathbb{R})\}, \end{aligned}$$

and

$$\mathrm{Ad}_{P^{-1}}(dL_{A^{-1}}(AY_{\bar{x}})) = P^{-1}Y_{\bar{x}}P.$$

Then T consists of matrices of the form:

$$D^{-1}XD - X = \begin{bmatrix} 0 & (\frac{\lambda_2}{\lambda_1} - 1)x_{12} & \cdots & (\frac{\lambda_n}{\lambda_1} - 1)x_{1n} \\ (\frac{\lambda_1}{\lambda_2} - 1)x_{21} & 0 & \cdots & (\frac{\lambda_n}{\lambda_2} - 1)x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{\lambda_1}{\lambda_n} - 1)x_{n1} & (\frac{\lambda_2}{\lambda_n} - 1)x_{n2} & \cdots & 0 \end{bmatrix},$$

where $X = (x_{ij})_{i,j=1}^n$. Since $\lambda_1, \ldots, \lambda_n$ are distinct and non-zero, T is exactly the set of matrices that have zeroes on the diagonal. The matrix $P^{-1}Y_{\bar{x}}$ is of the form:

$$\begin{bmatrix} f_1(\bar{x}) & 0 & 0 & \cdots & 0 \\ f_2(\bar{x}) & 0 & 0 & \cdots & 0 \\ f_3(\bar{x}) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n(\bar{x}) & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where for any non-zero $\bar{x} \in \mathbb{R}^n$ at least one of $f_1(\bar{x}), \ldots, f_n(\bar{x})$ is non-zero. Let

$$P = (p_{ij})_{i,j=0}^n = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix},$$

then

$$P^{-1}Y_{\bar{x}}P = \begin{bmatrix} f_1(\bar{x})p_{11} & f_1(\bar{x})p_{12} & f_1(\bar{x})p_{13} & \cdots & f_1(\bar{x})p_{1n} \\ f_2(\bar{x})p_{11} & f_2(\bar{x})p_{12} & f_2(\bar{x})p_{13} & \cdots & f_2(\bar{x})p_{1n} \\ f_3(\bar{x})p_{11} & f_3(\bar{x})p_{12} & f_3(\bar{x})p_{13} & \cdots & f_3(\bar{x})p_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n(\bar{x})p_{11} & f_n(\bar{x})p_{12} & f_n(\bar{x})p_{13} & \cdots & f_n(\bar{x})p_{1n} \end{bmatrix}.$$

Note that $P_1 \ldots, P_n$ are eigenvectors of A. By Lemma 2.3.3 applied to α and δ, each of $p_{11}, p_{12}, \ldots, p_{1n}$ is non-zero.[1] Hence for any non-zero $\bar{x} \in \mathbb{R}^n$, there is a $j \in \{1, \ldots, n\}$, such that $(P^{-1}Y_{\bar{x}}P)_{jj} = f_j(\bar{x})p_{1j} \neq 0$, which ends the proof of the lemma.

□

[1]Lemma 3.1.3 is only used once - and is applied to elements β_1 and $\alpha_1\beta_1\alpha_1^{-1}$. These elements have non-intersecting axes in the sense of [BCL20]. Hence we could use here [BCL20, Corollary 4.1] instead.

Theorem 3.1.4 (Higher Takeuchi theorem)**.** *Any Hitchin* $\mathrm{SL}_n(\mathbb{R})$*-representation can be approximated by a* $\mathrm{SL}_n(\mathbb{Q})$*-representation arbitrarily well.*

Proof. Take any Hitchin $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_n(\mathbb{R}))$. We can present $\pi_1(\Sigma_g)$ as

$$\left\langle \alpha_1, \beta_1, \dots \alpha_g, \beta_g \;\middle|\; \prod_{i=1}^{g} [\alpha_i, \beta_i] \right\rangle.$$

Let

$$A_i := \rho(\alpha_i); B_i := \rho(\beta_i).$$

Thanks to Lemma 2.3.1 and Lemma 1.1.5 we can replace ρ with a conjugate representation such that $B_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ with $|\lambda_1| > |\lambda_2| \cdots > |\lambda_n|$. We can denote

$$B_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Let us approximate $A_2, B_2, \dots, A_g, B_g \in \mathrm{SL}_n(\mathbb{R})$ by some $X_2, Y_2, \dots X_g, Y_g \in \mathrm{SL}_n(\mathbb{Q})$ and let us set $C := (\prod_{i=2}^{g} [A_i, B_i])^{-1}$ and $Z := (\prod_{i=2}^{g} [X_i, Y_i])^{-1}$. Our goal is to prove the existence of a $Y = Y_1 \in \mathrm{SL}_n(\mathbb{Q})$ approximating B_1 and such that there exists, $X = X_1 \in \mathrm{SL}_n(\mathbb{Q})$ approximating A_1, satisfying

$$[X_1, Y_1] \cdots [X_g, Y_g] = I.$$

We can multiply both sides of the equation by Z from the right, obtaining

$$[X_1, Y_1] = Z,$$

which is equivalent to

$$XY = ZYX.$$

As will later turn out, we only need to find a $Y \in \mathrm{SL}_n(\mathbb{Q})$ close to B_1, such that Y is conjugate to ZY in $\mathrm{GL}_n \mathbb{R}$. While it would be convenient to find a diagonal Y, we were unable to ensure it, however, it will be useful later to denote

$$Y' := \begin{bmatrix} y_1 & 0 & 0 & \cdots & 0 \\ 0 & y_2 & 0 & \cdots & 0 \\ 0 & 0 & y_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y_n \end{bmatrix}.$$

We will however achieve it partially, namely we will limit our search scope for Y to matrices with entries equal to zero except on the diagonal and in the first column i.e.

$$Y = \begin{bmatrix} y_1 & 0 & 0 & \cdots & 0 \\ q_2 & y_2 & 0 & \cdots & 0 \\ q_3 & 0 & y_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & 0 & \cdots & y_n \end{bmatrix}.$$

Claim. If

$$X_2, Y_2, X_3, Y_3, \ldots, X_g, Y_g \in \mathrm{SL}_n(\mathbb{Q})$$

are sufficiently close to

$$A_2, B_2, A_3, B_3, \ldots, A_g, B_g$$

and $y_1, \ldots, y_n \in \mathbb{Q}$ satisfying $\prod_{i=1}^{n} y_i = 1$ are sufficiently close to $\lambda_1, \ldots, \lambda_n$, then there exist unique $q_2, \ldots, q_n \in \mathbb{Q}$ such that Y lies in $\mathrm{SL}_n(\mathbb{Q})$ and $Y \sim_{\mathrm{GL}_n \mathbb{R}} ZY$. If

$$X_2, Y_2, X_3, Y_3, \ldots, X_g, Y_g$$

converge to

$$A_2, B_2, A_3, B_3, \ldots, A_g, B_g$$

and $y_1, \ldots, y_n$ converge to $\lambda_1, \ldots, \lambda_n$, then each of $q_2, \ldots, q_n$ converges to 0.

Proof of Claim. First observe that $\beta_1^+, \beta_1^-, (\alpha_1\beta_1\alpha_1^{-1})^+, (\alpha_1\beta_1\alpha_1^{-1})^-$ are 4 distinct points in $\partial_\infty(\pi_1(\Sigma_g))$. To see this consider any hyperbolic structure on Σ_g. The corresponding representation

$$\pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{R})$$

provides us with an action of $\pi_1(\Sigma_g)$ on $\mathbb{H}^2$. Any element of $\pi_1(\Sigma_g)$ is mapped to a hyperbolic element of $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2, \mathbb{R})$. Loops representing the standard

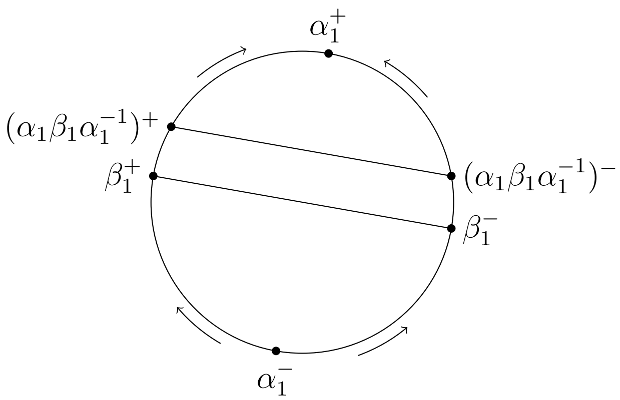


Figure 3.1: Action of α_1 on $\partial_\infty(\mathbb{H}^2)$.

generators α_1 and β_1 intersect transversely, hence by Proposition 1.3 and Proposition 1.7 in [FM11] the unique simple geodesic loops in their respective free homotopy classes also intersect once. But this means, that in the universal cover $\mathbb{H}^2$, the axes of α_1 and β_1 intersect. Hence $\alpha_1^+, \alpha_1^-, \beta_1^+, \beta_1^-$ are distinct points of $\partial_\infty(\mathbb{H}^2)$. The action of $\pi_1(\Sigma_g)$ on $\mathbb{H}^2$ extends to $\partial(\mathbb{H}^2)$. We have that

$$(\alpha_1\beta_1\alpha_1^{-1})^\pm = \alpha_1(\beta_1^\pm).$$

The element α_1 acts on the boundary by shifting every point (except the ends of its axis) towards the attracting point α_1^+ (see Figure 3.1). It follows that the axes β_1

and $\alpha_1\beta_1\alpha_1^{-1}$ do not intersect (even on the boundary), hence the endings of the axes are 4 different points.

Thanks to this observation we can invoke Lemma 3.1.3 for $\alpha_1\beta_1\alpha_1^{-1}$ and β_1, obtaining that two matrices

$$CB_1 = (\prod_{i=2}^{g}[A_i, B_i])^{-1}B_1 = A_1B_1A_1^{-1} = \rho(\alpha_1\beta_1\alpha_1^{-1})$$

and

$$CB_1E_{\bar{x}}$$

cannot be conjugate, unless $x_2, \ldots, x_n$ are all zero. We will now show that $CB_1E_{\bar{x}}$ is not conjugate to $B_1E_{\bar{x}}$, except for $x_2 = x_3 = \cdots = x_n = 0$. First note that B_1 is conjugate to CB_1 by the original surface group relation, namely $A_1B_1A_1^{-1} = CB_1$. Moreover, since

$$B_1E_{\bar{x}} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ \lambda_2x_2 & \lambda_2 & 0 & \cdots & 0 \\ \lambda_3x_3 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_nx_n & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

we note that $B_1E_{\bar{x}}$ has the same spectrum as B_1, and since it is a simple spectrum, they have the same diagonal form and are hence conjugate in $\mathrm{GL}_n\mathbb{R}$. Therefore $B_1E_{\bar{x}} \sim_{\mathrm{GL}_n\mathbb{R}} B_1 \sim_{\mathrm{GL}_n\mathbb{R}} CB_1 \sim_{\mathrm{GL}_n\mathbb{R}} CB_1E_{\bar{x}}$, with the last conjugacy holding only when $x_2, \ldots, x_n$ are all equal zero.

So far, in the proof of the Claim, we worked mainly with the original representation matrices, but now we will move to our approximations. Let us select $y_1, \ldots, y_{n-1} \in \mathbb{Q}$ close to $\lambda_1, \ldots, \lambda_{n-1}$ (the diagonal entries of B_1) and let $y_n := (y_1 \cdots y_{n-1})^{-1} \in \mathbb{Q}$. Note that y_n is close to λ_n and since $\lambda_1, \ldots, \lambda_n$ are non-zero, we can assume that $y_1, \ldots y_n$ are also non-zero. Plugging these $y_1, \ldots, y_n$ into the diagonal of Y' yields a matrix in $\mathrm{SL}_n(\mathbb{Q})$ approximating B_1. We will find $x_2, \ldots, x_n$ such that $Y'E_{\bar{x}}$ is conjugate to $ZY'E_{\bar{x}}$. We investigate the equality of characteristic polynomials:

$$\det(Y'E_{\bar{x}} - tI) = \det(ZY'E_{\bar{x}} - tI).$$

This yields a system of linear equations in $x_2, \ldots, x_n$. Since $\det(Y'E_{\bar{x}}) = \det(ZY'E_{\bar{x}}) = 1$, the equation for one of the coefficients is trivially true, so we obtain $n-1$ equations in $n-1$ variables. We consider a second, analogous linear system for

$$\det(B_1E_{\bar{x}} - tI) = \det(CB_1E_{\bar{x}} - tI).$$

The matrix $CB_1E_{\bar{x}}$ is conjugate to $B_1E_{\bar{x}}$ only if $x_2 = \cdots = x_n = 0$, hence the second system is homogeneous and has non-zero determinant. Since these two systems have close coefficients, the first system also has non-zero determinant, and hence there exists a unique solution $\bar{q} = (q_2, \ldots, q_n)$ for which $Y'E_{\bar{q}}$ has the same characteristic polynomial as $ZY'E_{\bar{q}}$. Moreover, because the second system is homogeneous, we can make $\bar{q}$ arbitrarily small, by getting better approximations of $A_2, B_2, A_3, B_3, \ldots, A_q, B_q$ and $\lambda_1, \ldots, \lambda_{n-1}$. This can be seen by applying the Cramer's rule. As these approximations improve, we have that $E_{\bar{q}} \to I$ and $ZY' \to CB_1$, hence $ZY'E_{\bar{q}} \to CB_1$. In particular, since CB_1 is an SSM, we can, by Corollary 1.1.3, assume that $ZY'E_{\bar{q}}$ is an SSM. The same argument applies to $Y'E_{\bar{q}}$ - but in fact it is an SSM for any $\bar{q} \in \mathbb{R}^n$, provided that Y' has distinct entries on the diagonal. Therefore we can invoke Lemma 1.1.5 for $ZY'E_{\bar{q}}$ and $Y'E_{\bar{q}}$ concluding that $Y'E_{\bar{q}}$ is conjugate to $ZY'E_{\bar{q}}$. The linear system for equality of characteristic polynomials of $Y'E_{\bar{x}}$ and $ZY'E_{\bar{x}}$ has rational coefficients, hence its solution $\bar{q}$ lies in $\mathbb{Q}^{n-1}$. We can then set $Y = Y'E_{\bar{q}}$, since it approximates B_1, lies in $\mathrm{SL}_n(\mathbb{Q})$, is conjugate to ZY and is of required form.

□

Back to the proof of the Main Theorem. Having already found the approximation Y of B_1, we now proceed to find an approximation X of A_1. Recall that B_1 is diagonal and $CB_1 = A_1B_1A_1^{-1}$. Since Y and ZY can be made arbitrarily close to B_1 and CB_1, respectively, we can apply Lemma 1.1.2 to each of them. Note that they have a common diagonal form equal to Y'. Hence, there are $R, S \in \mathrm{GL}_n \mathbb{R}$ such that

$$Y = RY'R^{-1} \quad \text{and} \quad ZY = SY'S^{-1}$$

with R and S close to I and A_1 respectively. For $X' := SR^{-1}$ we have

$$X'YX'^{-1} = (SR^{-1})Y(SR^{-1})^{-1} = S(R^{-1}YR)S^{-1} = SY'S^{-1} = ZY,$$

with $X' \in \mathrm{GL}_n \mathbb{R}$ close to A_1.

We will now construct a rational approximation of the conjugating matrix, that still conjugates Y to ZY. First, observe that there exists a matrix $Q \in \mathrm{GL}_n \mathbb{Q}$ satisfying $QY = ZYQ$. Indeed, this equation can be viewed as a system of linear equations in the entries of Q, with coefficients in $\mathbb{Q}$. Hence rational solutions are dense in the set of real solutions. Consequently, we can choose Q lying arbitrarily close to X', and therefore close to A_1. While the determinant of Q may not be exactly 1, it will be close to 1, because Q approximates $A_1 \in \mathrm{SL}_n(\mathbb{R})$. In the next step, we will replace Q with a nearby matrix lying in $\mathrm{SL}_n(\mathbb{Q})$. Recall that

$$Y = \begin{bmatrix} y_1 & 0 & 0 & \cdots & 0 \\ q_2 & y_2 & 0 & \cdots & 0 \\ q_3 & 0 & y_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & 0 & \cdots & y_n \end{bmatrix}.$$

Let

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{q_2}{y_1 - y_2} & 1 & 0 & \cdots & 0 \\ \frac{q_3}{y_1 - y_3} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{q_n}{y_1 - y_n} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$F = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\det(Q)} \end{bmatrix}.$$

We have that

$$Y = EY'E^{-1},$$

and

$$FY'F^{-1} = Y',$$

therefore

$$\begin{aligned} ZY &= QYQ^{-1} \\ &= QEY'E^{-1}Q^{-1} \\ &= QEFY'F^{-1}E^{-1}Q^{-1} \\ &= Q(EFE^{-1})\, EY'E^{-1}\, (EFE^{-1})^{-1}Q^{-1} \\ &= (QEFE^{-1})\, Y\, (QEFE^{-1})^{-1}. \end{aligned}$$

Furthermore, we have

$$\det(QEFE^{-1}) = \det(Q)\det(E)\det(F)\det(E)^{-1} = \det(Q) \cdot 1 \cdot \det(Q)^{-1} \cdot 1 = 1.$$

The matrix $QEFE^{-1}$ has rational entries and remains close to Q, provided that $q_2, \ldots, q_n$ are small relative to $y_1 - y_2, \ldots y_1 - y_n$, which can be made as small as desired, because $y_1, \ldots, y_n$ approximate distinct real numbers $\lambda_1, \ldots, \lambda_n$. Therefore, we can set $X_1 = X = QEFE^{-1}$.

□

Chapter 4

Simple Goldman twists

In [Gol86] Goldman discusses Hamiltonian flows on $\mathrm{Hom}(\pi_1(\Sigma_g), G)/G$, with G a Lie group satisfying some additional conditions, satisfied by both $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{PSL}_n(\mathbb{R})$. A special case of this construction is a Hamiltonian twist flow - a flow associated to a 'length function' - a function whose value at point $[\rho]$ is a conjugation invariant function of $\rho(\alpha)$, for a fixed closed curve α. For a simple α these flows lift to $\mathrm{Hom}(\pi_1(\Sigma_g), G)$. These will be of our central interest as they will allow us to rationally approximate representations.

4.1 Definition and properties

We restate the definition of the Goldman flow on $\mathrm{Hom}(\pi_1(\Sigma_g), G)$ (see [Gol86, Section 4]). Let G be any group and take any representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g, x_0), G)$ and an essential oriented simple closed curve α in Σ_g based at x_0. We consider two cases:

Non-separating case. Assume the curve α is non-separating. Then $\Sigma_g \mid \alpha$ is of type $(g-1, 2)$. Hence, the change of coordinates principle ([FM11, Section 1.3.3]) there is an orientation-preserving homeomorphism $f\colon \Sigma_g \to \Sigma_g$ mapping the standard generator α_1 to α. Since $f \circ \alpha_1 = \alpha$, we automatically have $f(x_0) = x_0$. Denote $\bar{\alpha}_i := f(\alpha_i)$ and $\bar{\beta}_i := f(\beta_i)$. Since f induces an automorphism of $\pi_1(\Sigma_g, x_0)$, the curves

$$\bar{\alpha}_1, \bar{\beta}_1, \ldots, \bar{\alpha}_g, \bar{\beta}_g$$

are generators of $\pi_1(\Sigma_g, x_0)$, subject only to the standard surface group relation

$$\prod_{i=1}^{g} [\bar{\alpha}_i, \bar{\beta}_i] = 1.$$

Take any Z from the group centralizer $Z_G(\rho(\alpha))$. Define a new representation ρ' by

$$\rho'(\bar{\alpha}_i) = \rho(\bar{\alpha}_i), \qquad \rho'(\bar{\beta}_i) = \begin{cases} \rho(\bar{\beta}_i)Z & \text{if } i = 1 \\ \rho(\bar{\beta}_i) & \text{if } i \neq 1 \end{cases} \tag{*}$$

where $i \in \{1, \ldots, g\}$, $Z \in Z_G\big(\rho(\alpha)\big)$ and $\bar{\alpha}_1 = \alpha$.

The surface group relation is still satisfied, because

$$\begin{aligned}\prod_{i=1}^{g}\left[\rho'(\bar\alpha_i),\rho'(\bar\beta_i)\right] &= \left[\rho'(\bar\alpha_1),\rho'(\bar\beta_1)\right]\prod_{i=2}^{g}\left[\rho'(\bar\alpha_i),\rho'(\bar\beta_i)\right]\\ &= \left[\rho(\bar\alpha_1),\rho(\bar\beta_1)Z\right]\prod_{i=2}^{g}\left[\rho(\bar\alpha_i),\rho(\bar\beta_i)\right]\\ &= \rho(\bar\alpha_1)\rho(\bar\beta_1)Z\rho(\bar\alpha_1)^{-1}Z^{-1}\rho(\bar\beta_1)^{-1}\prod_{i=2}^{g}\left[\rho(\bar\alpha_i),\rho(\bar\beta_i)\right]\\ &= \rho(\bar\alpha_1)\rho(\bar\beta_1)\rho(\bar\alpha_1)^{-1}\rho(\bar\beta_1)^{-1}\prod_{i=2}^{g}\left[\rho(\bar\alpha_i),\rho(\bar\beta_i)\right]\\ &= \prod_{i=1}^{g}\left[\rho(\bar\alpha_i),\rho(\bar\beta_i)\right] \;=\; 1.\end{aligned}$$

We will show that ρ' is independent of the choice of f. Let g be another orientation-preserving homeomorphism with $g(\alpha_1)=\alpha$, and denote by

$$\widetilde\alpha_i := g(\alpha_i), \qquad \widetilde\beta_i := g(\beta_i)$$

the images of the standard generators under g. Let ρ'' be the corresponding representation for g. We will show that $\rho'=\rho''$ by checking that they agree on all $\widetilde\alpha_i, \widetilde\beta_i$. The cut surface $\Sigma_g \mid \alpha$ contains 2 curves that map to α in Σ_g via the standard map $\iota\colon \Sigma_g \mid \alpha \to \Sigma_g$. Recall from Section 1.2, that we denote these curves by α^l, α^r, where α^l is the left copy of α. In the cut surface the curve $\bar\beta_1$ is no longer closed and connects two copies x_0^l, x_0^r of the base point x_0 for $\pi_1(\Sigma_g)$. The point x_0^l lies at the beginning of $\bar\beta_1$ and is contained in α^l. Observe that each of $\bar\alpha_2, \bar\beta_2, \ldots, \bar\alpha_g, \bar\beta_g$ is an image via ι of a unique loop based at x_0^l (see Figure 1.1), which we will, slightly abusively, denote by the same symbol $\bar\alpha_i$ or $\bar\beta_i$. Then $\pi_1(\Sigma_g \mid \alpha) = \pi_1(\Sigma_g \mid \alpha, x_0^l)$ is generated by $\alpha^l, \bar\beta_1\alpha^r\bar\beta_1^{-1}, \bar\alpha_2, \bar\beta_2, \ldots, \bar\alpha_g, \bar\beta_g$. Because each of

$$\widetilde\alpha_2, \widetilde\beta_2, \ldots, \widetilde\alpha_g, \widetilde\beta_g$$

is an image via ι of some loop in $\Sigma_g \mid \alpha$, each of them can be realized as a word in

$$\iota(\alpha^l), \iota(\bar\beta_1\alpha^r\bar\beta_1^{-1}), \bar\alpha_2, \bar\beta_2, \ldots, \bar\alpha_g, \bar\beta_g \in \pi_1\Sigma_g.$$

But $\rho(\gamma)=\rho'(\gamma)$ for any $\gamma \in \{\bar\alpha_1, \bar\beta_1\bar\alpha_1\bar\beta_1^{-1}, \bar\alpha_2, \bar\beta_2, \ldots, \bar\alpha_g, \bar\beta_g\} \subset \pi_1\Sigma_g$. Hence we have that

$$\rho''(\gamma)=\rho(\gamma)=\rho'(\gamma)$$

for $\gamma \in \{\widetilde\alpha_2, \widetilde\beta_2, \ldots, \widetilde\alpha_g, \widetilde\beta_g\}$. Since

$$\rho'(\alpha)=\rho(\alpha)=\rho''(\alpha)$$

and $\alpha=\widetilde\alpha_1$, it remains to show that

$$\rho''(\widetilde\beta_1)=\rho'(\widetilde\beta_1).$$

The loop $\widetilde{\beta}_1\bar{\beta}_1^{-1}$ is based homotopic to a curve disjoint from α, hence, as above, it is a word in $\bar{\alpha}_1, \bar{\beta}_1\bar{\alpha}_1\bar{\beta}_1^{-1}, \bar{\alpha}_2, \bar{\beta}_2, \ldots, \bar{\alpha}_g, \bar{\beta}_g$, and hence

$$\rho'(\widetilde{\beta}_1\bar{\beta}_1^{-1}) = \rho(\widetilde{\beta}_1\bar{\beta}_1^{-1}).$$

Therefore

$$\rho'(\widetilde{\beta}_1) = \rho'(\widetilde{\beta}_1\bar{\beta}_1^{-1}\bar{\beta}_1) = \rho'(\widetilde{\beta}_1\bar{\beta}_1^{-1})\rho'(\bar{\beta}_1) = \rho(\widetilde{\beta}_1\bar{\beta}_1^{-1})\rho(\bar{\beta}_1)Z = \rho(\widetilde{\beta}_1)Z = \rho''(\widetilde{\beta}_1).$$

Separating case. If α is separating, then $\Sigma_g \mid \alpha$ consists of two surfaces of types $(k, 1)$ (the component to the right of α) and $(g-k, 1)$ for some $k \in \{1, \ldots, g-1\}$. Let σ_k be the standard $(k, g-k)$-partitioning curve based at x_0. Recall that in the fundamental group $\sigma_k = \prod_{i=1}^{k} [\alpha_i, \beta_i]$. There exists an orientation-preserving homeomorphism $f : \Sigma_g \to \Sigma_g$ with $f(x_0) = x_0$ and $f(\sigma_k) = \alpha$. Let $\bar{\alpha}_i = f(\alpha_i), \bar{\beta}_i = f(\beta_i)$. We will define a new representation ρ'. For any $Z \in Z_G\big(\rho(\alpha)\big)$ and $i \in \{1, \ldots, g\}$ we set:

$$\rho'(\bar{\alpha}_i) = \begin{cases} Z\rho(\bar{\alpha}_i)Z^{-1} & \text{if } i \leq k \\ \rho(\bar{\alpha}_i) & \text{if } i > k \end{cases} \qquad \rho'(\bar{\beta}_i) = \begin{cases} Z\rho(\bar{\beta}_i)Z^{-1} & \text{if } i \leq k \\ \rho(\bar{\beta}_i) & \text{if } i > k \end{cases} \tag{**}$$

Since Z commutes with $\alpha = f(\sigma_k) = \prod_{i=1}^{k} \big[\rho(\bar{\alpha}_i), \rho(\bar{\beta}_i)\big]$ the following calculation shows that ρ' is a homomorphism:

$$\begin{aligned}
\prod_{i=1}^{g} \big[\rho'(\bar{\alpha}_i), \rho'(\bar{\beta}_i)\big] &= \prod_{i=1}^{k} \big[\rho'(\bar{\alpha}_i), \rho'(\bar{\beta}_i)\big] \prod_{i=k+1}^{g} \big[\rho'(\bar{\alpha}_i), \rho'(\bar{\beta}_i)\big] \\
&= \prod_{i=1}^{k} \big[Z\rho(\bar{\alpha}_i)Z^{-1}, Z\rho(\bar{\beta}_i)Z^{-1}\big] \prod_{i=k+1}^{g} \big[\rho(\bar{\alpha}_i), \rho(\bar{\beta}_i)\big] \\
&= Z \prod_{i=1}^{k} \big[\rho(\bar{\alpha}_i), \rho(\bar{\beta}_i)\big] Z^{-1} \prod_{i=k+1}^{g} \big[\rho(\bar{\alpha}_i), \rho(\bar{\beta}_i)\big] \\
&= \prod_{i=1}^{k} \big[\rho(\bar{\alpha}_i), \rho(\bar{\beta}_i)\big] \prod_{i=k+1}^{g} \big[\rho(\bar{\alpha}_i), \rho(\bar{\beta}_i)\big] \\
&= \prod_{i=1}^{g} \big[\rho(\bar{\alpha}_i), \rho(\bar{\beta}_i)\big] \;=\; 1.
\end{aligned}$$

Similarly to the non-separating case, we will show that ρ' does not depend on the choice of f. Let g be another orientation-preserving homeomorphism mapping σ_k to α and let ρ'' denote corresponding representation. As above, denote the images of the standard generators under g by $\widetilde{\alpha}_i, \widetilde{\beta}_i$. As this is similar to the previous case, we will simply state this time that each of

$$\widetilde{\alpha}_1, \widetilde{\beta}_1, \ldots, \widetilde{\alpha}_k, \widetilde{\beta}_k$$

can be realized as a word in

$$\bar{\alpha}_1, \bar{\beta}_1, \ldots, \bar{\alpha}_k, \bar{\beta}_k$$

and each of

$$\widetilde{\alpha}_{k+1} \widetilde{\beta}_{k+1}, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_g$$

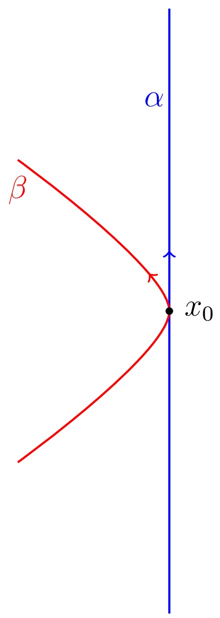
can be realized as a word in

$$\bar{\alpha}_{k+1}, \bar{\beta}_{k+1}, \ldots, \bar{\alpha}_g, \bar{\beta}_g.$$

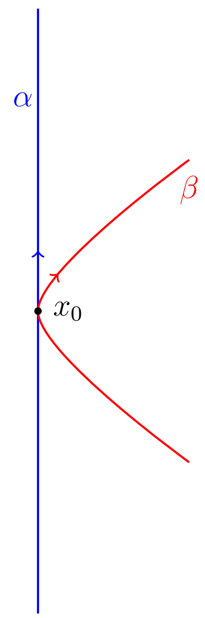
It follows that $\rho''(\gamma) = \rho'(\gamma)$ for any $\gamma \in \{\widetilde{\alpha}_1, \widetilde{\beta}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_g\}$ and hence for all $\gamma \in \pi_1(\Sigma_g)$.

Definition 4.1.1. We will call the representation ρ' given by (*) or (**) a *simple Goldman twist* of ρ along α by Z and we will denote it by $T^Z_\alpha(\rho)$.

It is easy to check that our definition is equivalent to one in [Gol86]. Goldman prove that these flows cover the Fenchel-Nielsen flows (see [Gol86, p. 4.11]).



(a) Curve β touches α on the left.



(b) Curve β touches α on the right.

Figure 4.1: Curves intersecting only once with $i(\alpha, \beta) = 0$.

Lemma 4.1.2. *Let α, β be essential oriented simple closed non-separating curves intersecting only at x_0. Let $i(\alpha, \beta)$ denote the algebraic intersection number of α and β. Then $i(\alpha, \beta) \in \{-1, 0, 1\}$ and*

$$\left(T^Z_\alpha(\rho)\right)(\beta) = \begin{cases} \rho(\beta)Z & \text{if} \quad i(\alpha,\beta) = 1 \\ Z^{-1}\rho(\beta) & \text{if} \quad i(\alpha,\beta) = -1 \\ \rho(\beta) & \text{if} \quad i(\alpha,\beta) = 0 \text{ and } \beta \text{ touches } \alpha \text{ on the left} \\ Z^{-1}\rho(\beta)Z & \text{if} \quad i(\alpha,\beta) = 0 \text{ and } \beta \text{ touches } \alpha \text{ on the right.} \end{cases}$$

Proof. Denote the representation after the twist by $\rho' = T^Z_\alpha(\rho)$. If the intersection number is ± 1, then, as in the definition of a simple Goldman twist, we can choose any orientation-preserving homeomorphism mapping α_1 to α. We choose one that also maps $\beta_1^\pm$ to β (the exponent depending on the sign of $i(\alpha, \beta)$). Such a homeomorphism exists by the change of coordinates principle - for details see [FM11, section 1.3.3]. Hence equation (*) dictates that $\rho'(\beta^{\pm 1}) = \rho(\beta^{\pm 1})Z$, which ends the case of non-zero intersection number. If the intersection number is 0 there are two cases - see Figure 4.1. In either case β is an image of a unique loop in $\Sigma_g \mid \alpha$. But only in the first case (β touches α on the left) this loop is based at x_0^l, hence represents an element of $\pi_1(\Sigma_g \mid \alpha, x_0^l)$. The first case is then a consequence of β being a word in generators of $\pi_1(\Sigma_g)$ that are not modified by the twist. For β touching α on the right, choose a curve λ intersecting α once with positive intersection number. Then $\lambda\beta\lambda^{-1}$ is the image, via the standard map $\iota \colon S \mid \alpha \to S$, of some loop based at x_0^l. Hence $\rho'(\lambda\beta\lambda^{-1}) = \rho(\lambda\beta\lambda^{-1})$. We again note that $\rho'(\lambda) = \rho(\lambda)Z$ and hence $\rho'(\lambda^{-1}) = Z^{-1}\rho(\lambda^{-1})$. Therefore

$$\begin{aligned}
\rho'(\beta) &= \rho'(\lambda^{-1})\rho'(\lambda\beta\lambda^{-1})\rho'(\lambda) \\
&= Z^{-1}\rho(\lambda^{-1})\rho(\lambda\beta\lambda^{-1})\rho(\lambda)Z \\
&= Z^{-1}\rho(\beta)Z.
\end{aligned}$$

□

Fact 4.1.3. *Let $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g, x_0), \mathrm{SL}_n(\mathbb{Q}))$ be Hitchin and let α be an essential oriented simple closed curved based at x_0. For $Z \in Z_{\mathrm{SL}_n(\mathbb{R})}(\rho(\alpha))$ and any neighborhood U of $T^Z_\alpha(\rho)$ in $\mathrm{Hom}(\pi_1(\Sigma_g, x_0), \mathrm{SL}_n(\mathbb{R}))$ we can find $Q \in Z_{\mathrm{SL}_n(\mathbb{Q})}(\rho(\alpha))$ such that $T^Q_\alpha(\rho) \in U \cap \mathrm{Hom}(\pi_1(\Sigma_g, x_0), \mathrm{SL}_n(\mathbb{Q}))$.*

Proof. By definition and continuity of Goldman twists it suffices to find a rational element of $Z_{\mathrm{SL}_n(\mathbb{R})}(\rho(\alpha))$ arbitrarily close to Z. But since ρ is Hitchin, $\rho(\alpha)$ is an SSM, and we can apply Lemma 1.1.4. □

Now that we have Fact 4.1.3 the main obstruction for approximating Hitchin representation is finding appropriate twists, so that we can reach any Hitchin representation from some rational representation. We were not able to do it in full generality, but we have partial results - namely case $n = 2$ and density of rational representations in some subset of $\mathrm{Hit}_3(\Sigma_2)$.

4.2 Applications in Teichmüller space

Corollary 4.2.4 shows that in the case $n = 2$ every Hitchin representation is rationally approximable via twists. The following Theorem is what provides us with an appropriate set of twists.

Theorem 4.2.1. *There is a collection of $9g - 9$ simple closed curves, such that every two points of the Teichmüller space can be connected by a finite number of Fenchel-Nielsen twists along these curves.*

Note that we do not assume that the curves are disjoint; as a result the corresponding Fenchel-Nielsen twists may not commute with each other. Theorem 4.2.1 is a straightforward consequence of the following two lemmas. Recall that for $t \in \mathbb{R}$ we denote by $\tau_\gamma^t(X)$ the Fenchel-Nielsen twist of X along γ by a parameter t.

Lemma 4.2.2. *There is a family of $9g-9$ simple closed curves $\kappa_1, \ldots, \kappa_{9g-9}$ in Σ_g such that*

$$\operatorname{span}\left\{\left(\frac{\partial}{\partial \tau_{\kappa_1}}\right)_X, \ldots, \left(\frac{\partial}{\partial \tau_{\kappa_{9g-9}}}\right)_X\right\} = T_X\mathcal{T}(\Sigma_g),$$

for any $X \in \mathcal{T}(\Sigma_g)$.

Proof. We begin by proving a strengthening of the classical $9g-9$ theorem. The proof of injectivity of this map is a repeat of the proof of Theorem 10.7 in [FM11], but the proof that the map is an immersion is our own. Let $X \in \mathcal{T}(\Sigma_g)$. For a closed curve α in Σ_g denote the length function by $\ell_\alpha(X)$.

Claim. There are $9g-9$ simple closed curves $\kappa_1, \ldots, \kappa_{9g-9}$ such that the map

$$X \mapsto (\ell_{\kappa_1}(X), \ldots, \ell_{\kappa_{9g-9}}(X))$$

is an injective **immersion** into $\mathbb{R}^{9g-9}$.

Proof of Claim. Let $\eta_1, \ldots, \eta_{3g-3}$ be any pants decomposition of Σ_g. We can choose simple closed curves $\gamma_1, \ldots, \gamma_{3g-3}$ so that the geometric intersection $i_{\text{geom}}(\eta_i, \gamma_j)$ is 0 if $i \neq j$ and non-zero for $i = j$. Let δ_i be the Dehn twist of γ_i about η_i, for $i \in \{1, \ldots, 3g-3\}$. Note that $i_{\text{geom}}(\eta_i, \gamma_j') = i_{geom}(\eta_i, \gamma_j)$. Let $\Theta\colon \mathcal{T}(\Sigma_g) \to \mathbb{R}^{9g-9}$ be defined by

$$\Theta(X) = (l_{\eta_1}(X), \ldots, l_{\eta_{3g-3}}(X), l_{\gamma_1}(X), \ldots, l_{\gamma_{3g-3}}(X), l_{\delta_1}(X), \ldots, l_{\delta_{3g-3}}(X)).$$

We will first show injectivity of Θ. Consider Fenchel-Nielsen coordinates of $\mathcal{T}(\Sigma_g)$ associated to the pants decomposition $\eta_1, \ldots, \eta_{3g-3}$. Take two distinct points $X, Y \in \mathcal{T}(\Sigma_g)$. We will show that $\Theta(X) \neq \Theta(Y)$. We can assume that the length coordinates of X and Y are equal, otherwise $\Theta(X)$ and $\Theta(Y)$ would differ on some of the first $3g-3$ entries. Assume that $\tau_{\eta_k}(X) \neq \tau_{\eta_k}(Y)$ for some k. We will show that either $\ell_{\gamma_k}(X) \neq \ell_{\gamma_k}(Y)$ or $\ell_{\delta_k}(X) \neq \ell_{\delta_k}(Y)$. Since γ_k and δ_k are disjoint from $\eta_1, \ldots, \hat{\eta}_k, \ldots, \eta_{3g-3}$, the twists around η_i (for $i \neq k$) do not change lengths of γ_k and δ_k. As we focus our attention on these two curves, we can therefore assume that Y differs from X only by a Fenchel-Nielsen twist along η_k. Let

$$C(t) = \ell_{\gamma_k}(\tau_{\eta_k}^t(X)), \quad D(t) = \ell_{\delta_k}(\tau_{\eta_k}^t(X)).$$

Let $s \neq 0$ be so that $\tau_{\eta_k}^s(X) = Y$. We want to show that either $C(0) \neq C(S)$ or $D(0) \neq D(s)$. By definition of δ_k we have that $D(t) = C(2\pi + t)$ for any $t \in \mathbb{R}$. Proposition 10.8 in [FM11] implies that C and D are strictly convex functions. Assume $C(0) = C(s) \leq D(0) = D(s)$ - the case $C(0) = C(s) > D(0) = D(s)$ is analogous. We will focus on a single function C, obtaining $C(0) = C(s) \leq C(2\pi) = C(2\pi + s)$. We consider multiple cases, each leading to a triple of points, with value at the middle point not strictly lower than values at the extreme points, violating strict convexity of C:

- If $s \leq -2\pi$, then $s < 2\pi + s < 2\pi$, but $C(s) \leq C(2\pi + s) = C(2\pi)$.
- If $-2\pi < s < 0$, then $0 < 2\pi + s < 2\pi$, but $C(0) \leq C(2\pi + s) = C(2\pi)$.
- If $s > 0$, then $0 < 2\pi < 2\pi + s$, but $C(0) \leq C(2\pi) = C(2\pi + s)$.

Therefore either $C(0) \neq C(s)$ or $D(0) \neq D(s)$, which proves that Θ is injective.
Our goal now is to prove that Θ is an immersion at every point $X \in \mathcal{T}(\Sigma_g)$. Since each $\ell_\gamma(X)$ is a smooth function on $\mathcal{T}(\Sigma_g)$ the map Θ is smooth. Using the Fenchel-Nielsen coordinates associated to the pants decomposition $\eta_1, \ldots, \eta_{3g-3}$ we can express the Jacobian matrix of J_Θ as a block matrix:

$$J_\Theta\left(\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3}\right) = \begin{bmatrix} I & 0 \\ * & A \\ * & B \end{bmatrix},$$

where $\ell_1, \ldots, \ell_{3g-3}$ and $\tau_1, \ldots, \tau_{3g-3}$ are the length and twist coordinates respectively and each of the entries is a $(3g-3) \times (3g-3)$ matrix. Note that in the upper left corner we have an identity matrix, because it expresses derivative of an identity map. The matrix in the upper right corner is zero, because twisting along η_i does not change the length of any of $\eta_1, \ldots, \eta_{3g-3}$. We bring our focus to matrices A and B. We have that

$$A_{ij} = \frac{d\ell_{\gamma_i}}{d\tau_j} = \frac{d\ell_{\gamma_i}}{d\tau_{\eta_j}} \quad \text{and} \quad B_{ij} = \frac{d\ell_{\delta_i}}{d\tau_j} = \frac{d\ell_{\delta_i}}{d\tau_{\eta_j}},$$

for any $1 \leq i, j \leq 3g-3$. Since twisting along η_j does not change γ_i and δ_i for $i \neq j$, both A and B are diagonal. In an analogy to C and D in the 'injectivity' part of the proof denote

$$C_k(t) := \ell_{\gamma_k}(\tau^t_{\eta_k}(X)), \quad D_k(t) := \ell_{\delta_k}(\tau^t_{\eta_k}(X)).$$

Each C_k and D_k is a strictly convex smooth function, hence $C''_k(t) > 0$ and $D''_k(t) > 0$ for any $t \in \mathbb{R}$. Therefore C'_k and D'_k can have only one root each. Since for any $t \in \mathbb{R}$ we have $C_k(2\pi + t) = D_k(t)$, we conclude that the roots of C_k and D_k are at the distance 2π from each other. Hence $C'_k(0)$ and $D'_k(0)$ cannot both be zero. This means that for each k either A_{kk} or B_{kk} is non-zero, implying that $3g-3$ last columns of $J_\Theta(\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3})$ are linearly independent, spanning some $3g-3$-dimensional $V < \mathbb{R}^{9g-9}$. Moreover first $3g-3$ columns are linearly independent and span $3g-3$-dimensional $W < \mathbb{R}^{9g-9}$, with $V \cap W = \{0\}$. Hence $\dim(V+W) = 6g-6$ and the Claim is proved. $\square$

Let $\kappa_1, \ldots, \kappa_{9g-9}$ be as in the Claim. Then at each $X \in \mathcal{T}(\Sigma_g)$ we have

$$\operatorname{span}\left\{(d\ell_{\kappa_1})_X, \ldots, (d\ell_{\kappa_{9g-9}})_X\right\} = T^*_X\mathcal{T}(\Sigma_g).$$

To convert these covectors into vectors, we require a Riemannian structure on $\mathcal{T}(\Sigma_g)$. Fortunately, Weil introduced a Kählerian metric on $\mathcal{T}(\Sigma_g)$ called the Weil-Petersson metric. We will not dig into the details, referring the reader to a survey by Wolpert in [Wol09]. Citing a result from another paper of Wolpert, namely Theorem 2.10 in [Wol82], we obtain, for a simple closed curve α, that

$$\left(\frac{\partial}{\partial \tau_\alpha}\right)^* = -i\, d\ell_\alpha,$$

where the dual is with respect to the Weil-Petersson metric. Note that this is duality on the level of real vectors and covectors and $-i$ is simply a smooth family of linear automorphisms of fibers of $T\mathcal{T}(\Sigma_g)$. Applying this result to $(d\ell_{\kappa_1})_X, \dots, (d\ell_{\kappa_{9g-9}})_X$ we obtain

$$\operatorname{span}\left\{\left(\frac{\partial}{\partial \tau_{\kappa_1}}\right)_X, \dots, \left(\frac{\partial}{\partial \tau_{\kappa_{9g-9}}}\right)_X\right\} = T_X\mathcal{T}(\Sigma_g).$$

□

Corollary 3.5 in [Wol82] states a result similar to Lemma 4.2.2, requiring only $6g-6$ curves, but in turn spanning the tangent space only locally. While this would be sufficient for a slightly modified version of Theorem 4.2.1, we felt that Lemma 4.2.2 has some value of its own.

Lemma 4.2.3. *Let M be a smooth, connected, n-dimensional manifold and let $X^1, \dots, X^k$ be vector fields spanning the tangent space of M at any point $p \in M$:*

$$\operatorname{span}\left\{X^1(p), \dots, X^k(p)\right\} = T_pM.$$

Let $\Phi^1, \dots, \Phi^k$ denote local smooth flows on M induced from $X^1, \dots, X^k$ respectively. Then for any two points $p, q \in M$, there exists a finite sequence of indices $i_1, \dots, i_m \in \{1, \dots, k\}$ and parameters $t_1, \dots, t_m \in \mathbb{R}$ such that each Φ^{i_j} is defined at $\left(\Phi^{i_{j-1}}_{t_{j-1}} \circ \dots \circ \Phi^{i_1}_{t_1}\right)(p)$ for t_j and

$$q = \left(\Phi^{i_m}_{t_m} \circ \Phi^{i_{m-1}}_{t_{m-1}} \circ \dots \circ \Phi^{i_1}_{t_1}\right)(p).$$

In other words, any point of M can be reached from any other by successively following the local flows $\Phi^1, \dots, \Phi^k$ a finite number of times.

Proof. Denote the domain of Φ^i by $D^i \subset \mathbb{R} \times M$. For $p \in M$ we define the reachable set of p as

$$\mathcal{R}(p) := \left\{\Phi^{i_m}_{t_m} \circ \dots \circ \Phi^{i_1}_{t_1}(p) \;\middle|\; m \geq 0,\ (t_j, q_{j-1}) \in D^{i_j} \text{ for all } j\right\},$$

where $q_0 := p$ and $q_j := \Phi^{i_j}_{t_j}(q_{j-1})$. For any $q \in M$ we can choose $X^{j_1}, \dots, X^{j_n}$ than span T_qM. Then the map

$$(t_1, \dots, t_n) \mapsto \left(\Phi^{j_n}_{t_n} \circ \Phi^{j_{n-1}}_{t_{n-1}} \circ \dots \circ \Phi^{j_1}_{t_1}\right)(q)$$

has an invertible derivative at 0. Therefore, by the inverse function theorem, its image contains a neighborhood of q, which shows that $\mathcal{R}(p)$ is open. On the other hand let $q \in M$ lie in closure of $\mathcal{R}(p)$. Since $\mathcal{R}(q)$ is open too, we have that $\mathcal{R}(p) \cap \mathcal{R}(q) \neq \emptyset$. Take any $r \in \mathcal{R}(p) \cap \mathcal{R}(q)$. There is finite sequence of flows mapping p to r and another mapping q to r. Reversing the path from q to r (using negative flow times) we can reach q from r and hence from p, so $q \in \mathcal{R}(p)$. Because $\mathcal{R}(p)$ is both open and closed, the connectedness of M implies that $\mathcal{R}(p) = M$.

□

Proof of Theorem 4.2.1. Take the curves $\kappa_1, \ldots, \kappa_{3g-3}$ provided by Lemma 4.2.2. By Lemma 4.2.3 any two points of $\mathcal{T}(\Sigma_g)$ can be joined by a finite composition of the corresponding Fenchel–Nielsen twists. □

Corollary 4.2.4. *Let $X \in \mathcal{T}(\Sigma_g)$ and let $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{Q}))$, with $[\rho] \in \mathcal{T}(\Sigma_g)$. Then, for any open $U \subset \mathcal{T}(\Sigma_g)$ containing X, there exists a finite sequence $T_{\gamma_1}^{Q_1}, \ldots, T_{\gamma_k}^{Q_k}$ of simple Goldman twists, each by some rational matrix Q_i, such that $\left[\left(T_{\gamma_k}^{Q_k} \circ \cdots \circ T_{\gamma_1}^{Q_1}\right)(\rho)\right] \in U$.*

Proof. By Theorem 4.2.1, there exists a sequence

$$\tau_{\gamma_1}^{t_1}, \ldots, \tau_{\gamma_k}^{t_k}$$

of Fenchel-Nielsen twists such that

$$\left(\tau_{\gamma_k}^{t_k} \circ \cdots \circ \tau_{\gamma_1}^{t_1}\right)([\rho]) = X.$$

Any Fenchel-Nielsen twist τ_γ^t lifts to a simple Goldman $\mathrm{PSL}_2(\mathbb{R})$ twist $T_{\bar\gamma}^{Z'}$ for some $\bar\gamma \in \pi_1(\Sigma_g)$ and $Z' \in Z_{\mathrm{PSL}_2(\mathbb{R})}(\rho(\bar\gamma))$ - for details see 4.11 in [Gol86]. We can again lift this $\mathrm{PSL}_2(\mathbb{R})$ twist to a simple Goldman twist in $\mathrm{SL}_2(\mathbb{R})$. Indeed Z' lifts to some $Z \in \mathrm{SL}_2(\mathbb{R})$, as we have that $\left[Z_{\mathrm{SL}_2(\mathbb{R})}(A)\right] = Z_{\mathrm{PSL}_2(\mathbb{R})}([A])$ for any $A \in \mathrm{SL}_2(\mathbb{R})$.[1] Let $U_k := U$. By continuity of Fenchel-Nielsen twists, we can choose open $U_{k-1}, \ldots, U_1$, so that

$$\tau_{\gamma_i}^{t_i}[U_{i-1}] \subseteq U_i, \quad \tau_{\gamma_1}^{t_1}(\rho) \in U_1$$

for $i \in \{1, \ldots, k\}$. Let us take the lift $T_{\bar\gamma_1}^{Z_1}$ of $\tau_{\gamma_1}^{t_1}$ and apply Fact 4.1.3 to $\rho, T_{\bar\gamma_1}^{Z_1}$ and open set $V_1 := \{\sigma \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R})) \mid [\sigma] \in U_1\}$. This yields a matrix $Q_1 \in Z_{\mathrm{SL}_2(\mathbb{Q})}(\rho(\bar\gamma_1))$ approximating Z_1 and $\mathrm{SL}_2(\mathbb{Q})$-representation

$$\rho_1 := T_{\bar\gamma_1}^{Q_1}(\rho),$$

whose projection to $\mathcal{T}(\Sigma_g)$ lies in U_1. Let us now take a lift $T_{\bar\gamma_2}^{Z_2}$ of $\tau_{\gamma_2}^{t_2}$. Since $\left[T_{\bar\gamma_2}^{Z_2}(\rho_1)\right] \in U_2$, we have that $V_2 := \{\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R})) \mid [\rho] \in U_2\}$ is an open set containing $T_{\bar\gamma_2}^{Z_2}(\rho_1)$. Hence, by Fact 4.1.3 there is a rational Q_2 such that the representation

$$\rho_2 := T_{\bar\gamma_2}^{Q_2}(\rho_1) \in V_2$$

is rational. We repeat the process $k-2$ more times, reaching an $\mathrm{SL}_2(\mathbb{Q})$-representation ρ_k with $[\rho_k] \in U$. □

Since every point of $\mathcal{T}(\Sigma_g)$ can be lifted to a representation in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R}))$, Corollary 4.2.4 shows that every point of $\mathcal{T}(\Sigma_g)$ is rationally approximable - compare [Tak71] and Theorem 3.1.4.

[1]The only obstruction for $\left[Z_{\mathrm{SL}_n(\mathbb{K})}(A)\right] = Z_{\mathrm{PSL}_n(\mathbb{K})}([A])$ is the existence of a $Z \in \mathrm{SL}_n(\mathbb{K})$ such that $[Z, A] = kI \in \mathrm{SL}_n(\mathbb{K})$ for $k \neq 1$. For $n = 2$ and $K = \mathbb{R}$ it would mean that $[Z, A] = -I$. Such a Z does not exist for any $A \in \mathrm{SL}_2(\mathbb{R})$ - the choice of the field and the dimension is crucial, because a pair A, Z with $[A, Z] = -I$ exists for $\mathrm{SL}_4(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$. In our case, proving that there is no Z with $[Z, A] = -I$ is particularly simple, as the image of ρ contains only matrices with non-zero trace. If $ZAZ^{-1} = -A$ we would have that $\operatorname{tr} ZAZ^{-1}$ is equal to both $\operatorname{tr} A$ and $-\operatorname{tr} A$ contradicting $\operatorname{tr} A \neq 0$.

An invariant for $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(K))/\mathrm{SL}_2(K)$, called the Witt class, was introduced by Nekovar in [Nek91]. This class is invariant under simple Goldman twists by a matrix with entries in K (see the discussion at the beginning of Section 5 and Corollary 5.6 in [DJ24a]), hence we obtain:

Corollary 4.2.5. *The* $\mathrm{SL}_2(\mathbb{R})$ *Witt class is constant on every component of*

$$\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R}))^{Hit}.$$

Let

$$Q = \left(\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R}))^{Hit}\right) \cap \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{Q}))\right)$$

If the $\mathrm{SL}_2(\mathbb{Q})$ *Witt class attains a value v on some $\rho \in Q$, then it is equal to v on a dense subset of the component of* $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R}))^{Hit}$ *containing ρ.*

We note that the first part is already known since the Witt class for $\mathrm{SL}_2(\mathbb{R})$ is equal to the Euler class (see [DJ24b]).

Proof. Denote $R_0 = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R}))^{Hit}$. Take any $\rho, \sigma \in R_0$ and apply Theorem 4.2.1 to their images $[\rho], [\sigma] \in \mathcal{T}(\Sigma_g)$. The composition of simple Goldman twist covering the obtained Fenchel-Nielsen twists, maps ρ to some $\sigma' \in R_0$ with $[\sigma'] = [\sigma]$ in $\mathcal{T}(\Sigma_g)$. Let φ be the projection $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R})) \to \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))$. We have that

$$\varphi(\sigma) = A\varphi(\sigma')A^{-1}$$

for some $A \in \mathrm{PSL}_2(\mathbb{R})$. For a lift $\bar{A}$ of A to $\mathrm{SL}_2(\mathbb{R})$, we have

$$\varphi(\sigma) = \varphi(\bar{A}\sigma'\bar{A}^{-1}).$$

This means that

$$\sigma(\alpha_i) = \pm\bar{A}\sigma'(\alpha_i)\bar{A}^{-1}$$

and

$$\sigma(\beta_j) = \pm\bar{A}\sigma'(\beta_j)\bar{A}^{-1}$$

for all α_i, β_i. But we can then map σ' to σ applying a composition of simple Goldman twists by $-I$ around α_i (resp. β_i) for every i such that $\sigma(\beta_i) \neq \bar{A}\sigma'(\beta_i)\bar{A}^{-1}$ (resp. $\sigma(\alpha_i) \neq \bar{A}\sigma'(\alpha_i)\bar{A}^{-1}$). Since the Witt class is invariant under twists and conjugating we have proved the real case.

We now move to the rational case. Take a rational Hitchin representation (i.e. lying in Q) ρ and any Hitchin representation σ. Select an open set U around σ. By Lemma 4.2.2 the tangent space at $[\sigma]$ is spanned by derivatives of some $6g - 6$ Fenchel-Nielsen twists. Therefore these twists parametrize some neighborhood V of $[\sigma]$. Moreover we can lift to ρ each of these twists - so that we obtain $6g - 6$ simple Goldman twist curves passing through ρ. Since the projection map is smooth their derivatives at ρ also span a $6g-6$-dimensional subspace of $T_\rho \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}_2(\mathbb{R}))$. We can, locally, choose a $6g-6$-dimensional submanifold M tangent to this subspace. For M small enough, there is the projection map $M \to W \subset V$ is a diffeomorphism. By possibly shrinking M, we can assume that $M \subseteq U$. We then invoke Lemma 4.2.4

for ρ, σ and W obtaining a rational Hitchin representation ρ' with the same Witt class as ρ and with $[\rho'] \in V$. Therefore there is $\rho'' \in M$ with

$$\rho'' = A\rho' A^{-1}$$

for some $A \in \mathrm{SL}_2(\mathbb{R})$. By Lemma 3.1 we can take $\mathrm{SL}_n(\mathbb{Q})$-approximation Q of A such that $Q\rho' Q^{-1} \in U$. The Witt classes of $Q\rho' Q^{-1}$ and ρ are equal.

□

Chapter 5

Explicit computation in $\mathrm{Hit}_3(\Sigma_2)$

Long, Reid and Thistlewaite have discovered a remarkable family of triangle group representations

Theorem 5.0.1 ([LRT11], Theorem 1.1)**.** *The family of representations of the triangle group*

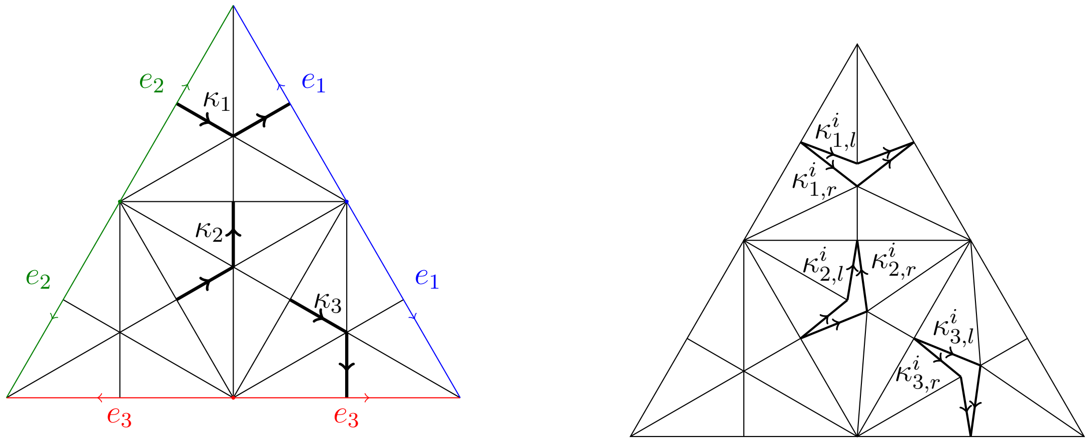
$$\Delta(3,3,4) \;=\; \left\langle\, a,b \;\middle|\; a^3 = b^3 = (ab)^4 = 1 \right\rangle,$$

given by

$$a \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad b \mapsto \begin{bmatrix} 1 & 2-t+t^2 & 3+t^2 \\ 0 & -2+2t-t^2 & -1+t-t^2 \\ 0 & 3-3t+t^2 & (-1+t)^2 \end{bmatrix}$$

are discrete and faithful for every $t \in \mathbb{R}$. Moreover, for all integral values of t the image groups are nonconjugate subgroups of $\mathrm{SL}_3(\mathbb{Z})$ *which are Zariski dense in* $\mathrm{SL}_3(\mathbb{R})$.

It will be of our interest, as for rational t this defines a representation into $\mathrm{SL}_3(\mathbb{Q})$.



(a) Triangulation of a tetrahedron showing the curves κ_1, κ_2, and κ_3.

(b) Triangulation of T_i - surface of type $(0,3)$.

Figure 5.1: Construction of a triangulation of S.

We will compute a monomorphism $\pi_1(\Sigma_g) \to \Delta(3,3,4)$. Let us triangulate the boundary of a tetrahedron as in Figure 5.1a. Note that both edges e_i in Figure 5.1a are identified for each i. We now take two copies T_1, T_2 of this tetrahedron cut along open oriented paths $\kappa_1, \kappa_2, \kappa_3$. Denote the closure of the left (right) copy of κ_i in T_j as $\kappa^j_{i,l}$ ($\kappa^j_{i,r}$) - see Figure 5.1. Let S denote the space obtained by taking $T_1 \sqcup T_2$ and identifying each closed path $\kappa^j_{i,l}$ with $\kappa^{1-j}_{i,r}$.

Routine check shows that this is a compact two-dimensional closed oriented manifold. Moreover

$$\chi(S) = \chi(T_1) + \chi(T_2) - \chi\left(\bigcup_{\substack{i\in\{1,2,3\}\\ s\in\{l,r\}}} \kappa^1_{i,s}\right) = -1 - 1 - 0 = -2 = 2 - 2\cdot 2,$$

hence S is homeomorphic to Σ_2. The triangulations of T_1 and T_2 give rise to a triangulation of S via 48 triangles. Each of these triangles has two vertices of degree 6 and one vertex of degree 8. There is, up to hyperbolic isometry, a unique (geodesic) triangle in $\mathbb{H}^2$ with angles $(\pi/3, \pi/3, \pi/4)$. Thus the triangulation of S provides us with 48-hyperbolic triangles and a pasting scheme to apply Theorem 1.3.5 from [Bus10], hence obtaining a complete hyperbolic structure on S. We choose a basepoint x_0 as in Figure 5.2. We can then lift the triangulation to a tiling of the universal cover $\mathbb{H}^2$ of S by $(\pi/3, \pi/3, \pi/4)$-triangles. Choose a lift $\bar{R}$ of the triangle R containing x_0, and denote the lifts of vertices A, B (see Figure 5.4) lying in $\bar{R}$ as $\bar{A}, \bar{B}$. Denote the rotation by $+\frac{2}{3}\pi$ (we assume the covering map is orientation-preserving) around $\bar{A}$ by a and the rotation by $+\frac{2}{3}\pi$ around $\bar{B}$ by b. The group of orientation-preserving symmetries of this tiling is

$$\Delta(3,3,4) \;=\; \left\langle\, a, b \;\middle|\; a^3 = b^3 = (ab)^4 = 1 \right\rangle,$$

with a, b in the presentation corresponding to (resp.) a, b described above geometrically.

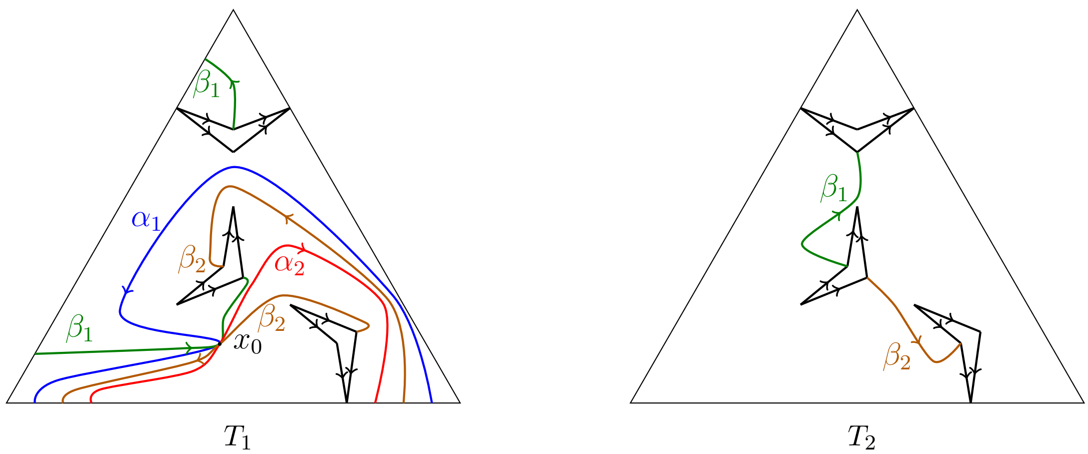


Figure 5.2: Curves $\alpha_1, \beta_1, \alpha_2, \beta_2$ on the surface S.

But the fundamental group of S acts on $\mathbb{H}^2$ by orientation preserving isometries, while also preserving the tiling, hence giving us the required monomorphism.

Select curves $\alpha_1, \beta_1, \alpha_2, \beta_2$ as in Figure 5.2. We will show that

$$\pi_1(S) = \langle \alpha_1, \beta_1 \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] \rangle \,,$$

with α_i, β_i in the presentation corresponding to (the homotopy classes of) the curves in Figure 5.2. Consider the curve δ that starts at a point p near x_0 and travels parallel to α_1 on its left side - see Figure 5.3. Because the surface S is orientable

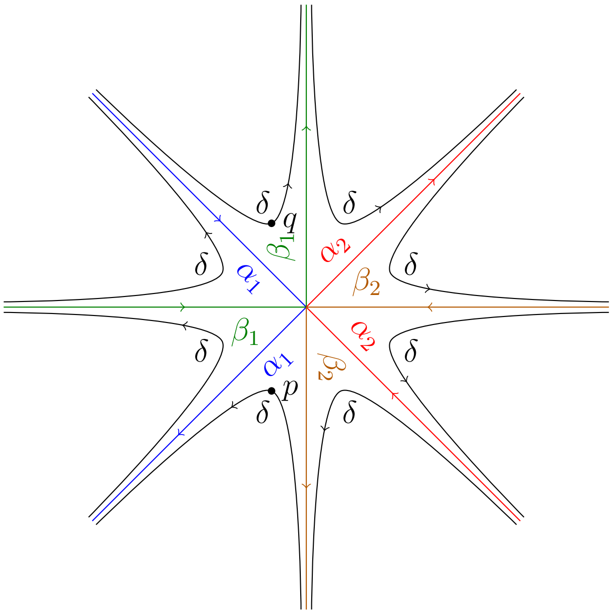


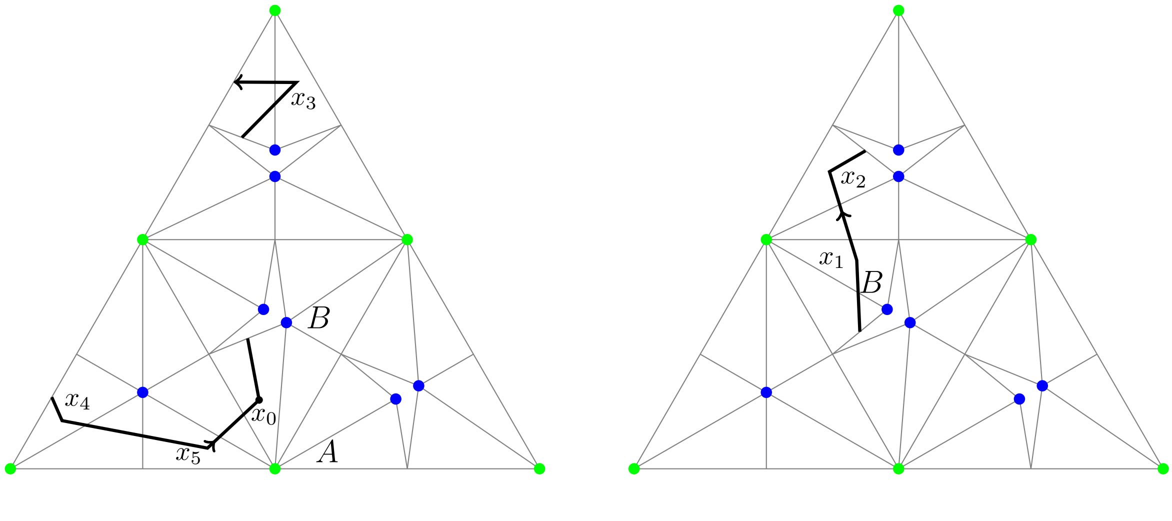
Figure 5.3: Curves $\alpha_1, \beta_1, \alpha_2, \beta_2$ and δ near x_0.

and $\alpha_1, \beta_1, \alpha_2, \beta_2$ intersect only at x_0 the curve δ can travel this way up until reaching a point q near x_0 and then begins to travel on the left side of β_1. This construction leads to a curve δ looking as in Figure 5.3. It runs, in sequence, along the left side of $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \alpha_2, \beta_2, \alpha_2^{-1}, \beta_2^{-1}$ and separates S into some surface S' on its left side and a union U of tubular neighborhoods of $\alpha_1, \beta_1, \alpha_2, \beta_2$. The surface U is orientable, has one boundary component and, since it is homotopy equivalent to a bouquet of 4 circles, has Euler characteristic equal to -3. A simple calculation of the Euler characteristic shows that S' is a two dimensional disk. Therefore, by van Kampen's theorem

$$\begin{aligned}\pi_1(S) &= \pi_1(S') *_\delta \pi_1(U) \\ &= \{1\} *_\delta F_{\{\alpha_1,\beta_1,\alpha_2,\beta_2\}} \\ &= \big\langle \alpha_1, \beta_1, \alpha_2, \beta_2 \;\big|\; [\alpha_1, \beta_1][\alpha_2, \beta_2] = 1 \big\rangle,\end{aligned}$$

where $F_{\{\alpha_1,\beta_1,\alpha_2,\beta_2\}}$ denotes the free group with free generators $\alpha_1, \beta_1, \alpha_2, \beta_2$.

We now need to determine the images of $\alpha_1, \beta_1, \alpha_2, \beta_2$ in the triangle group $\Delta(3, 3, 4)$. We will show a part of the computation for β_1 and leave the rest for the reader to verify. Let $\pi \colon \mathbb{H}^2 \to \Sigma_g$ denote the covering map. Choose a lift $\bar{x}_0$ of x_0 and lift β_1 to $\bar{\beta}_1$ beginning at $\bar{x}_0$. We need to find the (unique) element of $\Delta(3, 3, 4)$ mapping the beginning of $\bar{\beta}_1$ (i.e. $\bar{x_0}$) to the end of $\bar{\beta}_1$. We color the degree 6 vertices of the triangulation of S by two colors: blue and green as in Figure 5.4. It is a unique coloring such that A is green, B is blue and every triangle has exactly one green and one blue vertex. We also lift this coloring to a coloring of degree 6 vertices of the tiling of $\mathbb{H}^2$. For a degree 6 vertex of either the triangulation of S or



T_1 T_2

Figure 5.4: Vertices A, B and path β.

the tiling of $\mathbb{H}^2$, denote by D_v the union of triangles containing v - note that D_v is convex. The covering map π restricted to $D_{\bar{v}}$ is an isometry onto D_v, for any lift $\bar{v}$ of a colored vertex v. Let σ_v denote the rotation of D_v around v by $2\pi/3$. Since π preserves orientation we have

$$\pi \circ \sigma_v = \sigma_{\pi(v)}$$

for any colored vertex of the tiling. We have found points $x_1, \dots, x_5 \in S$ (see Figure 5.4 such that:

- each x_i lies in an interior of some triangle
- the triangles containing x_i and x_{i+1} share exactly one colored vertex v
- $x_{i+1} = \sigma_v(x_i)$ or $x_{i+1} = \sigma_v^2(x_i)$

for each $i \in \{0, 1, \dots, n\}$ and $x_6 = x_0$. Moreover the curve β obtained by concatenating unique geodesics between x_i and x_{i+1} is based homotopic to β_1. Now that we have β, we can consider its lift $\bar{\beta}$ starting at $\bar{x}_0$, and let $\bar{x}_i$ denote the lift of x_i contained in $\bar{\beta}$ with $\bar{x}_6$ being the endpoint. Since β is based homotopic to β_1, the curves $\bar{\beta}$ and $\bar{\beta}_1$ both begin at $\bar{x}_0$ and end at $\bar{x}_6$. Notice that $x_1 = \sigma_B^2(x_0)$. Since $\bar{B}$ is the blue vertex of the unique triangle containing $\bar{x}_0$, the lift of the geodesic segment between x_0 and x_1 starting at $\bar{x}_0$ is a segment inside $D_{\bar{B}}$. By the definition of $\bar{\beta}$ the endpoint of this segment is exactly $\bar{x}_1$, but is also equal to $\sigma_{\bar{B}}(x_0)$, because $\pi(\sigma_{\bar{B}}^2(x_0)) = \sigma_B^2(x_0) = x_1$. But $\sigma_{\bar{B}}$ is a restriction of $b \in \Delta(3, 3, 4)$, hence

$$\bar{x}_1 = b^2(\bar{x}_0).$$

Now let A_1 be the green vertex of the triangle containing x_1 and note that $x_2 = \sigma_{A_1}(x_1)$. Denote the green vertex of the triangle containing $\bar{x}_1$ by $\bar{A}_1$. Similarly as before we reason that $\bar{x}_2$ lies in $D_{\bar{A}_2}$ and is equal to $\sigma_{\bar{A}_1}(\bar{x}_1)$. But since b^2 maps $\bar{x}_0$ to

$\bar{x}_1$ it maps the whole triangle R (containing $\bar{x}_0$) to the triangle containing $\bar{x}_1$, hence $\bar{A}_1 = b^2(\bar{A})$ and $\sigma_{\bar{A}_1} = \sigma_{b^2\bar{A}} = b^2\sigma_{\bar{A}}b^{-2}$. Since $\sigma_{\bar{A}}$ is a restriction of a, we have that

$$\bar{x}_2 = \sigma_{\bar{A}_1}(\bar{x}_1) = b^2\sigma_{\bar{A}}(\bar{x}_1)b^{-2} = b^2ab^{-2}(\bar{x}_1) = b^2ab^{-2}b^2\bar{x}_0 = b^2a\bar{x}_0.$$

We continue this process up until we reach $x_6 = x_0$ again, obtaining

$$\bar{x}_6 = b^2ab^2a^2ba^2\bar{x}_0.$$

This means that our monomorphism maps β_1 to $b^2ab^2a^2ba^2 \in \Delta(3,3,4)$. The word $b^2ab^2a^2ba^2$ can be determined from the sequence $x_0, x_1, \dots, x_5$ by sequentially looking at the color of the common vertex v_i of the triangles containing x_i and x_{i+1}, with exponent corresponding to the oriented angle between segments connecting v_i to x_i and x_{i+1}. Conversely, a word in $\{a, b\}$ defines an analogous sequence (uniquely if we demand that the color of the common vertex alternates). Using such procedure we calculated the images of generators of $\pi_1(\Sigma_g)$ in $\Delta(3,3,4)$:

$$\begin{aligned}
\alpha_1 &\mapsto ab^2a^2bab^2ab^2a^2 \\
\beta_1 &\mapsto b^2ab^2a^2ba^2 \\
\alpha_2 &\mapsto bab^2aba^2 \\
\beta_2 &\mapsto ab^2a^2bab^2a^2ba^2baba^2
\end{aligned}$$

Denote the obtained monomorphism $\pi_1(S) \to \Delta(3,3,4)$ by f. Let $\rho_\Delta \colon \Delta(3,3,4) \to \mathrm{SL}_3(\mathbb{R})$ denote any representation from the family described by Theorem 5.0.1. The authors of [LRT11] assure that every such ρ_Δ is Hitchin in the sense of [CG05]. Theorem B in [CG05] then implies that there exists an immersion (called a developing map)

$$\mathrm{dev} \colon \mathbb{H}^2 \to \mathbb{RP}^2,$$

with $\mathrm{dev}[\mathbb{H}^2]$ a convex subset of an affine patch of $\mathbb{RP}^2$, equivariant with respect to ρ_Δ, i.e. such that

$$\rho_\Delta(g) \circ \mathrm{dev} = \mathrm{dev} \circ g,$$

for any $g \in \Delta(3,3,4)$. But then dev is also a developing map for S (treated as an orbifold with trivial local groups) because

$$\rho_\Delta(f(\gamma)) \circ \mathrm{dev} = \mathrm{dev} \circ f(\gamma) = \mathrm{dev} \circ \gamma,$$

where the last equality holds, because $f(\gamma)$ acts on $\mathbb{H}^2$ the same way as γ. Applying Theorem B from [CG05] again we obtain that $\rho_\Delta \circ f$ is Hitchin, in the usual (hyperbolic surface) sense.

Let $\rho_{\Delta,t}$ denote the curve of triangle group representations as described by Theorem 5.0.1 and let $\rho_t := \rho_{\Delta,t} \circ f$. Now that we have an explicit family of representations in a Hitchin component, we will use the FriCAS computer algebra software to investigate the tangent space at these points, obtaining the following result:

Theorem 5.0.2. *There exist simple oriented closed curves*

$$\gamma_1, \dots, \gamma_9$$

in S, based at x_0, and smooth functions

$$X_1^1, X_2^1, \ldots, X_1^9, X_2^9 \colon \operatorname{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_3(\mathbb{R}))^{Hit} \to \mathfrak{sl}_n(\mathbb{R})$$

with $X_j^i(\rho) \in \operatorname{Lie}(Z_{\mathrm{SL}_3(\mathbb{R})}(\rho(\gamma_i)))$, such that for every $t \in \mathbb{R}$ the map:

$$(s_1, r_1, \ldots, s_9, r_9) \mapsto \left[\left(T_{\gamma_9}^{\exp(s_1 X_1^9 + r_1 X_2^9)} \circ \cdots \circ T_{\gamma_i}^{\exp(s_1 X_1^1 + r_1 X_2^1)}\right)(\rho_t)\right] \in \operatorname{Hit}_3(\Sigma_2)$$

is a submersion at $0 \in \mathbb{R}^{18}$.

As the following proof is dependent on calculations made in FriCAS, we supply the appropriate code in the appendix.

Proof. We explicitly define the curves:

$$(\gamma_1, \ldots, \gamma_9) = (\alpha_1, \beta_1^{-1}, \alpha_2, \alpha_1\beta_1^{-1}, \alpha_2^{-1}\beta_1, \alpha_2\beta_2, \alpha_1^{-1}\beta_2, \alpha_2^{-1}\beta_1\alpha_1, \alpha_2^{-1}\beta_1\alpha_1\beta_1^{-1}).$$

A cautious reader will note these curves are not simple, but we will soon homotope them to simple ones. Fix $t \in \mathbb{R}$ and set $\rho = \rho_t$. We will be representing representations as quadruples of matrices

$$(\rho(\alpha_1), \rho(\beta_1), \rho(\alpha_2), \rho(\beta_2)) \in \mathrm{SL}_3(\mathbb{R})^4.$$

Since we are concerned with conjugacy classes of representations, we first compute the linear subspace of $T_\rho \operatorname{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_3(\mathbb{R}))$ corresponding to deforming the whole representation by continuous conjugation of the whole representation. We choose a basis for $\mathfrak{sl}_3(\mathbb{R})$. For each element X of this basis, we calculate the derivative of the map

$$s \mapsto \begin{bmatrix} e^{sX}\rho(\alpha_1)e^{-sX} \\ e^{sX}\rho(\beta_1)e^{-sX} \\ e^{sX}\rho(\alpha_2)e^{-sX} \\ e^{sX}\rho(\beta_2)e^{-sX} \end{bmatrix}$$

at 0. This happens to be

$$\begin{bmatrix} X\rho(\alpha_1) - \rho(\alpha_1)X \\ X\rho(\beta_1) - \rho(\beta_1)X \\ X\rho(\alpha_2) - \rho(\alpha_2)X \\ X\rho(\beta_2) - \rho(\beta_2)X \end{bmatrix} = \begin{bmatrix} \rho(\alpha_1)(\rho(\alpha_1)^{-1}X\rho(\alpha_1) - X) \\ \rho(\beta_1)(\rho(\beta_1)^{-1}X\rho(\beta_1) - X) \\ \rho(\alpha_2)(\rho(\alpha_2)^{-1}X\rho(\alpha_2) - X) \\ \rho(\beta_2)(\rho(\beta_2)^{-1}X\rho(\beta_2) - X) \end{bmatrix}.$$

We normalize each coordinate via left-invariant vector fields, so that it lies in $\mathfrak{sl}_3(\mathbb{R})$ Hence we express the derivative of conjugation by e^{sX} by a vector in $\mathfrak{sl}_3(\mathbb{R})^4$ equal to

$$\begin{bmatrix} \rho(\alpha_1)^{-1}X\rho(\alpha_1) - X \\ \rho(\beta_1)^{-1}X\rho(\beta_1) - X \\ \rho(\alpha_2)^{-1}X\rho(\alpha_2) - X \\ \rho(\beta_2)^{-1}X\rho(\beta_2) - X \end{bmatrix}.$$

Since we had 8 vectors in the basis of $\mathfrak{sl}_3(\mathbb{R})$, we obtain 8 vectors tangent to representation space normalized to lie in $\mathfrak{sl}_3(\mathbb{R})^4$. We put each of them into a single row vector of length 36. Then we combine these row vectors into a (8×36) matrix.

We will extend this matrix to a (26×36) matrix M by calculating derivatives of simple Goldman twists, and then we will investigate the rank of M. Our goal is to show that the rank of M is equal to the dimension of the space of all Hitchin representations. Since $\mathrm{Hit}_3(\Sigma_2)$ is 16-dimensional and the stabilizer of the action by conjugation is discrete (see property 2.1 of osculating flag), this dimension is equal to 24.

Each γ_i is based homotopic to a simple non-separating curve, intersecting each $\alpha_1, \beta_1, \alpha_2, \beta_2$ only at x_0 - indeed, you can homotope each γ_i to an edge or a diagonal of the fundamental octagon of S - see Figure 5.5. This is the homotopy we spoke at the beginning of the proof and take this simple curve as the definition of γ_i. Let $C = \rho(\gamma_i)$ and define:

$$X_1 := 3C - \mathrm{tr}(C)I \text{ and } X_2 := 3C^2 - \mathrm{tr}(C^2)I.$$

Both X_1, X_2 are traceless and commuting with C, hence they lie in $\mathrm{Lie}(Z_{\mathrm{SL}_3(\mathbb{R})}(C))$. Note that $\dim \mathrm{Lie}(Z_{\mathrm{SL}_3(\mathbb{R})}(C)) = 2$. Since C is an SSM X_1 and X_2 are not colinear and span the whole $\mathrm{Lie}(Z_{\mathrm{SL}_3(\mathbb{R})}(C))$. We will determine how a twist around γ_i changes the values of ρ on $\alpha_1, \beta_1, \alpha_2, \beta_2$. Since γ_i intersects each $\eta \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ only once, at x_0, we can apply Lemma 4.1.2 obtaining:

$$\left(T_{\gamma_i}^{e^{sX}}(\rho)\right)(\eta) = \begin{cases} \rho(\eta)e^{sX} & \text{if} \quad i(\gamma_i, \eta) = 1 \\ e^{-sX}\rho(\eta) & \text{if} \quad i(\gamma_i, \eta) = -1 \\ \rho(\eta) & \text{if} \quad i(\gamma_i, \eta) = 0 \text{ and } \eta \text{ intersects } \gamma_i \text{ on the left} \\ e^{-sX}\rho(\eta)e^{sX} & \text{if} \quad i(\gamma_i, \eta) = 0 \text{ and } \eta \text{ intersects } \gamma_i \text{ on the right.} \end{cases}$$

This implies that the tangent vector $\frac{d}{ds}\left(T_{\gamma_i}^{e^{sX}}(\rho)\right)(\eta)\Big|_{s=0}$ is equal to

$$\rho(\eta) \cdot \begin{cases} X & \text{if} \quad i(\gamma_i, \eta) = 1 \\ \rho(\eta)^{-1}(-X)\rho(\eta) & \text{if} \quad i(\gamma_i, \eta) = -1 \\ 0 & \text{if} \quad i(\gamma_i, \eta) = 0 \text{ and } \eta \text{ intersects } \gamma_i \text{ on the left} \\ \rho(\eta)^{-1}(-X)\rho(\eta) + X & \text{if} \quad i(\gamma_i, \eta) = 0 \text{ and } \eta \text{ intersects } \gamma_i \text{ on the right.} \end{cases}$$

As an example we perform explicit calculation for γ_8. We first determine the algebraic intersection numbers:

$$\begin{aligned} i(\gamma_8, \alpha_1) &= i(\alpha_2^{-1}\beta_1\alpha_1, \alpha_1) = i(\alpha_2^{-1}, \alpha_1) + i(\beta_1, \alpha_1) + i(\alpha_1, \alpha_1) = 0 - 1 + 0 = -1 \\ i(\gamma_8, \beta_1) &= i(\alpha_2^{-1}\beta_1\alpha_1, \beta_1) = i(\alpha_2^{-1}, \beta_1) + i(\beta_1, \beta_1) + i(\alpha_1, \beta_1) = 0 + 0 + 1 = 1 \\ i(\gamma_8, \alpha_2) &= i(\alpha_2^{-1}\beta_1\alpha_1, \alpha_2) = i(\alpha_2^{-1}, \alpha_2) + i(\beta_1, \alpha_2) + i(\alpha_1, \alpha_2) = 0 + 0 + 0 = 0 \\ i(\gamma_8, \beta_2) &= i(\alpha_2^{-1}\beta_1\alpha_1, \beta_2) = i(\alpha_2^{-1}, \beta_2) + i(\beta_1, \beta_2) + i(\alpha_1, \beta_2) = -1 + 0 + 0 = -1. \end{aligned}$$

Since α_2 intersects γ_8 on the left (see Figure 4.1) the simple Goldman twist maps:

$$\begin{aligned} \rho(\alpha_1) &\mapsto e^{-sX}\rho(\alpha_1) \\ \rho(\beta_1) &\mapsto \rho(\beta_1)e^{sX} \\ \rho(\alpha_2) &\mapsto \rho(\alpha_2) \\ \rho(\beta_2) &\mapsto e^{-sX}\rho(\beta_2). \end{aligned}$$

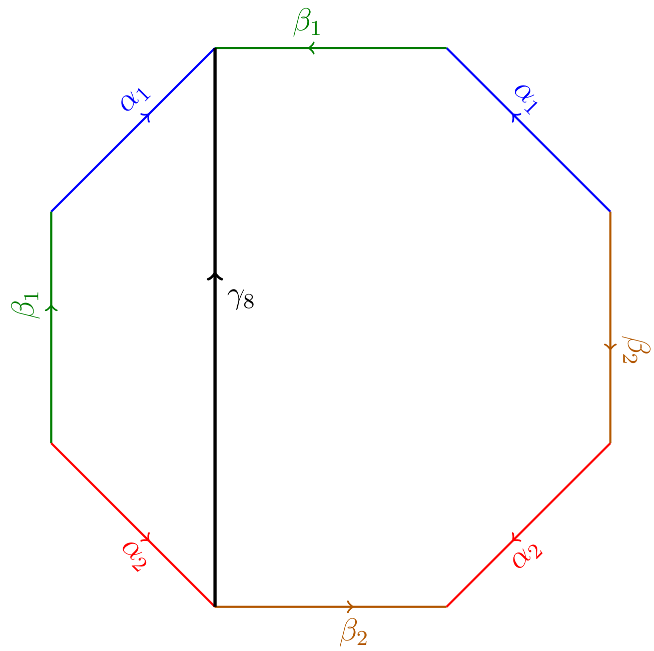


Figure 5.5: Curve α_2 touches γ_8 (homotopic to $\alpha_2^{-1}\beta_1\alpha_1$) on the left.

As a sanity check, we verify the relation:

$$\begin{aligned}
&\left[e^{-sX}\rho(\alpha_1), \rho(\beta_1)e^{sX}\right]\left[\rho(\alpha_2), e^{-sX}\rho(\beta_2)\right] &=\\
&e^{-sX}\rho(\alpha_1)\rho(\beta_1)e^{sX}\rho(\alpha_1)^{-1}e^{sX}e^{-sX}\rho(\beta_1)^{-1}\rho(\alpha_2)e^{-sX}\rho(\beta_2)\rho(\alpha_2)^{-1}\rho(\beta_2)e^{sX} &=\\
&e^{-sX}\rho(\alpha_1)\rho(\beta_1)e^{sX}\rho(\alpha_1)^{-1}\rho(\beta_1)^{-1}\rho(\alpha_2)e^{-sX}\rho(\beta_2)\rho(\alpha_2)^{-1}\rho(\beta_2)e^{sX} &=\\
&e^{-sX}\rho(\alpha_1)\rho(\beta_1)\rho(\alpha_1)^{-1}\rho(\beta_1)^{-1}\rho(\alpha_2)\rho(\beta_2)\rho(\alpha_2)^{-1}\rho(\beta_2)e^{sX} &=\\
&e^{-sX}Ie^{sX} = I,
\end{aligned}$$

where the third equality holds, because e^{sX} commutes with $\rho(\alpha_2^{-1}\beta_1\alpha_1)$. This means that the simple Goldman twist around γ_8 contributes two tangent vectors to the matrix M, namely

$$\begin{bmatrix} \rho(\alpha_1)^{-1}(-X_1)\rho(\alpha_1) \\ X_1 \\ 0 \\ \rho(\beta_2)^{-1}(-X_1)\rho(\beta_1) \end{bmatrix} \text{ and } \begin{bmatrix} \rho(\alpha_1)^{-1}(-X_2)\rho(\alpha_1) \\ X_2 \\ 0 \\ \rho(\beta_2)^{-1}(-X_2)\rho(\beta_1) \end{bmatrix},$$

both vectorized to row vectors of length 36. We have constructed the (26×36) matrix M for any $t \in \mathbb{R}$. Define $M(t)$ as the effect of this construction for t. Note that $M(t)$ is a matrix of polynomials of variable t. We want to ensure that

$$\operatorname{rank} M(t) = 24$$

for any t. For our choice of curves the matrix $M(t)$ became quite sparse. Randomly looking for non-vanishing minors would take a long time. We fix this issue by multiplying M from both sides by a random invertible square matrices of integers, obtaining $M_1(t)$. The rank of $M_1(t)$ is the same as the rank of $M(t)$ for any t We calculated two minors of $d_1(t), d_2(t)$ of M_1 and they turned out to be non-zero polynomials. Moreover their GCD turned out to be equal to $(t^2-2t+4)^{15}$, which clearly has no real roots. Hence for any $t \in \mathbb{R}$ the rank of the matrix M is 24. This means that derivatives of twists around $\gamma_1, \ldots, \gamma_9$ and continuous conjugation span $T_{\rho_t} \operatorname{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_3(\mathbb{R}))$ for any t.

□

We also note that derivatives of twists around $\{\gamma_1, \ldots, \gamma_8\}$ span $T_{\rho_t} \operatorname{Hit}_3(\Sigma_2)$ for almost all $t \in \mathbb{R}$. The instructions on how to verify this fact are included in the comments inside the code provided in the Appendix.

Corollary 5.0.3. *There exists an open set in $U \subseteq \operatorname{Hit}_3(\Sigma_2)$, containing curve ρ_t, of rationally approximable points.*

Proof. Consider the function $\Phi_\rho \colon \mathbb{R}^{26} \to \operatorname{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_3(\mathbb{R}))$ mapping

$$(c_1, \ldots, c_8, s_1, r_1, \ldots, s_9, r_9)$$

to

$$e^{c_1E_1+\cdots+C_8E_8}\left(T_{\gamma_9}^{\exp(s_1X_1^9+r_1X_2^9)} \circ \cdots \circ T_{\gamma_1}^{\exp(s_1X_1^1+r_1X_2^1)}\right)(\rho)e^{-(c_1E_1+\cdots+C_8E_8)}$$

where $\{E_1, \ldots, E_8\}$ is a basis for $\mathfrak{sl}_n(\mathbb{R})$ and γ_i and X_1^i, X_2^i are as in proof of Theorem 5.0.2. The proof of Theorem 5.0.2 shows that Φ_ρ is a submersion at 0 for any ρ lying in the curve ρ_t. Since Φ_ρ depends smoothly on ρ, it is a submersion at 0 for any ρ lying in some neighborhood of the curve ρ_t. Denote by U such a neighborhood with additional requirement that U is connected. Take any rational point of ρ_t, for example ρ_1. Then, by Lemma 4.2.3 any point of U can be reached from ρ_1 by successive simple Goldman twists $T_{\gamma_i}^{exp(s_iX_j^i)}$ and conjugation. We approximate each twist by a rational one by Fact 4.1.3 and each conjugation like in the proof of Lemma 3.1.2, obtaining a rational representation in U. □

Appendix A

FriCAS code for Theorem 5.0.2

To run the code, first install FriCAS on your system, copy the code below to a text file called "tangents.input", run FriCAS, type

```
)cd [location of tangents.input]
```

then press 'Enter' and type

```
)read tangents
```

and press 'Enter' again. The last printed value should be the GCD of two minors of M_1.

```
--Verified in FriCAS 1.3.8 for Windows.

--The images of generators of (3,3,4) triangle group as a function of parameter t.
a:=matrix[_
[0,0,1],_
[1,0,0],_
[0,1,0]_
]::Matrix(Polynomial(Integer))
b:=matrix[_
[1,2-t+t^2,3+t^2],_
[0,-2+2*t-t^2,-1+t-t^2],_
[0,3-3*t+t^2,(-1+t)^2]_
]::Matrix(Polynomial(Integer))

--The images of standard generators of surface group in SL(3,R)
--as a function of parameter t.
a1:=a*b^2*a^2*b*a*b^2*a*b^2*a^2
b1:=b^2*a*b^2*a^2*b*a^2
a2:=b*a*b^2*a*b*a^2
b2:=a*b^2*a^2*b*a*b^2*a^2*b*a^2*b*a*b*a^2

--Sanity check for surface group relation.
rel:=a1*b1*inverse(a1)*inverse(b1)*_
a2*b2*inverse(a2)*inverse(b2)

--Define identity, zero matrix and trace.
id:=diagonalMatrix([1,1,1])
null:=diagonalMatrix([0,0,0])
tr M == reduce(+, [M(i,i) for i in 1..nrows(M)])
```

```
--Define the functions calculating linearly independent (for SSM matrices)
--elements of the Lie algebra (sl3) centralizer of c.
cent1 c == 3*c-tr(c)*id
cent2 c == 3*c*c-tr(c*c)*id

--Define the helper function calculating the (inverse/right)
--Adjoint action of a on the Lie algebra.
Adi(a,X)==inverse(a)*X*a

--Define matrices linearly spanning the Lie algebra sl3.
el(r,k)== matrix([[(if r=j and k=i then 1 else 0) for i in 1..3] for j in 1..3])
conjugation_tangents:=[el(1,2),el(1,3),el(2,1),el(2,3),_
el(3,1),el(3,2),el(1,1)-el(2,2),el(2,2)-el(3,3)]

--Define a function 'conjugation_derivative' that takes a matrix X from
--the Lie algebra and calculates the derivative
--(in the space of representations) of conjugation by exp(sX) at s=0.
--The derivative is initially expressed as a vectorized element of
--the Lie algebra for each standard generator of the surface group.
--Then all these vectorized matrices are concatenated into a
--single row vector of length 36.
conjugation_derivative X == concat(_
[concat(Adi(a1,X)-X),_
 concat(Adi(b1,X)-X),_
 concat(Adi(a2,X)-X),_
 concat(Adi(b2,X)-X)])

--The variable 'prematrix' is a (eventually) very long list of
--vectorized Lie algebra matrices, representing tangent vectors
--of (continuous) conjugation and simple Goldman twists.
--Note that this is a polynomial of variable t.
--The following line invokes the function 'conjugation_derivative' on
--each element of 'conjugation_tangents' and
--appends the result to the prematrix variable.
prematrix:=map(conjugation_derivative,conjugation_tangents)


--In each of the several following blocks of code we will first select a
--simple closed curve gamma in the fundamental group and
--calculate the value g of representation on gamma.
--Then we define (in the first block, in next blocks we redefine)
--a function 'derivative', that calculates the derivative of
--simple Goldman twist along the curve gamma in the direction X.
--The format of output tangent vector is as for 'conjugation_derivative'
--and each block will contribute two new vectors to 'prematrix'.

--gamma1
g:=a1
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(null),_
 concat(X),_
 concat(null),_
 concat(null)]
```

```
prematrix:=append([derivative X1, derivative X2],prematrix)

--gamma2
--We take the inverse element, because we need the generators that
--have trivial intersection number with beta1
--to touch beta1 on the left. We will do this a few times in following cases.
g:=inverse(b1)
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(X),_
 concat(null),_
 concat(null),_
 concat(null)]
prematrix:=append([derivative X1, derivative X2],prematrix)

--gamma3
g:=a2
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(null),_
 concat(null),_
 concat(null),_
 concat(X)]
prematrix:=append([derivative X1, derivative X2],prematrix)

--gamma4
g:=a1*inverse(b1)
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(X),_
 concat(X),_
 concat(null),_
 concat(null)]
prematrix:=append([derivative X1, derivative X2],prematrix)

--gamma5
g:=inverse(a2)*b1
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(Adi(a1,-X)),_
 concat(null),_
 concat(null),_
 concat(Adi(b2,-X))]
prematrix:=append([derivative X1, derivative X2],prematrix)

-gamma6
g:=a2*b2
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(null),_
 concat(null),_
```

```
 concat(Adi(a2,-X)),_
 concat(X)]
prematrix:=append([derivative X1, derivative X2],prematrix)

--gamma7
g:=inverse(a1)*b2
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(null),_
 concat(Adi(b1,-X)),_
 concat(Adi(a2,-X)),_
 concat(null)]
prematrix:=append([derivative X1, derivative X2],prematrix)

--gamma8
g:=inverse(a2)*b1*a1
X1:=cent1(g)
X2:=cent2(g)
derivative X == concat_
[concat(Adi(a1,-X)),_
 concat(X),_
 concat(null),_
 concat(Adi(b2,-X))]
prematrix:=append([derivative X1, derivative X2],prematrix)

--gamma9 -- remove/comment out the following block and then look at det1
--to see that twists around gamma1-gamma8 generate the tangent space
--of Hitchin component at almost all
--points of the chosen representation curve.
g:=inverse(a2)*b1*a1*inverse(b1)
X1:=cent1(g)
X2:=cent2(g)
derivative X ==concat_
[concat(Adi(a1,-X)+X),_
 concat(X),_
 concat(null),_
 concat(Adi(b2,-X))]
prematrix:=append([derivative X1, derivative X2],prematrix)


--We now convert the prematrix (list of lists) to a matrix
--so that we can use the tools of the linear algebra.
m:=matrix(prematrix)

--We define a random matrix.
randmat:=matrix([_
[1,0,0,0,0,1,1,1,1,0,0,0,0,1,1,0,1,1,1,0,1,1,0,1,0,1,1,1,1,1,1,1,0,1,0,1],_
[1,1,1,0,0,0,0,0,1,1,1,1,1,1,0,1,1,0,1,1,0,0,0,0,1,1,0,0,1,1,1,0,0,1,1,0],_
[1,0,1,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,1,1,0,0,1,1,1,0,0,1,0,1,0,0,1,1],_
[1,0,0,1,1,1,1,0,1,0,1,1,0,1,1,0,0,0,0,0,1,1,0,0,0,0,1,0,1,1,0,0,0,0,1,0],_
[1,1,1,0,0,0,1,0,0,1,1,0,0,1,1,0,1,1,1,1,0,0,1,1,0,1,1,1,0,0,1,1,1,1,0,0],_
[0,1,0,1,0,1,0,1,1,1,0,0,1,1,0,0,1,0,0,1,0,1,1,0,1,1,0,0,1,1,1,1,1,0,0,0],_
[1,1,0,0,1,0,0,1,1,1,0,0,0,1,1,1,1,1,0,0,0,0,1,1,1,1,1,0,0,1,0,0,1,1,1,1],_
[0,1,1,0,1,0,0,0,0,0,0,1,0,1,1,0,1,0,0,1,0,1,1,0,0,0,1,1,1,0,1,0,0,1,0,1],_
[1,1,0,0,0,1,0,0,1,1,0,1,1,0,0,0,1,1,1,1,0,0,1,0,1,0,0,0,0,0,0,1,0,0,1,1],_
```

```
[0,1,1,0,1,0,1,1,0,1,1,0,0,0,0,0,1,1,0,0,1,1,1,0,0,1,0,0,1,1,1,1,1,1,1,1],_
[1,1,0,1,0,1,1,0,1,0,1,0,0,0,0,1,1,0,1,1,1,0,1,0,0,1,1,0,0,1,0,0,1,1,0,1],_
[0,1,0,1,0,0,0,0,1,1,0,0,0,1,1,1,0,1,0,1,1,0,0,1,1,0,1,1,0,1,1,0,0,1,0,1],_
[1,0,0,0,0,1,1,1,1,1,1,0,0,1,0,0,1,0,0,1,1,0,1,1,0,1,0,1,1,0,1,0,0,0,0,0],_
[1,0,0,0,1,0,0,0,0,0,0,1,0,0,0,1,1,1,0,1,0,1,0,0,1,1,1,1,1,1,0,1,1,0,0,0],_
[1,1,0,0,0,1,0,1,1,1,0,0,0,1,1,0,1,1,0,1,1,0,1,0,0,1,0,0,1,0,0,1,1,1,0,1],_
[1,0,0,1,0,0,1,0,0,0,1,0,0,0,1,0,0,1,0,0,1,1,0,0,1,0,1,1,1,1,1,1,0,1,0,1],_
[1,0,0,1,0,0,1,1,1,1,0,1,1,0,1,1,1,1,0,0,0,0,0,1,1,0,1,1,0,1,1,1,1,0,1,0],_
[0,1,1,0,0,0,1,1,1,1,1,0,0,1,0,1,1,1,1,1,0,0,0,1,1,0,0,1,1,0,0,1,0,0,1,0],_
[1,0,1,1,1,0,1,1,0,1,1,1,0,1,0,0,1,1,1,1,0,0,1,1,1,0,1,1,1,0,1,1,0,1,0,0],_
[0,0,0,1,1,1,1,0,0,0,1,1,0,0,1,1,0,0,0,0,1,0,0,0,0,1,1,1,1,0,1,1,1,0,0,1],_
[1,1,1,0,0,1,0,1,1,1,0,1,0,0,1,1,1,1,0,0,0,1,1,1,1,0,0,1,1,1,1,0,0,0,1,1],_
[0,1,1,0,1,1,0,1,0,1,0,1,1,1,0,1,0,1,0,1,0,0,1,1,1,0,0,1,0,1,0,1,1,1,1,1],_
[0,1,0,1,1,0,0,1,1,1,0,1,1,0,0,1,0,0,0,1,1,1,0,1,0,0,0,0,0,0,1,0,0,1,0,0],_
[0,1,1,0,0,1,1,1,1,0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0,1,1,0,0,0,1,1,1,0,1],_
[1,0,0,1,0,1,1,1,0,1,0,0,0,1,1,0,1,0,1,1,0,1,1,1,0,1,1,1,0,1,1,0,0,1,0,1],_
[1,0,0,1,0,1,0,1,1,0,1,1,1,0,1,1,0,0,1,1,1,1,1,1,0,1,0,0,1,1,1,0,1,1,0,1],_
[1,0,0,1,0,1,1,1,1,0,0,0,1,1,0,0,0,0,0,1,1,1,0,1,1,0,0,1,1,1,1,0,0,0,0,0],_
[1,0,0,1,0,0,0,1,1,0,0,0,1,0,1,1,0,0,0,1,0,0,0,0,0,0,1,0,1,0,1,1,0,1,0,1],_
[0,1,0,0,1,1,0,0,1,1,0,0,0,1,0,1,0,1,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,0,1,1],_
[0,0,1,0,1,1,1,1,1,0,0,1,0,0,1,1,1,0,1,0,1,1,0,0,0,1,1,0,0,1,0,1,0,1,1,1],_
[1,0,0,0,1,1,0,0,1,1,0,0,0,1,1,1,0,0,0,0,1,1,0,0,0,1,0,0,1,0,0,1,0,1,1,1],_
[0,0,0,0,0,1,1,1,0,0,0,1,1,0,1,0,1,1,1,1,0,0,0,0,1,1,0,1,0,0,0,1,0,1,0,0],_
[1,1,0,0,0,1,0,0,0,1,0,0,0,0,0,1,1,1,1,0,1,0,0,0,1,0,1,1,1,0,1,0,0,0,1,0],_
[0,0,1,0,1,0,0,0,1,0,1,0,1,1,1,1,0,1,1,1,0,0,0,0,0,0,0,1,0,0,1,0,1,1,0,0],_
[1,0,1,0,1,1,1,1,0,0,1,0,1,0,0,0,1,0,1,1,0,0,1,0,0,1,0,0,0,1,0,0,0,0,1,1],_
[1,1,1,0,0,0,1,1,0,1,1,1,0,1,0,1,1,0,0,1,1,1,0,1,1,0,0,0,0,0,1,0,1,0,0,0] _
])

--If one prefers to generate amother random matrix,
--below is a (inactive) line that does that.
--randsqmat n== matrix [[randnum 2 for i in 1..n] for j in 1..n]
randsqmat n == randmat(1..n,1..n)

--We multiply our matrix m by random square matrices from both sides,
--obtaining m1.
--The rank of m1 cannot exceed the rank of m.
m1:=randsqmat(nrows m)*m*randsqmat(ncols m)

--Sanity check - it should not be possible for m1 to have
--rank greater than 24 if all calculations are correct.
det25:=determinant m1(1..25,1..25)

--Calculate two minors of m1.
det1:=determinant m1(1..24,1..24)
det2:=determinant m1(2..25,2..25)

--Convert from Fraction(Polynomial(Integer)) to Polynomial(Fraction(Integer))
--so that we can properly use GCD.
d1:=det1::Polynomial(Fraction(Integer))
d2:=det2::Polynomial(Fraction(Integer))

--Calculate the GCD of minors. This should be (t^2-2t+4)^15.
factor(gcd(d1,d2))
```

Bibliography

[AZ23] Jacques Audibert and Michael Zshornack. "Rational approximation for Hitchin representations". In: *arXiv preprint arXiv:2310.15121* (2023).

[BCL20] Martin Bridgeman, Richard Canary, and François Labourie. "Simple length rigidity for Hitchin representations". In: *Advances in Mathematics* 360 (2020), p. 106901.

[BD14] Francis Bonahon and Guillaume Dreyer. "Parameterizing Hitchin components". In: *Duke Mathematical Journal* 163.15 (Dec. 2014). ISSN: 0012-7094.

[Bus10] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Springer Science & Business Media, 2010.

[Can22] Richard Canary. "Hitchin representations of Fuchsian groups". In: *EMS Surveys in Mathematical Sciences* 9.2 (2022).

[CG05] Suhyoung Choi and William M Goldman. "The deformation spaces of convex $\mathbb{RP}^2$-structures on 2-orbifolds". In: *American journal of mathematics* 127.5 (2005), pp. 1019–1102.

[DJ24a] Jan Dymara and Tadeusz Januszkiewicz. "Tautological characteristic classes II: the Witt class". In: *Mathematische Annalen* 390.3 (2024), pp. 4463–4496.

[DJ24b] Jan Dymara and Tadeusz Januszkiewicz. "Tautological characteristic classes, I". In: *Algebraic & Geometric Topology* 24.5 (2024), pp. 2889–2932.

[FM11] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Vol. 49. Princeton university press, 2011.

[Gol86] William M Goldman. "Invariant functions on Lie groups and Hamiltonian flows of surface group representations". In: *Inventiones mathematicae* 85.2 (1986), pp. 263–302.

[Gol90] William M Goldman. "Convex real projective structures on compact surfaces". In: *Journal of Differential Geometry* 31.3 (1990), pp. 791–845.

[Hit92] Nigel J Hitchin. "Lie groups and Teichmüller space". In: *Topology* 31.3 (1992), pp. 449–473.

[Lab04] François Labourie. "Anosov flows, surface groups and curves in projective space". In: *arXiv preprint math/0401230* (2004).

[Lab17] François Labourie. "Lectures on representations of surface groups". In: *Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich* (2017).

[LRT11] Darren D Long, Alan W Reid, and Morwen Thistlethwaite. "Zariski dense surface subgroups in SL (3, Z)". In: *Geometry & Topology* 15.1 (2011), pp. 1–9.

[Nek91] Ya Nekovar. "Maslov index and Clifford algebras". In: *Functional Analysis and Its Applications* 24.3 (1991), pp. 196–204.

[PRR93] Vladimir Platonov, Andrei Rapinchuk, and Rachel Rowen. *Algebraic groups and number theory.* Vol. 139. Academic press, 1993.

[RBC96] Andrei S Rapinchuk, Valerii V Benyash-Krivetz, and Vladimir I Chernousov. "Representation varieties of the fundamental groups of compact orientable surfaces". In: *Israel Journal of Mathematics* 93.1 (1996), pp. 29–71.

[Tak71] Kisao Takeuchi. "Fuchsian groups contained in SL2 (Q)". In: *Journal of the Mathematical Society of Japan* 23.1 (1971), pp. 82–94.

[Wie18] Anna Wienhard. "An invitation to higher Teichmüller theory". In: *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*. World Scientific. 2018, pp. 1013–1039.

[Wol09] Scott A Wolpert. "The Weil-Petersson metric geometry". In: *Handbook of Teichmüller theory* 2 (2009), pp. 47–64.

[Wol82] Scott Wolpert. "The Fenchel-Nielsen deformation". In: *Annals of Mathematics* 115.3 (1982), pp. 501–528.