Uniwersytet Wrocławski

Wydział Matematyki i Informatyki

Instytut Matematyczny

Jakub Gogolok

O teorii modeli ciał z operatorami

praca doktorska
napisana pod opieką
prof. dr. hab. Piotra Kowalskiego

Wrocław 2024

Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny

Jakub Gogolok

On model theory of fields with operators

doctoral thesis
supervised by
prof. dr hab. Piotr Kowalski

Wrocław 2024

Table of Contents

Acknowledgements

I would like to express my deepest gratitude to Piotr Kowalski for his guidance during my master and doctoral studies, for his unbounded patience towards me and for the plethora of thing I learned from him. I would like also to thank the whole model theory group at the University of Wrocław. In particular, I would like to thank Ludomir Newelski and Krzysztof Krupiński for all the things they taught me and for being a very strong inspiration for me.

I would like to thank my high-school teachers, Joachim Jałowiecki and Michał Rał, for the many important lessons they gave me and for their guidance.

I would like to thank my family, friends and my girlfriend for being a constant support. Last but not least, I would like to thank my cat Lilith for never failing to distract me while I was writing this thesis.

Introduction

This thesis is about fields with operators, through the point of view of algebra and model theory.

The two prototypical examples of fields with operators are differential fields (i. e. fields with a distinguished derivation) and difference fields (i. e fields with a distinguished endomorphism). These classes are immensely interesting, on one hand because they have many interesting model-theoretic properties and on the other hand because of their applications. Just to name a few applications:

(1) Various results in algebraic dynamics (see **[9]**),
(2) Hrushovski's proof of the Manin-Mumford conjecture (see **[21]**),
(3) Ax-Lindemann-Weierstrass theorems for uniformizers of Fuchsian groups (see **[8]**).

Both differential and difference fields are instances of the notion of *$\mathcal{D}$-rings structures* introduced by Moosa and Scanlon in **[34]** (an equivalent set-up of *B-operators* was given in **[2]** by Beyarslan, Hoffmann, Kamensky and Kowalski). What this means is that both these classes are governed by a certain k-algebra in the following manner. Fix some base ring k. For a k-algebra R, let $R[\varepsilon]$ be the ring of dual numbers over R, i. e. $R[\varepsilon] = R[X]/(X^2)$ and ε is the coset of X. Then, the map $\partial\colon R \to R$ on a k-algebra R is a k-derivation (i. e. a derivation vanishing on k) if and only if the map

$$R \to R[\varepsilon],\ x \mapsto x + \partial(x)\varepsilon,$$

is a morphism of k-algebras. Since $R[\varepsilon] = R \otimes_k k[\varepsilon]$, we may say that ($k$-)derivations are governed by the algebra $k[\varepsilon]$. Similarly, endomorphisms (fixing k) are governed by the algebra $k \times k$.

In Chapter 2 we expand this idea by replacing the algebra k-algebra B (above we have: $B = k[\varepsilon]$ or $B = k \times k$), or rather the functor $-\otimes_k B$, by an appropriate functor $\mathcal{B}\colon \mathrm{Alg}_k \longrightarrow \mathrm{Alg}_k$. The functors we deem fit for this purpose are *coordinate k-algebra schemes*, a notion we introduce and develop in Section 2.1. After that, we define the notion of a *$\mathcal{B}$-operator* and develop some basic results of *$\mathcal{B}$-algebra*. We then generalize the notion of *prolongations* to the context of $\mathcal{B}$-operators, which will be crucial in the model-theoretic analysis done in Chapter 3.

Chapter 3 is devoted to the analysis of $\mathcal{B}$-fields (i. e. fields with a $\mathcal{B}$-operators) which are existentially closed, possibly in a generalized sense, e. g. only in regular extensions. The fundamental question is whether being existentially closed (in a generalized sense) is an elementary property. This question belongs to a well-established line of research in model theory, and in particular entails the pursuit of model companions in algebraic model theory. We prove a very general result in this direction (Theorem 3.13), which entails and simplifies many results from the literature (see Remark 3.17). We also analyse the model-theoretic properties of the resulting theories.

Authorship of results

Let us comment on the authorship of the results in this thesis.

(1) **Chapter 1**: This Chapter is only a review of classical material and contains no original work.

(2) **Chapter 2**: Sections 2.1, 2.2 and 2.3.2 are based on the joint paper [**13**] by Kowalski and the author. Section 2.3.1, 2.3.3 and 2.4 are by the author.

(3) **Chapter 3**: Sections 3.2.3, 3.2.4 are again based on [**13**]. Section 3.2.2 is based on the paper [**12**] by the author. The rest of the Chapter is the sole work of the author and is yet to be published.

CHAPTER 1

Preliminaries

In this chapter we gathered for the convenience of the reader some classical results from algebra, geometry and model theory, which we will use throughout this thesis. We also fix some notation and conventions here. Besides classical notions, we also included a discussion on $\mathcal{D}$-ring structures, where we take the equivalent B-operators approach (Section 1.8), as they are the object we want to generalize in this thesis.

All theories we consider are first-order. All rings are unital and homomorphisms of rings preserve the identity. We have to accept the zero ring as a ring. Since $0 = 1$ in the zero ring, there is no homomorphism from the zero ring to a non-zero ring. By default rings are commutative - the only exception is the ring of skew-polynomials $k\,[\mathrm{Fr}]$ and the matrix rings over it (see Section 1.5).

1.1. Field theory

We refer to the book [**11**] by Fried and Jarden for this section. We assume for convenience that all fields under consideration are contained in one big algebraically closed field Ω. For a field K we denote by K^{alg} and K^{sep} respectively the algebraic and separable closure (inside Ω) of K. Let $K \subseteq L, M$ be subfields of Ω. We say that L and M are **free over** K (or **algebraically independent**) if every tuple of elements $x_1, \ldots, x_n \in L$ which is algebraically independent over K is also algebraically independent over M. Analogously, we say that L and M are **linearly disjoint over** K if every tuple of elements $x_1, \ldots, x_n \in L$ which is linearly independent over K is also linearly independent over M. An important property is the following: L and M are linearly disjoint over K if and only if the natural map $L \otimes_K M \to LM$ is injective, where LM is the compositum of L and M.

We say that a field extension $K \subseteq L$ is **separable** if K has characteristic zero or if $\mathrm{char}\, K = p > 0$ and K and $L^p (= \{a^p : a \in L\})$ are linearly disjoint over K^p. On the other extreme, an algebraic extension $K \subseteq L$ of fields of characteristic $p > 0$ is called **purely inseparable** if for every $a \in L$ there is some $n > 0$ such that $a^{p^n} \in K$.

Let K be a field of characteristic $p > 0$. Then K naturally becomes a vector space over the field K^p. A subset $B \subseteq K$ is called **p-independent** (in K) if the set

$$M = \left\{ b_1^{i_1} \dots b_n^{i_n} : b_1, \dots, b_n \in B,\ 0 \leqslant i_1, \dots, i_n \leqslant p-1, n > 0 \right\}$$

is linearly independent over K^p. If furthermore M is a basis of the K^p-vector space K, then we say that B is a **p-basis** (of K). It is easy to see that a p-basis is just a maximal p-independent set. One can prove that any two p-bases of K have the same cardinality, which we call the **imperfection degree of** K.

Let K be a field of characteristic $p > 0$. For every $n \in \omega$ fix an enumeration $m_{n,i}(x)$ (where $i = 1, \dots, p^n$) of monomials $x_1^{i_1} \cdots x_n^{i_n}$ where $i_1, \dots, i_n \in \{0, \dots, p-1\}$. We define the **$\lambda$-functions of** K as follows. The functions $\lambda_{n,i} \colon K^n \times K \to K$ (where $i = 1, \dots, p^n$) are the unique functions satisfying the following properties for any $(\bar{a}, b) \in K^n \times K$:

(1) if the n-tuple $\bar{a}$ is p-dependent or the $(n+1)$-tuple $(\bar{a}, b)$ is p-independent, then $\lambda_{n,i}(\bar{a}, b) = 0$ for any i;

(2) otherwise $\lambda_{n,i}(\bar{a}, b)$ (where $i = 1, \dots, p^n$) are determined by the equality

$$b = \sum_{i=1}^{p^n} \lambda_{n,i}(\bar{a}, b)^p \, m_{n,i}(\bar{a}).$$

Thus, morally λ-functions are coordinates of an element in a given p-basis (modulo some boundary cases to make them total functions). Clearly the λ-functions are definable in any field K and moreover this definition is uniform (i. e. given by the same formula for every field) once one fixes the characteristic p. We suppress K in the notation $\lambda_{n,i}$, though their value depends of course on the field K we are working in. Of particular importance is the case $n = 0$. Since there is precisely one monomial in 0 variables (namely the constant monomial 1), there is only one λ-function for $n = 0$, which we denote by λ_0 (instead of $\lambda_{0,1}$). We have that $\lambda_0 \colon K \to K$ is the inverse of the Frobenius map on K^p and zero on $K \setminus K^p$.

An extension of fields $K \subseteq L$ is called **regular** if L and K^{alg} are free over K. This is the same as saying that L is a separable extension of K and $L \cap K^{\mathrm{sep}} = K$, or that L is a separable extension of K and L and K^{sep} are linearly disjoint over K.

Fact 1.1. *Let $K \subseteq M, N$ be extensions of fields with $K \subset M$ regular. Then the tensor product $M \otimes_K N$ is a domain.*

Fact 1.2. *Let $K \subseteq L$ be an extension of fields. Let $a = (a_1, \dots, a_n)$ be a tuple of elements of L and assume that there are some polynomials $f_1, \dots, f_n \in K[X_1, \dots, X_n]$ such that*

(1) $f_1(a) = \ldots = f_n(a) = 0$,

(2) *the Jacobian of* $(f_1, \ldots, f_n)$ *is nonzero.*

Then the extension $K \subseteq K(a)$ *is separably algebraic.*

1.2. Formally smooth and étale algebras

We refer to Matsumura's book [**32**] for this section. A ring homomorphism $f\colon R \to S$ is called **formally smooth** if for some (equivalently: any) $e > 1$ the following property hold: for any ring T and an ideal $N \trianglelefteq T$ with $N^e = 0$, any commutative solid diagram (consisting of ring homomorphisms) of the form

$$\begin{tikzcd} S \arrow[r] \arrow[dr, dashed] & T/N \\ R \arrow[u, "f"] \arrow[r] & T \arrow[u] \end{tikzcd}$$

can be extended via a dashed arrow to a commutative diagram. If this extension is always unique, we say that $f\colon R \to S$ is **étale**. Any separable field extension is formally smooth and separably algebraic extensions are étale.

1.3. Scheme theory

Schemes should be introduced as certain locally ringed spaces, which generalize algebraic varieties from classical algebraic geometry. However, for our purposes we may take a "functor of points" approach. We send the reader to [**31**] for a proper approach to scheme theory. For the approach presented below we refer to the book [**49**].

Let k be a ring and denote by Alg_k the category of k-algebras. An **affine scheme over** k is a representable functor $\mathcal{S}\colon \mathrm{Alg}_k \longrightarrow \mathrm{Set}$, where Set is the category of sets. This means, that there is a k-algebra A and a natural isomorphism between the functor $\mathcal{S}$ and the functor $\mathrm{Hom}(A, -)$. By the Yoneda lemma, A is (up to an isomorphism) uniquely determined by $\mathcal{S}$ and we will sometimes refer to it as the **coordinate ring** of $\mathcal{S}$ and denoted by $k[\mathcal{S}]$. In such a situation, $\mathcal{S}$ is sometimes denoted by $\mathrm{Spec}(A)$. In what follows instead of "an affine scheme over k" we will just write "k-scheme" or even just "scheme". If R is an k-algebra, then elements of $\mathcal{S}(R)$ are called **R-rational points**.

A scheme $\mathcal{S}$ is a **scheme of finite type over** k if its coordinate ring is finitely generated as a k-algebra. We say that $\mathcal{S}$ is **reduced** if $k[\mathcal{S}]$ reduced and **connected** if $k[\mathcal{S}]$ has no (nontrivial) idempotent elements.

A **morphism of schemes** $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ is simply a natural transformation of functors $\mathcal{S}_1 \to \mathcal{S}_2$. Again by Yoneda lemma, this corresponds to a k-algebra homomorphism

$f^*\colon k[\mathcal{S}_2] \to k[\mathcal{S}_1]$. The class of all affine schemes together with morphisms forms a category Aff_k, which is (by Yoneda) equivalent to the opposite category of Alg_k. Since Alg_k has coproducts (namely, tensor products of algebras), the category Aff_k has products. We denote the product of two schemes $\mathcal{S}_1, \mathcal{S}_2$ by $\mathcal{S}_1 \times_k \mathcal{S}_2$ and by the previous sentence $k[\mathcal{S}_1 \times_k \mathcal{S}_2] = k[\mathcal{S}_1] \otimes_k k[\mathcal{S}_2]$. We have also an explicit description using rational points, namely for any k-algebra R we have $(\mathcal{S}_1 \times_k \mathcal{S}_2)(R) = \mathcal{S}_1(R) \times \mathcal{S}_2(R)$.

1.4. Algebraic geometry à la Weil

We refer the reader to Chapter III of Lang's book **[30]** for this subsection. We again work inside some big algebraically closed field Ω, in particular all fields under consideration are subfields of Ω. Let K be a field and m, n be positive integers. A subset of the affine space Ω^n is called a **Zariski K-closed set** if it the set of common zeroes of some family of polynomials with coefficients in K. The family of all Zariski K-closed subsets of Ω^n forms a noetherian topology, called the **K-Zariski topology**. In particular, every Zariski K-closed set V is the set of common zeroes of a finite family of polynomials over K. We say that V is **K-irreducible** (or a **K-variety**) if it cannot be written as the union of two proper K-closed subsets. Every K-closed set V is the union of K-varieties $V_1 \cup \ldots \cup V_m$ and this decomposition is unique up to permutation indices, provided one assumes that V_i and V_j are not contained in each other for $i \neq j$.

Let V be a K-variety. For a field extension $K \subseteq L$ we define the **set of L-rational points of** V as $V(L) = V \cap L^n$. We define the **ideal of** V **over** K to be the ideal $K[X]$ defined as

$$I_K(V) = \{f \in K[X] : (\forall a \in V(K))\, f(a) = 0\}.$$

Then V is a K-variety if and only if $I_K(V)$ is prime. From now on we assume V is a K-variety. We associate to V the **ring of regular functions on** V **over** K (or the **coordinate ring of** V **over** K) equal to $K[V] = K[X]/I_K(V)$. This ring is equal to the ring functions $V \to K$ which are restrictions of (K-)polynomial maps $K^n \to K$, hence the name "regular functions". We also define the **field of K-rational functions on** V as the fraction field $K(V)$ of the integral domain $K[V]$. If the field K will be clear from the context, we will simply say e. g. "variety" instead of "K-variety".

To any point $a \in \Omega^n$ we can naturally associate a K-variety, namely the **locus of** a **over** K. It is defined as the smallest K-variety containing a, i. e. the intersection of all

K-varieties containing a, and we denote it by $\text{locus}_K(a)$. It is easy to see that

$$I_K(\text{locus}_K(a)) = \{f \in K[X] : f(a) = 0\}$$

and thus $K[V]$ and $K(V)$ are isomorphic as K-algebras to $K[a]$ and $K(a)$ respectively. In fact, every K-variety V is of the form $V = \text{locus}_K(a)$ for some a. We say sometimes in this situation that a is a **generic point** of V (over K).

Given two K-varieties $W \subseteq \Omega^m, V \subseteq \Omega^n$, a **$K$-morphism** between W and V is a map $f: W \to V$ which is the restriction to W of some K-polynomial map $\Omega^m \to \Omega^n$. One sees that K-morphisms of varieties correspond to K-algebra homomorphisms $f^*: K[V] \to K[W]$. We say that a K-morphism $f: W \to V$ is **dominant** if its (set-theoretic) image is dense in the K-Zariski topology on V. On the level of coordinate rings this means that the associated K-algebra homomorphism f^* is injective. Since $[V]$ and $K[W]$ are domains, this yield a field extension $f^*(K(V)) \subseteq K(W)$, which we are free to think is just a field extension $K(V) \subseteq K(W)$. We sat that that a K-morphism $f: W \to V$ is **separable** if it is dominant and the extension $K(V) \subseteq K(W)$ is separable.

Suppose that $W \subseteq \Omega^m, V \subseteq \Omega^n$ are K-varieties, $m > n$ and the projection onto the first n coordinates yields a dominant K-morphism $W \to V$. If $V = \text{locus}_K(a)$, then we can find some $b \in \Omega^{m-n}$ such that $W = \text{locus}_K(a, b)$.

Any K-variety V corresponds naturally to an scheme over K, namely the functor $\text{Alg}_k \longrightarrow \text{Set}$ represented by $K[V]$. For any field $L \supseteq K$ the L-rational points of V in the sense of varieties are in natural bijection with the L-rational points in the sense of schemes.

1.5. Group schemes

We refer to the book **[49]** by Waterhouse for this section. An **affine group scheme over** k is a scheme $\mathcal{S}$ together with a lift to a functor $\mathcal{S}: \text{Alg}_k \longrightarrow \text{Grps}$, where Grps is the category of groups. Once again by Yoneda lemma, this is the same as a scheme $\mathcal{S}$ together with a morphisms $m: \mathcal{S} \times_k \mathcal{S} \to \mathcal{S}$ (multiplication), $\text{inv}: \mathcal{S} \to \mathcal{S}$ (inverse) and $e: * \to \mathcal{S}$ (identity element, where $*$ is the functor sending every k-algebra to a fixed single element set, represented by k) obeying the usual group-law axioms expressed properly using commutative diagrams. This in turn corresponds to giving $k[\mathcal{S}]$ the structure of a **Hopf algebra**, but we will not use this point of view in this thesis.

Two standard examples are the **additive group scheme** $\mathbb{G}_{\mathrm{a},k}$ which sends any k-algebra R to its additive group $\mathbb{G}_{\mathrm{a},k}(R) = (R, +)$, and the **multiplicative group scheme** $\mathbb{G}_{\mathrm{m},k}$

which sends any k-algebra R to its group of invertible elements $\mathbb{G}_{\mathrm{m},k}(R) = (R^\times, \cdot\,)$. The former is represented by $k[X]$ and the latter by $k[X, X^{-1}]$. Since we will be working over a fixed base ring k, we will usual skip k in the index and simply write $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$.

A **morphism of group schemes** $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ is a natural transformation $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ where we consider $\mathcal{S}_1, \mathcal{S}_2$ are considered as functors $\mathrm{Alg}_k \longrightarrow \mathrm{Grps}$. This is the same as demanding that f is a morphism of schemes and for every k-algebra R the map $f_R\colon \mathcal{S}_1(R) \to \mathcal{S}_2(R)$ is a morphism of groups.

Products of group schemes (regarded as affine schemes) have a natural group scheme structure. The ring of endomorphisms of $\mathbb{G}_\mathrm{a}$ is (isomorphic to) k in characteristic zero and to $k[\mathrm{Fr}]$ in positive characteristics. Here elements of $k[\mathrm{Fr}]$ are k-polynomials of the from $a_0 \mathrm{Fr}^0 + \ldots + a_m \mathrm{Fr}^m$ where $\mathrm{Fr}\colon \mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{a}$ is the morphism coming from the k-algebra map $k[X] \to k[X]$ determined by $X \mapsto X^p$. Multiplication in this ring is simply composition. In particular, this ring is not commutative unless $k = \mathbb{F}_p$. Analogously, the ring of endomorphisms of $\mathbb{G}_\mathrm{a}^e$ is isomorphic to the ring of $e \times e$ matrices with coefficients in $k[\mathrm{Fr}]$. An important fact (for our purposes) is the fact that right division with remainder is always possible in $k[\mathrm{Fr}]$ and left division with remainder is possible if k is perfect (see [**38**]). By a slight abuse we will also think that the ring of endomorphisms of $\mathbb{G}_\mathrm{a}$ in characteristic zero is $k[\mathrm{Fr}]$, but now Fr is interpreted as the identity $\mathrm{id}\colon \mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{a}$.

An **affine algebraic group** is an affine group scheme which is of finite type as a scheme over k.

We will need some facts about unipotent algebraic groups, for which we refer to [**46**] and [**49**]. An affine group $\mathcal{G}$ is called **elementary unipotent** if it is isomorphic to a closed subgroup of some $\mathbb{G}_\mathrm{a}^e$. We say that an (elementary) unipotent group $\mathcal{G}$ is k**-split** if it admits a subnormal series whose quotients are isomorphic to $\mathbb{G}_\mathrm{a}$.

Fact 1.3.
(1) *An affine group scheme is an elementary unipotent group if and only if it is so after a base change to* k^{alg}.
(2) *Every connected commutative algebraic group of exponent p is isomorphic to $\mathbb{G}_\mathrm{a}^e$ after some base change (in particular: after a base change to k^{alg}).*
(3) *If k is perfect then any elementary unipotent group scheme is k-split.*
(4) *Any elementary unipotent k-split group scheme is isomorphic to $\mathbb{G}_\mathrm{a}^e$.*

Fact 1.4 (Special case of Corollary 14.4.2 in [**46**])**.** *Let $\mathcal{G}$ be connected, elementary unipotent and assume that $\mathbb{G}_\mathrm{a}$ acts on $\mathcal{G}$ via group scheme automorphisms. Assume moreover that $e \in \mathcal{G}(k)$ is the only k-rational point fixed by $\mathbb{G}_\mathrm{a}$. Then $\mathcal{G}$ is k-split.*

1.6. Model companions

We refer to Hodges' book [**15**] for this section. Let L be a first-order language and let T, T^* be L-theories. We say that T **model complete** if any inclusion of models of T is elementary or equivalently, every model of T is existentially closed. We say that T^* is a **companion** of T if every model of T embeds into a model of T and vice-versa. A **model companion of** T is a companion which is model complete. A model companion is unique, if it exists. We say that T is **companionable** if it has a model companion. Recall that a theory T is **inductive** if the sum of any chain of models of T is a model of T; syntactically this means that T can be axiomatized using $\forall\exists$-sentences. There is a convenient characterisation of companionable inductive theories, namely: an inductive theory T is companionable if and only if the class of existentially closed models of T is elementary. In this situation, a model companion of T is precisely an axiomatization of the class of existentially closed models of T.

1.7. Stability and forking

By convention, we say that an incomplete theory is **stable** if all of its completions are. There is very practical characterisation of stability which we will rely on (see [**26**, Fact 2.1.4]). Namely, a complete theory T is stable if and only if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on small subsets of a monster model $\mathfrak{C} \models T$, which satisfies the following conditions:

(P1) (invariance) The relation $\mathop{\smile\!\!\!\!\!\!|}$ is $\mathrm{Aut}(\mathfrak{C})$-invariant.

(P2) (symmetry) For every small $A, B, C \subset \mathfrak{C}$, it follows that

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \qquad \iff \qquad B \mathop{\smile\!\!\!\!\!\!|}_C A.$$

(P3) (monotonicity and transitivity) For all small $A \subseteq B \subseteq C \subset \mathfrak{C}$ and small $D \subset \mathfrak{C}$, it follows that

$$D \mathop{\smile\!\!\!\!\!\!|}_A C \qquad \iff \qquad D \mathop{\smile\!\!\!\!\!\!|}_A B \quad \text{and} \quad D \mathop{\smile\!\!\!\!\!\!|}_B C.$$

(P4) (existence) For every finite $a \subset \mathfrak{C}$ and every small $A \subseteq B \subset \mathfrak{C}$, there exists $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(a) \mathop{\smile\!\!\!\!\!\!|}_A B$.

(P5) (local character) For every finite $a \subset \mathfrak{C}$ and every small $B \subset \mathfrak{C}$, there is $B_0 \subseteq B$ such that $|B_0| \leqslant \omega$ and $a \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.

(P6) (finite character) For every small $A, B, C \subset \mathfrak{C}$, we have:

$$A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{C} B \text{ if and only if } a \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{C} B \text{ for every finite } a \subseteq A.$$

(P7) (uniqueness over a model) Any complete type over a model is stationary, i. e. whenever $K \preccurlyeq M \prec \mathfrak{C}$ are small models and p is a complete type over K, then there is a unique $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$-independent extension of p to M (i. e. an extension $p \subseteq q$ such that for any/every $a \models p$ we have $a \mathop{\smile\!\!\!\!\!\!\!\;\mid}_K M$).

Moreover, if $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ satisfies the above properties, then it coincides with forking independence in T.

1.8. B-operators

$\mathcal{D}$-rings structures were introduced by by Moosa and Scanlon in [**34**]. We take here an equivalent approach of B-operators from [**2**]. Fix some base field k, a finite k-algebra B (finite means that B has finite dimension as a vector space over k) and a k-algebra homomorphism $\pi\colon B \to k$. A **B-operator** on a k-algebra R is a k-algebra homomorphism $\partial\colon R \to R \otimes_k B$ such that $(\mathrm{id}_R \otimes \pi) \circ \partial = \mathrm{id}_R$.[1] Let us fix a basis $e_0, \dots, e_n$ of B such that $\pi(e_0) = 1$ and $\pi(e_i) = 0$ for $i = 1, \dots, n$. Then, a B-operator on R corresponds to a sequence of maps $\partial_1, \dots, \partial_n\colon R \to R$ such that the map

$$R \ni x \longmapsto x \otimes e_0 + \partial_1(x) \otimes e_1 + \ldots + \partial_n(x) \otimes e_n \in R \otimes B$$

is a homomorphism of k-algebras. This translates to the following: $\partial_1, \dots, \partial_n$ are k-linear and for every i there is some law of the form

$$\partial_i(xy) = F_i(x, y, \partial_1(x), \partial_1(y), \dots, \partial_n(x), \partial_n(y))$$

for every $x, y \in R$, where F_i are some specific k-polynomials (in fact, precisely the polynomials defining the multiplication on B, relative to the basis $e_0, \dots, e_n$). In such a situation we will also simply say that the tuple $(\partial_1, \dots, \partial_n)$ is a B-operator.

As mentioned in the Introduction, there are two prototypical examples of B-operators, namely endomorphisms and derivations. Endomorphism correspond to $(k \times k)$-operators where $\pi\colon k \times k \to k$ is the projection on the first coordinate. If we take the basis $e_0 = (1, 0), e_1 = (0, 1)$, then ∂_1 in the above notation is a B-operator if and only if ∂_1 is a

[1]The notion of a B-operator visibly depends on π, which is not reflected in the name. Whenever we say "B-operators" we have to have some fixed π in the background.

ring endomorphism fixing k. Derivations on the other hand arise as $k[X]/(X^2)$-operators, where $\pi\colon k[X]/(X^2) \to k$ is defined by setting $X \mapsto 0$. The appropriate basis is now $e_0 = 1 + (X^2), e_1 = X + (X^2)$ and it is easy to check that ∂_1 is a $k[X]/(X^2)$-operator if and only if ∂_1 is a derivation vanishing on k.

After adding to the language of fields unary function symbols for $\partial_1, \ldots, \partial_n$ and constant symbols for elements of k, we can consider B-fields (i. e. fields with a B-operator) as first-order structures. Moosa and Scanlon proved in [**36**] that in characteristic zero the resulting theory of B-fields has a model companion, which has many nice properties, e. g. it is simple and satisfies the Zilber dichotomy for finite-dimensional minimal types.

Model theory of B-fields of positive characteristic was analysed by Beyarslan-Hoffmann-Kamensky-Kowalski in [**2**]. In this case the theory of B-fields has a model companion if and only if[2] one of the following holds:

(1) The k-algebra B is separable. In this case the model companion is bi-interpretable with $\mathrm{ACFA}_{p,n}$ (the model companion of the theory of characteristic p fields with n automorphisms), where $n = \dim B - 1$. This theory is simple.
(2) The nilradical of B coincides with the kernel of the Frobenius morphism on B (i. e. for any nilpotent element $\varepsilon \in B$ we have $\varepsilon^p = 0$). In [**2**] it was show that the resulting model companion is stable, not super stable, and eliminates quantifies after adding λ-functions to the language (we will see in Chapter 2 that adding λ_0 is enough).

[2]Actually the non-existence part was tackled already in [**36**].

CHAPTER 2

$\mathcal{B}$-operators

Using the notion of B-operators (see Preliminaries, Section 1.8) one can describe many uniformly interesting classes of operators on fields (or rings), including the prototypical ones like derivations and endomorphisms. Unfortunately, there are interesting operators, e. g. derivations of the Frobenius map, which are not B-operators. A **derivation of the Frobenius map** on a ring R of prime characteristic $p > 0$ is an additive map $\partial\colon R \to R$ satisfying the following "twisted" Leibniz rule

$$\partial\left(xy\right) = x^p \partial\left(y\right) + \partial\left(x\right) y^p$$

for all $x, y \in R$. Of course, if δ is a derivation on R then $\mathrm{Fr}_R \circ \delta$ is a derivation of the Frobenius - that is where the adjective "twisted" comes from. Fortunately, not every derivation of the Frobenius comes from an actual derivation and derivation of the Frobenius. They can not be described as B-operators, but we still can describe them in a functorial manner. Define a functor $\mathcal{F}\colon \mathrm{Alg}_{\mathbb{F}_p} \longrightarrow \mathrm{Alg}_{\mathbb{F}_p}$ in the following manner: the additive group scheme of $\mathcal{F}$ is $\mathbb{G}_{\mathrm{a}}^2$, and multiplication is given via the formula

$$\left(x, x'\right) \cdot \left(y, y'\right) := \left(xy, x^p y' + y^p x'\right).$$

It is easy to check that $\partial\colon R \to R$ is a derivation of the Frobenius if and only if $\left(\mathrm{id}, \partial\right) : R \to \mathcal{F}\left(R\right)$ is a morphism of rings.

A natural question arises: since B-operators and derivation of the Frobenius can be described "functorially" is there some common framework including both of these notions? As we will prove in this Chapter, the answer is yes. The idea is to replace the functor $- \otimes_k B$ used to define B-operators by a more general functor. These functors which we deem fit for our purposes are certain ring schemes, which we call *coordinate k-algebra schemes.* We definite them in Section 2.1 and classify them, at least over perfect fields.

Given a coordinate k-algebra schemes $\mathcal{B}$ we can define the notion of a *$\mathcal{B}$-operators* on a ring. In Section 2.2 we investigate the basic algebraic properties of such structures.

In Section 2.3 construct for any algebraic set V over a $\mathcal{B}$-field (K, ∂) another algebraic set $\tau^\partial V$ called the *prolongation of* V. This prolongation will us allow us to capture the interaction of V with ∂. Actually our construction works for any scheme over K and τ^∂ turns out to be a left-adjoint functor to certain specialization of $\mathcal{B}$. We also define in this section the notion of a $\mathcal{B}$-variety, which will be crucial in our model-theoretic analysis of $\mathcal{B}$-fields.

Finally, in Section 2.4 we talk about iterative $\mathcal{B}$-operators. We give many examples relating to notions existing in the literature and extend the algebraic results from 2.2 to the case of iterative operators.

2.1. Ring schemes

In this section, we define certain functors "governing" the class of operators, which will be introduced in the next section. We also prove a classification result (Theorem 2.17) extending the main theorem of [**28**]. For the rest of the thesis we fix a base field k (although for some of the results it is enough that k is a ring)

2.1.1. A categorical set-up

We will consider k-algebra schemes with some extra data. The notion of a k-algebra scheme is natural, but it does not seem to be very well established. For possible references, we could find only [**44**, page 148] and [**36**, Section 3] (it is called an "$\mathbb{S}$-algebra scheme" in [**36**]). Therefore, we will recall this notion below.

We start from the notion of an affine ring scheme over k, which is completely analogous the the notion of an affine group scheme.

Definition 2.1. An **affine ring scheme over** k is a representable functor from the category of k-algebras to the category of rings.[1]

Example 2.2. The affine line, represented by the polynomial ring $k[X]$, is a ring scheme over k. As a functor it is simply the forgetful functor from the category of k-algebras to the category of rings. We will denote this ring scheme by $\mathbb{S}_k$.

Definition 2.3. A k**-algebra scheme** is a ring scheme $\mathcal{B}$ together with a morphism $\iota : \mathbb{S}_k \longrightarrow \mathcal{B}$ of ring schemes over k. As in the case of k-algebras, we often say simply "$\mathcal{B}$ is a k-algebra scheme", suppressing ι from the notation.

[1] Since we assume that k is a field and most of the time consider ring schemes of finite type over k (as schemes), we could also use the name "algebraic ring" as in [**14**].

Let us mention that there are ring schemes, which do not have a k-algebra scheme structure. The main example is the ring scheme of n-truncated Witt vectors W_n for $n > 1$ (see e.g. [**37**, Lecture 26, Section 2]).

Remark 2.4. Let $\mathcal{B}$ be a k-algebra scheme and let R be a k-algebra. The structure map evaluated on R:

$$\iota_R : \mathbb{S}_k(R) = R \longrightarrow \mathcal{B}(R)$$

gives $\mathcal{B}(R)$ the structure of an R-algebra.

Example 2.5. Any finite k-algebra B (that is: B is a finite dimensional vector space over k) yields a k-algebra scheme, which we denote by $B_{\otimes}$, in the following way:

$$B_{\otimes} : \mathrm{Alg}_k \longrightarrow \mathrm{Alg}_k, \quad B_{\otimes}(R) = R \otimes_k B$$

(see [**35**, Remark 2.3]). In particular, we have $k_{\otimes} = \mathbb{S}_k$.

We state below an important result about the additive group scheme of a k-algebra scheme. If k is an algebraically closed field, then this result appears in [**14**] and it can be easily transferred to the case of a perfect field k, using for example the theory described in [**46**]. However, in the case of an arbitrary base field k, we could not find such a result in the literature. We will heavily rely on the classical results gathered in the Preliminaries, Section 1.5.

Proposition 2.6. *Let $\mathcal{B}$ be a k-algebra scheme, which as a scheme over k is of finite type and connected. Then the additive group scheme of $\mathcal{B}$ is isomorphic to $\mathbb{G}_{\mathrm{a}}^e$ for some positive integer e.*

Proof. We consider two cases.

Case 1 $\mathrm{char}(k) = 0$.
Greenberg proved this theorem in the case when k is algebraically closed (see [**14**]), so let us use base change: we get that for some positive integer e

$$(\mathcal{B}_{k^{\mathrm{alg}}}, +) \cong \mathbb{G}_{\mathrm{a}}^e.$$

Therefore, $(\mathcal{B}, +)$ is an elementary unipotent group by Fact 1.3(1). Since k is perfect, the elementary unipotent group $(\mathcal{B}, +)$ is k-split by Fact 1.3(3). Thus by Fact 1.3(4) we get that $(\mathcal{B}, +) \cong \mathbb{G}_{\mathrm{a}}^e$ for some e.

Case 2 $\mathrm{char}(k) = p > 0$.

For any k-algebra R, the group $(\mathcal{B}(R), +)$ has exponent p, since it has a structure of a vector space over k thanks to Remark 2.4. Therefore, $(\mathcal{B}, +)$ is an elementary unipotent group again by Fact 1.3(2). The k-algebra scheme structure on $\mathcal{B}$ yields an action of $\mathbb{G}_{\mathrm{m}}$ on $(\mathcal{B}, +)$ by group scheme automorphisms such that 0 is the only rational point of $(\mathcal{B}, +)$ fixed by $\mathbb{G}_{\mathrm{m}}$. Since $(\mathcal{B}, +)$ is connected, we are exactly in the situation from Fact 1.4 and we get that $(\mathcal{B}, +)$ is k-split. Therefore, we can conclude the proof as in Case 1 above. □

From now on, for any k-algebra scheme $\mathcal{B}$ satisfying the assumptions from Proposition 2.6, we just assume that $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$ (for some positive integer e). Our main definition is below.

Definition 2.7. A **coordinate k-algebra scheme** is a triple $(\mathcal{B}, \iota, \pi)$ where:

(1) $\mathcal{B}$ is a connected scheme of finite type over k,
(2) $(\mathcal{B}, \iota)$ is a k-algebra scheme,
(3) $\pi : \mathcal{B} \longrightarrow \mathbb{S}_k$ is a morphism of k-algebra schemes (that is: a morphism of ring schemes satisfying $\pi \circ \iota = \mathrm{id}$) such that under the above identification $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$, the morphisms π is the projection on the first coordinate.

Remark 2.8. Let us compare our approach to operators on rings with other set-ups from the literature:

(1) If we make an extra assumption that for each k-algebra R, the R-module structure (from Remark 2.4) on $\mathcal{B}(R) = R^e$ coincides with the product R-module structure (in other words, the isomorphism from Proposition 2.6 is an isomorphism of "k-module schemes"), then we get exactly "the basic data" from [**36**, page 5]. Therefore, preservation of the scalar multiplication by the isomorphism from Proposition 2.6 gives the main dividing line between the approach from [**36**] and the approach here.
(2) In the situation from [**36**], the entire ring scheme data is given by the finite k-algebra $B := \mathcal{B}(k)$ and the use of ring schemes can be avoided (see again [**36**, page 5]). It is not the case here, since there is no way to encode the functor $\mathcal{B}$ from Definition 2.7 using only the k-algebra $\mathcal{B}(k)$. However, if k is a perfect field, then the functor $\mathcal{B}$ *is* given by $\mathcal{B}(k)$ and a certain sequence of powers of Frobenius maps (see Theorem 2.17).
(3) There is also a related set-up by Kamensky (see [**25**] for details). The above comments on a finite k-algebra B controlling the corresponding functor apply to [**25**] as well. The main difference between [**36**] and [**25**] lies in the choice of the affine

scheme over k corresponding to the k-algebra B: the finite k-scheme $\mathrm{Spec}(B)$ is considered in [**25**], in particular, there are no ring schemes in [**25**].

Example 2.9. Assume that B is as in Example 2.5 and that we have a k-algebra map $\pi_B : B \to k$. Then, the functor $B_\otimes$ from Example 2.5 is a coordinate k-algebra scheme and this is exactly the type of ring schemes, which is considered in [**36**]. To identify $(B_\otimes, +)$ with $\mathbb{G}_\mathrm{a}^e$ it is enough to fix a basis $v_1, \ldots, v_e$ of B over k such that $\pi_B(v_1) = 1$ and $\pi_B(v_i) = 0$ for $i > 1$, which we will always tacitly do. In Proposition 2.20, we give several equivalent conditions characterizing this type of (coordinate) k-algebra schemes.

Example 2.10. The functor $\mathcal{F}$ from the introduction to this chapter is clearly a ring scheme and moreover has a natural structure of a coordinate $\mathbb{F}_p$-algebra scheme. The $\mathbb{F}_p$-algebra scheme structure on $\mathcal{F}$ is given by the following morphism:

$$\iota : \mathbb{S}_k \longrightarrow \mathcal{F}, \qquad \iota(x) = (x, 0).$$

Defining π as the projection on the first coordinate makes $\mathcal{F}$ a coordinate $\mathbb{F}_p$-algebra scheme. It is clear that $\mathcal{F}$ is not of the form $B_\otimes$. If it where, we would have $B = \mathcal{F}(k) = \mathbb{F}_p[X]/(X^2)$, hence for any $\mathbb{F}_p$-algebra R

$$\mathcal{F}(R) = \mathbb{F}_p[X]/(X^2) \otimes_{\mathbb{F}_p} R \cong R[X]/(X^2)$$

which is not true for every R. For example, if R is a field of is infinite imperfection degree, then $\dim_R \mathcal{F}(R)$ is infinite as well.

Remark 2.11. Similarly as in Example 2.10 above, we get a ring scheme structure $\mathcal{B}$ on $\mathbb{A}_k^2$ associated to any *jet operator* on k in the sense of Buium (see [**7**] and [**28**, Definition 1.1]). In the set-up of jet operators k is allowed to be a ring. In this case, there is still a morphism $\pi : \mathcal{B} \to \mathbb{S}_k$, but $\mathcal{B}$ need *not* be a k-algebra scheme (see [**7**, Example(d)]). However, if k is a field, then by a classification result from [**28**, Theorem 2.1], we get that $\mathcal{B}$ has a k-algebra scheme structure ι compatible with π, so $\mathcal{B}$ becomes a coordinate k-algebra scheme.

2.1.2. Transports

Example 2.10 shows that not every (coordinate) k-algebra scheme is of the form $B_\otimes$ for a finite k-algebra B. However, the coordinate $\mathbb{F}_p$-algebra scheme $\mathcal{F}$ from this example still can be constructed starting from a ring scheme of the from $B_\otimes$. More precisely, set $B = \mathbb{F}_p[X]/(X^2)$ and define $\phi\colon B_\otimes \to \mathcal{F}$ via $\phi(x, y) = (x, y^p)$. One easily checks that ϕ is a morphism of ring schemes. Thus, in a sense the multiplication on $\mathcal{F}$ is a formal transport of the multiplication on $B_\otimes$ via the morphism ϕ (and an actual transport on K-point if $K \supseteq \mathbb{F}_p$ is perfect).

This can be generalized. Suppose B is a finite k-algebra with a fixed basis, so that $(B_\otimes, +) = \mathbb{G}_\mathrm{a}^e$. Let $\left(c^i_{j,k}\right)_{i,j,k\leqslant e}$ be the structure constants of B relative to this basis. This means that the multiplication on Bs is given via the formula $(x_i)_{ik\leqslant e} \cdot_B (y_i)_{i\leqslant e} = (z_i)_{i\leqslant e}$ where

$$z_i = \sum_{j,k\leqslant e} c^i_{j,k} x_j y_k.$$

Let $\phi\colon (B_\otimes, +) \to \mathbb{G}_\mathrm{a}^e$ be morphism of group schemes of the form $\phi = (\mathrm{Fr}^{n_1}, \ldots, \mathrm{Fr}^{n_e})$. Then, the formal transport of $\cdot_B$ via ϕ is given by

$$(x_i)_{i\leqslant e} \cdot_\phi (y_i)_{i\leqslant e} = \phi\left(\phi^{-1}(x_i)_{i\leqslant e} \cdot_B \phi^{-1}(y_i)_{i\leqslant e}\right) = (z_i)_{i\leqslant e}$$

where

$$z_i = \sum_{j,k\leqslant e} c^i_{j,k} x_j^{p^{n_i-n_j}} y_k^{p^{n_i-n_k}}$$

If in the above formula all exponents are non-negative, then we get a well-defined multiplication on $\mathbb{G}_\mathrm{a}^e$, making it into a ring scheme. In this case we denote the resulting ring scheme by $B_{(n_1,\ldots,n_e)}$ and call it the **transport of** B **via** ϕ. Moreover, ϕ becomes a morphism of ring schemes and $B_{(n_1,\ldots,n_e)}$ is naturally a k-algebra scheme with the structural morphism being $\phi \circ \iota_B$. If $\mathcal{B}$ was a coordinate k-algebra scheme, then we get also a coordinate k-algebra scheme structure on $B_{(n_1,\ldots,n_e)}$.

If $n_1 = \ldots = n_e = n$, then we see that $B_{(n_1,\ldots,n_e)}$ always exists and we denote it by B^{Fr^n}. There is an amusing internal description of B^{Fr^n} not referring to any formal transport (see [**13**, Section 2.2]).

2.1.3. Classification of k-algebra schemes

Our goal now is to prove that over a perfect base field k *any* (coordinate) k-algebra scheme is of the form $B_{(n_1,\ldots,n_e)}$ for some finite k-algebra B and some tuple of natural numbers $(n_1, \ldots, n_e)$.

We fix a positive integer e. In the next few result we will rely on the facts about endomorphisms of $\mathbb{G}_\mathrm{a}^e$ given in the Preliminaries, Section 1.5.

Lemma 2.12. *Assume that k is perfect and $\phi\colon \mathbb{G}_\mathrm{a}^e \to \mathbb{G}_\mathrm{a}^e$ is a morphism of group schemes. Then, there are $\alpha, \beta \in \mathrm{Aut}(\mathbb{G}_\mathrm{a}^e)$ such that the composition morphism $\alpha\phi\beta\colon \mathbb{G}_\mathrm{a}^e \to \mathbb{G}_\mathrm{a}^e$ is given* *by*

$$(x_1, \ldots, x_e) \mapsto (s_1(x_1), \ldots, s_e(x_e))$$

for some monic additive polynomials $s_1, \ldots, s_e \in k[\mathrm{Fr}]$.

Proof. The morphism ϕ corresponds to an $e \times e$ matrix M with coefficients in $k[\mathrm{Fr}]$. There is an algorithm (similar to Gaussian elimination) putting M into the diagonal form, which yields the desired representation of ϕ. We present the details for $e = 2$, as the general case is analogous.

Let

$$M = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

for some $s_{ij} \in k[\mathrm{Fr}]$. Firstly, we will get rid of the $(2,1)$-entry. Assume that the degree of s_{21} is not lower than the degree of s_{11}; if this is not the case, we swap the rows of M using multiplication on the left by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Recall that right division with remainder is possible in $k[\mathrm{Fr}]$ (perfectness of k does not matters here), hence there are some $p, r \in k[\mathrm{Fr}]$ such that $s_{21} = ps_{11} + r$ with $\deg r < \deg s_{21}$. By setting $\alpha = \begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix}$, we get that $\alpha\phi = \begin{bmatrix} s_{11} & s_{12} \\ r & -ps_{12} + s_{22} \end{bmatrix}$. We reduced the degree of the $(2,1)$-entry and by iterating this process we annihilate this entry.

Left division with remainder is possible in $k[\mathrm{Fr}]$ (perfectness of k matters here) and we get rid of the $(1,2)$-entry in a similar way. Therefore, we can assume that ϕ corresponds to a diagonal matrix. By composing with a morphism of the form

$$(x_1, \ldots, x_e) \mapsto (x_1/a_1, \ldots, x_e/a_e),$$

we obtain that the additive polynomials from the diagonal of M are monic. $\square$

Lemma 2.13. *Let*

$$\phi = (s_1, \ldots, s_e) \colon \mathcal{A} \longrightarrow \mathcal{B}$$

be a morphism of ring schemes over a field k such that the additive group schemes of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic to $\mathbb{G}_{\mathrm{a}}^{\mathrm{e}}$, and $s_1, \ldots, s_e$ are as in Lemma 2.12. Then, for each i we have that s_i is a power of the Frobenius morphism or $s_i = 0$.

Proof. By the nature of the statement we are proving, we can assume that the field k is algebraically closed. By [**14**, Proposition 2.1], the ideal $\ker(\phi)$ is a connected algebraic variety. If there is i such that s_i is neither a power of Frobenius nor $s_i = 0$, then s_i is not a monomial and $\ker(s_i)$ is a finite non-trivial group. Since we have:

$$\ker(\phi) = \ker(s_1) \times \ldots \times \ker(s_e),$$

we get that $\ker(\phi)$ is not connected, which is a contradiction. $\square$

Theorem 2.14. *Assume that the field k is perfect and $\mathcal{B}$ is a k-algebra scheme such that $(\mathcal{B}, +) \cong \mathbb{G}_{\mathrm{a}}^e$. Then, we have:*

$$\dim_k(\mathcal{B}(k)) = e.$$

Proof. For any $b \in \mathcal{B}(k)$ considered as a scheme morphism $\mathrm{Spec}(k) \to \mathcal{B}$, we denote by s_b the following composition morphism of group schemes ($\mathcal{B}$ is considered below with the additive group scheme structure):

$$\mathbb{G}_{\mathrm{a}} \xrightarrow{\cong} \mathbb{G}_{\mathrm{a}} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k) \xrightarrow{\iota \times b} \mathcal{B} \times_{\mathrm{Spec}(k)} \mathcal{B} \xrightarrow{m} \mathcal{B},$$

where m is the ring scheme multiplication in $\mathcal{B}$. For any finite tuple $\bar{b} = (b_1, \ldots, b_n)$ of elements of $\mathcal{B}(k)$, we denote by $s_{\bar{b}}$ the following morphism of group schemes:

$$\mathbb{G}_{\mathrm{a}}^n \xrightarrow{s_{b_1} \times \ldots \times s_{b_n}} \mathcal{B}^n \xrightarrow{\quad + \quad} \mathcal{B},$$

where $+$ is the ring scheme addition in $\mathcal{B}$.

Claim 1

$$\dim_k(\mathcal{B}(k)) \geqslant e.$$

Proof of Claim 1. Suppose not. Then, there is $\bar{b} \in \mathcal{B}(k)^{\times(e-1)}$ such that for the following group scheme morphism (defined above):

$$s_{\bar{b}} : \mathbb{G}_{\mathrm{a}}^{e-1} \longrightarrow \mathcal{B},$$

the group homomorphisms on k-rational points:

$$(s_{\bar{b}})_k : \mathbb{G}_{\mathrm{a}}^{e-1}(k) \longrightarrow \mathcal{B}(k)$$

is onto. Since $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$, it is impossible (it is clear for a finite k; if k is infinite, then we get a dominant morphism $\mathbb{A}_k^{e-1} \to \mathbb{A}_k^e$ giving a contradiction). $\square$

Claim 2

$$\dim_k(\mathcal{B}(k)) \leqslant e.$$

Proof of Claim 2. Suppose not. Then, there is a tuple $\bar{b} \in \mathcal{B}(k)^{\times(e+1)}$ such that the homomorphism on k-rational points:

$$s := (s_{\bar{b}})_k : \mathbb{G}_{\mathrm{a}}^{e+1}(k) \longrightarrow \mathcal{B}(k)$$

is one-to-one. As in the proof of Lemma 2.12, s is given by an $e \times (e+1)$ matrix M with coefficients in $k[\mathrm{Fr}]$ and we can apply the "Gaussian elimination" over $k[\mathrm{Fr}]$ to M. Since the matrix M has less rows than columns, we can transform M into a matrix M' having at least one zero column. Since the map s corresponding to M is one-to-one, the map corresponding to M' is one-to-one as well, a contradiction. $\square$

Remark 2.15. The perfectness assumption in Theorem 2.14 was used only for the inequality in Claim 2, however, this assumption can not be dropped. For example, if k is a non-perfect field of characteristic $p > 0$, then for $\lambda \in k \setminus k^p$ and the group scheme morphism:

$$\Psi : \mathbb{G}_{\mathrm{a}}^2 \longrightarrow \mathbb{G}_{\mathrm{a}}, \qquad \Psi(x, y) = x^p + \lambda y^p,$$

we get that $\Psi_k : \mathbb{G}_{\mathrm{a}}^2(k) \to \mathbb{G}_{\mathrm{a}}(k)$ is one-to-one.

We introduce below a particular morphism of k-algebra schemes, which will play the role of ϕ from Lemma 2.13.

Notation 2.16. We assume that $\mathcal{B}$ is a k-algebra scheme such that $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$ and k is a perfect field. By Theorem 2.14, we have $\dim_k \mathcal{B}(k) = e$, so $\mathcal{B}(k)_\otimes$ is a k-algebra scheme by Example 2.5. Then, there is a natural transformation:

$$\Theta : \mathcal{B}(k)_\otimes \longrightarrow \mathcal{B}, \qquad \Theta_R : \mathcal{B}(k) \otimes_k R \to \mathcal{B}(R);$$

where for a k-algebra R, the map Θ_R is given by the universal property of the tensor product and the k-algebra homomorphisms $\mathcal{B}(k) \to \mathcal{B}(R), R \to \mathcal{B}(R)$.

We are ready now to show our main classification result.

Theorem 2.17. *Assume that k is perfect and $\mathcal{B}$ is a k-algebra scheme such that $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$. Then, $\mathcal{B}$ is isomorphic as a k-algebra scheme to $B_{(n_1, \ldots n_e)}$ where $n_1, \ldots, n_e$ are non-negative integers and $B = \mathcal{B}(k)$. Moreover, if $\mathcal{B}$ is a coordinate k-algebra scheme, then $n_1 = 0$.*

Proof. We use Lemma 2.13 in order to change the the identifications $(B_\otimes, +) = \mathbb{G}_{\mathrm{a}}^e$ and $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$ so that the morphism $\Theta\colon B_\otimes \to \beta$ of the form $\Theta = (s_1, \ldots, s_e)$ where each

s_i is either a Frobenius or zero, but since Θ_k is an isomorphism $\mathcal{B}(k)$, no zero morphism can appear. Thus $\Theta = (\mathrm{Fr}^{n_1}, \ldots, \mathrm{Fr}^{n_e})$. By the construction of the transport, this means precisely that β is the transport of B via Θ, as desired.

For the moreover part, we notice that if $\mathcal{B}$ is a coordinate k-algebra scheme, then Θ commutes with the projection on the first coordinate (since π is a natural map between $\mathcal{B}$ and the identity functor) implying $n_1 = 0$. □

We immediately get the following consequence of Theorem 2.17 saying that in the case of characteristic 0, there are no "new" k-algebra schemes.

Corollary 2.18. *Suppose that* $\mathrm{char}(k) = 0$ *and* $\mathcal{B}$ *is as in Theorem 2.17. Then, we have the following isomorphism of* k*-algebra schemes:*

$$\mathcal{B} \cong \mathcal{B}(k)_{\otimes}.$$

We also obtain the following.

Corollary 2.19. *Let* $k, \mathcal{B}$ *be as in Theorem 2.17 and suppose that* $k \subseteq K$ *is an extension of perfect fields. Then, we have the following isomorphism of* K*-algebras:*

$$\mathcal{B}(K) \cong \mathcal{B}(k) \otimes_k K.$$

Proof. By Theorem 2.17, the map

$$\Theta_K : \mathcal{B}(k) \otimes_k K \to \mathcal{B}(K)$$

(see Notation 2.16) is one-to-one. Since both the fields k and K are perfect, by Theorem 2.14 we get:

$$\dim_k(\mathcal{B}(k)) = e = \dim_K(\mathcal{B}(K)),$$

hence the map Θ_K is an isomorphism. □

We are able now to give algebraic conditions explaining when a k-algebra scheme is of the form $B_{\otimes}$. Because of Corollary 2.18, it is natural to assume that the characteristic of k is positive.

Proposition 2.20. *Assume that* k *is a perfect field of positive characteristic and* $\mathcal{B}$ *is a* k*-algebra scheme such that* $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$*. Then, the following are equivalent.*

(1) *The natural map* Θ *from Notation 2.16 is an isomorphism.*

(2) *The functor $\mathcal{B}$ is isomorphic (as a k-algebra scheme) to the functor $B_{\otimes}$ as in Example 2.5 for some k-algebra B (then, necessarily $B \cong \mathcal{B}(k)$).*

(3) *There is a "k-module scheme" isomorphism:*

$$\mathcal{B} \cong (\mathbb{S}_k)^{\times e}.$$

In particular, for each k-algebra R, we have $\mathcal{B}(R) \cong R^{\times e}$ as R-modules.

(4) *For any field extension $k \subseteq K$, we have:*

$$\dim_K(\mathcal{B}(K)) = e.$$

(5) *There is a field extension $k \subseteq K$ such that K is not perfect and:*

$$\dim_K(\mathcal{B}(K)) = e.$$

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

To show the implication (5) $\Rightarrow$ (1), we use Theorem 2.17 and obtain that up to an isomorphism, we have:

$$\Theta = (\mathrm{Fr}^{n_1}, \mathrm{Fr}^{n_2}, \ldots, \mathrm{Fr}^{n_e})$$

for some $n_1, \ldots, n_e \in \mathbb{N}$. We need to show that under the assumption of Item (5), we have $n_i = 0$ for all i. We note first, that for each field extension $k \subseteq K$, the map

$$\Theta_K = (\mathrm{Fr}_K^{n_1}, \mathrm{Fr}_K^{n_2}, \ldots, \mathrm{Fr}_K^{n_e}) : K^{\times e} \to \mathcal{B}(K)$$

is a K-linear isomorphism, where $(\mathcal{B}(K), +) = (K^{\times e}, +)$. It follows that there is the following isomorphism of vector spaces over K:

$$\mathcal{B}(K) \cong K \times (\mathrm{Fr}_K^{n_2})_*(K) \times \ldots (\mathrm{Fr}_K^{n_e})_*(K),$$

where for any field endomorphism $\rho : K \to K$, $\rho_*(K)$ denotes the K-vector space $(K, +)$ with the scalar multiplication given by:

$$a * x := \rho(a)x.$$

Therefore, if $n_i \neq 0$ for some i and K is not perfect, then we get

$$\dim_K(\mathcal{B}(K)) > e,$$

which contradicts the assumption from Item (5). $\square$

2.2. $\mathcal{B}$-operators and $\mathcal{B}$-algebra

In this section, we introduce our notion of operators on rings, called *$\mathcal{B}$-operators*. This class of operators contains both B-operators (discussed in the Preliminaries) and derivations of the Frobenius map, but also much more. In fact, as we will see in Remark 2.24, all "$\mathcal{B}$-operators laws" are "B-operator laws twisted by some Frobenius endomorphisms".

Let us fix a coordinate k-algebra scheme $(\mathcal{B}, \iota, \pi)$ with $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$.

Definition 2.21. Let R and S be k-algebras. A **$\mathcal{B}$-operator on R** is a k-algebra map $\partial\colon R \to \mathcal{B}(R)$ such that $\pi_R \circ \partial = \mathrm{id}$. More generally, given a k-algebra map $f\colon R \to S$ a **$\mathcal{B}$-operator from R to S (of f)** is a k-algebra map $\partial\colon R \to \mathcal{B}(S)$ such that $\pi_S \circ \partial = f$. The **ring of constants** of such a $\partial\colon R \to \mathcal{B}(S)$ is defined as:

$$R^{\partial} := \{r \in R \mid \partial(r) = \iota_S(f(r))\}.$$

Definition 2.22. A field with a $\mathcal{B}$-operator is called a **$\mathcal{B}$-field**. Similarly, we define **$\mathcal{B}$-rings**, **$\mathcal{B}$-field extensions**, etc.

Example 2.23. Let us give some examples of $\mathcal{B}$-operators. We fix a k-algebra R.

(1) There is always the "zero $\mathcal{B}$-operator" on R given by the structure homomorphism ι_R.

(2) Assume that $\mathcal{B} = B_{\otimes}$ for a finite k-algebra B. Then a $\mathcal{B}$-operator is the same thing as a B-operator. In particular, endomorphisms, derivations and higher derivations (and any finite sequences of them) are examples of $\mathcal{B}$-operators.

(3) Take $k = \mathbb{F}_p$. An $\mathcal{F}$-operator on R (where $\mathcal{F}$ was defined in the beginning of this chapter) is a map $R \to \mathcal{F}(R)$ of the form (id, ∂) where ∂ is a derivation of the Frobenius map.

(4) Combining items (2) and (3) above, we can consider "B-operators of Frobenius" by considering $\mathcal{B}$-operators where:

$$\mathcal{B} := (B_{\otimes})^{\mathrm{Fr}}.$$

Setting $B = k[X]/(X^2)$ we recover derivations of the Frobenius map.

Remark 2.24. Let us unwind the definition of a $\mathcal{B}$-operator. Since $(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^e$, we see that $\mathcal{B}$-operators from R to S correspond to certain sequences of maps $\partial = (\partial_1, \ldots, \partial_e)$ from R to S. More precisely, there are polynomials $F_1, \ldots, F_e \in k[X_1, Y_1, \ldots, X_e, Y_e]$ such that ∂

as above is a $\mathcal{B}$-operator if and only if for each i, the map ∂_i is additive and satisfies for any $x, y \in R$ the following identity:

$$\partial_i(xy) = F_i\left(\partial_1(x), \partial_1(y), \ldots, \partial_e(x), \partial_e(y)\right).$$

Of course, the polynomials $F_1, \ldots, F_e$ are just the polynomials defining the multiplication law on $\mathcal{B}$. From the description above, there is a clear relation between our $\mathcal{B}$-operators and Buium's *jet operators* from [**7**]. If k is a field, then jet operators coincide with $\mathcal{B}$-operators for $e = 2$. If k is an arbitrary ring, then a jet operator need not be additive, for example π-derivations from [**7**, Example(d)] are not additive.

Also, using this description, Theorem 2.17 has a natural interpretation, namely: over a perfect field k, any "$\mathcal{B}$-operator law" is a "$\mathcal{B}(k)$-operator law" twisted by a sequence of Frobenius morphisms.

Remark 2.25. Enjoyers of abstract nonsense[2] will recognize that a k-algebra R with a $\mathcal{B}$-operator is a coalgebra for the endofunctor $\mathcal{B}\colon \mathrm{Alg}_k \to \mathrm{Alg}_k$ satisfying a "counit condition". This point of view leads naturally to a framework for *iterative* $\mathcal{B}$-operators, which can be described by equipping $\mathcal{B}$ with a structure of a comonad, see Section 2.4.1. In the case of $\mathcal{D}$-ring structures this is discussed in [**35**] above Remark 2.18.

Let us briefly discuss a first order language which is appropriate for a model-theoretic analysis of $\mathcal{B}$-fields. Let $\mathcal{L}_k$ be the language of k-algebras, that is: there are constant symbols for elements of k, two binary function symbols, and a unary function symbol for each element of k. The language $\mathcal{L}_{\mathcal{B}}$ is the language $\mathcal{L}_k$ expanded by $e-1$ unary function symbols. By Remark 2.24, each $\mathcal{B}$-operator on a k-algebra R naturally gives R an $\mathcal{L}_{\mathcal{B}}$-structure. Remark 2.24 also implies that there is an $\mathcal{L}_{\mathcal{B}}$-theory $\mathcal{B}$-F, whose class of models coincides with the class of $\mathcal{B}$-fields. Additionally, if $\mathcal{L}_{\mathcal{B}}^{\lambda}$ is the language $\mathcal{L}_{\mathcal{B}}$ expanded by countably many function symbols for λ-functions (see Preliminaries, Section 1.1), then a $\mathcal{B}$-field is naturally an $\mathcal{L}_{\mathcal{B}}^{\lambda}$-structure. We will also need the expansion by only λ_0, for which the corresponding language will be denoted by $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$. By abuse of notation, we denote by $\mathcal{B}$-F the theory whose class of models coincides with the class of $\mathcal{B}$-fields, regardless which of the mentioned languages we consider (which will be either clear from the context or explicitly mentioned).

Remark 2.26. Let $(\mathcal{B}, \iota, \pi)$ and $(\mathcal{B}', \iota', \pi')$ be coordinate k-algebra schemes and suppose that $\Psi : \mathcal{B} \to \mathcal{B}'$ is a ring scheme isomorphism "over π", that is such that $\pi = \pi' \circ \psi$. Then,

[2]e. g. the author of this thesis.

for any k-algebra R, Ψ_R yields a bijection (given by polynomials over k) between the set of $\mathcal{B}$-operators on R and the set of $\mathcal{B}'$-operators on R. Therefore, the theories $\mathcal{B}$-F and $\mathcal{B}'$-F are (quantifier-freely) inter-definable.

Let us investigate some basic algebraic properties of $\mathcal{B}$-rings. First we need to isolate a certain property of $\mathcal{B}$ which roughly speaking "gives $\mathcal{B}$-operators a differential flavour". Let us consider the kernel of π as the following scheme of ideals:

$$\ker(\pi)(R) := \ker\left(\pi_R : \mathcal{B}(R) \longrightarrow R\right).$$

If $\mathcal{B} = B_{\otimes}$ for some finite k-algebra B, then it is easy to see that $\ker(\pi) = (\ker(\pi_B))_{\otimes}$.[3] Recall (Preliminaries, Section 1.8) that B is local if and only if $\ker(\pi_B)^e = 0$. The condition "$\ker(\pi)^e = 0$" makes sense for ring schemes as well and it means that the e-fold multiplication morphism:

$$m_e : \mathcal{B}^{\times e} \longrightarrow \mathcal{B}$$

vanishes on the closed subscheme $\ker(\pi)^{\times e}$.

Definition 2.27. Let $\mathcal{B}$ be a coordinate k-algebra scheme. We say that $\mathcal{B}$ is **local** if $\ker(\pi_{\mathcal{B}})^e = 0$ in the above sense.

Remark 2.28. Since the condition "$\ker(\pi)^e = 0$" can be checked on any infinite field extending k, saying that the coordinate k-algebra scheme $\mathcal{B} = B_{\otimes}$ is local is the same as saying that B is local. In particular, derivations are described using a local ring scheme. We will experience in the sequel that in general a local coordinate k-algebra scheme $\mathcal{B}$ yields operators which have a differential (as opposed to difference) flavour. This is especially true in positive characteristic p for $\mathcal{B}$ satisfying $\mathrm{Fr}\left(\ker \pi_{\mathcal{B}}\right) = 0$ (i. e. we can reduce e to p in Definition 2.27).

2.2.1. Some extension properties

It is now time to establish some basic extension properties of $\mathcal{B}$-operators, generalizing some of the results from [**2**], where the same was done for the special case of B-operators.

Lemma 2.29. *Assume that $\mathcal{B}$ is local. Suppose $f\colon R \to S$, $g\colon S \to T$ are morphisms of k-algebras and $\partial\colon R \to \mathcal{B}(T)$ is a $\mathcal{B}$-operator of $g \circ f$. If f is a formally smooth, then*

[3]Formally we defined $B_{\otimes}$ only for k-algebras, but of course it makes sense to consider $M_{\otimes}$ for any k-module M and is it to be understood as the functor $-\otimes_k M$.

∂ extends to a $\mathcal{B}$-operator $\tilde{\partial}\colon S \to \mathcal{B}(T)$ of g. If f is moreover étale, then this extension is unique.

Proof. Note that the composition of ι_T with the canonical projection

$$\mathcal{B}(T) \to \mathcal{B}(T) / \ker \pi_T$$

yields an isomorphism. Thus, using the fact that ∂ is a $\mathcal{B}$-operator of $g \circ f$, we get the following solid commutative diagram:

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, dashed] & \mathcal{B}(T) / \ker \pi_T \\
R \arrow[u, "f"] \arrow[r, "\partial"'] & \mathcal{B}(T) \arrow[u]
\end{tikzcd}$$

Since $\ker(\pi_T)^e = 0$, we are precisely in the situation from the definition of a formally smooth and étale morphism and thus there exists a dashed arrow making the above diagram commutative (which is moreover unique in the étale case) and it is easy to check that it is a $\mathcal{B}$-operator. $\square$

We have the following useful result, which follows purely formally from Lemma 2.29.

Corollary 2.30. *Assume that (N, ∂) is a $\mathcal{B}$-fields, $K \subseteq L \subseteq M \subseteq N$ is a tower of (pure) fields, ∂ restricts to a $\mathcal{B}$-operator of the inclusion $K \subseteq M$ and the extension $K \subseteq L$ is separably algebraic. Then ∂ restrict to a $\mathcal{B}$-operator of the inclusion $L \subseteq M$.*

Proof. Since separably algebraic extensions are étale, Lemma 2.29 says that there is an unique extension of $\partial|_K\colon K \to \mathcal{B}(M)$ to a $\mathcal{B}$-operator $\partial_1\colon L \to \mathcal{B}(M)$, which composed with the inclusion $\mathcal{B}(M) \subseteq \mathcal{B}(N)$ yields an extension of $\partial|_K$ to a $\mathcal{B}$-operator ∂_2 of the inclusion $K \subseteq N$. On the other hand, again by Lemma 2.29 there is a unique extension of $\partial|_K$ considered as a map $K \to \mathcal{B}(N)$ to a $\mathcal{B}$-operator $L \to \mathcal{B}(N)$. By uniqueness, this extension has to be equal ∂_2, thus the image of $\partial|_L$ is contained in $\mathcal{B}(M)$, as desired. $\square$

Corollary 2.31. *Assume that $\mathcal{B}$ is local. Suppose that $\partial\colon R \to \mathcal{B}(S)$ is a $\mathcal{B}$-operators of an embedding f and that R and S are domains. Then ∂ extends uniquely to a $\mathcal{B}$-operators between the fraction fields of R and S.*

Proof. Note that the composition ∂ with the map $\mathcal{B}(S) \to \mathcal{B}(\mathrm{Frac}(S))$ is still a $\mathcal{B}$-operator. Since localization morphisms are étale, we are again in the situation from Lemma 2.29 for $f\colon R \to \mathrm{Frac}(R)$ and $g = \mathrm{Frac}(f) : \mathrm{Frac}(R) \to \mathrm{Frac}(S)$. $\square$

Lemma 2.32. *Assume that* $\operatorname{char} k = p > 0$*,* $\mathcal{B}$ *is local and* $\partial \colon R \to \mathcal{B}(S)$ *is a* $\mathcal{B}$*-operator (of some* $f \colon R \to S$*). Then, for any* $n \in \omega$ *such* $p^n \geq e$ *we have* $R^q \subseteq R^{\partial}$*, where* $q = p^n$*. If moreover* $\operatorname{Fr}(\ker \pi_{\mathcal{B}}) = 0$*, then already* $R^p \subseteq R^{\partial}$*.*

Proof. Since $\pi_S \circ \partial = f$ and $\pi_S \circ \iota_S = \mathrm{id}_S$, we see that for any $r \in R$:

$$\partial(r) - \iota_S(f(r)) \in \ker(\pi_S).$$

Since $\mathcal{B}$ is local and $q \geq e$, we have $\ker(\pi)^q = 0$, thus $(\partial(r) - \iota_S(f(r)))^q = 0$, i. e. $\partial(r^q) = \iota_S(f(r^q))$, hence $r^q \in R^{\partial}$. The moreover part follows in the same way. $\square$

Lemma 2.32 tells that in positive characteristic $\mathcal{B}$-fields (for local $\mathcal{B}$) have big fields of constants, in the sense that the extension $K^{\partial} \subseteq K$ is algebraic. This is a very convenient property as we will see later on. One application of this is the following "squeezing lemma".

Lemma 2.33. *Assume that* $\operatorname{char} k = p > 0$*,* $\mathcal{B}$ *is local and let* $k \subseteq M \subseteq N \subseteq \Omega$ *be a tower of fields. Let* ∂ *be a* $\mathcal{B}$*-operator of the inclusion* $M \subseteq N$*, which extends to a* $\mathcal{B}$*-operator* ∂_{Ω} *of the inclusion* $N \subseteq \Omega$*. Then, there is some field* N_0 *intermediate between* M *and* N *such that the extension* $N_0 \subseteq N$ *is purely inseparable and* $\partial_{\Omega}[N_0] \subseteq \mathcal{B}(N)$*.*

Proof. Let S be a transcendence basis of N over M. If q as in Lemma 2.32, then S^q is still a transcendence basis and we have that $\partial_{\Omega}[M(S^q)] \subseteq \mathcal{B}(N)$. Let $N_0 \subseteq N$ be the relative separable closure of $M(S^q)$ in N, so that $M(S^q) \subseteq N_0$ is separably algebraic and $N_0 \subseteq N$ is purely inseparable. By Lemma 2.30 we have that $\partial_{\Omega}[N_0] \subseteq \mathcal{B}(N)$. $\square$

Proposition 2.34. *Suppose that either* $\operatorname{char} k = 0$ *and* B *is local or that* $\operatorname{char} k = p > 0$ *and* $\operatorname{Fr}(\ker \pi_{\mathcal{B}}) = 0$*. Let* $k \subseteq M \subseteq N \subseteq \Omega$ *be a tower of fields and let* ∂ *be a* $\mathcal{B}$*-operator of the inclusion* $M \subseteq N$ *which extends to a* $\mathcal{B}$*-operator* ∂_{Ω} *of the inclusion* $N \subseteq \Omega$*. Then,* ∂ *already extends to a* $\mathcal{B}$*-operator on* N*.*

Proof. If $\operatorname{char} k = 0$ and B is local then the extension $M \subseteq N$ is an extension of fields of characteristic zero, thus it is separable, hence we are done by Lemma 2.29 (Ω and the rest of the assumptions are irrelevant in this case). So assume that $\operatorname{char} k > 0$ and $\operatorname{Fr}(\ker \pi) = 0$. Let N_0 be as in Lemma 2.33. Take any $t \in N \setminus N_0$ and take the minimal $n > 0$ be such that $t^{p^n} \in N_0$. By Lemma 2.32 we have that

$$\partial_{\Omega}(t)^{p^n} = \partial_{\Omega}\left(t^{p^n}\right) = \iota\left(t^{p^n}\right) = \iota(t)^{p^n}$$

thus we may set $\partial_1(t) := \iota(t)$ to obtain a $\mathcal{B}$-operator from $\partial_1\colon N_0(t) \to \mathcal{B}(N)$ extending $\partial_\Omega|_{N_0}$. The field extension $N_0(t) \subseteq N$ is still purely inseparable, thus we may repeat this procedure till we arrive at a $\mathcal{B}$-operator $\partial_1\colon N_0(t) \to \mathcal{B}(N)$. □

Remark 2.35. Proposition 2.34 is a general version of one of the fundamental properties differentiating difference and differential algebra, namely: given a field extension $M \subseteq N$ and a derivation $\partial\colon M \to N$, if one can extend ∂ to a derivation with domain N and values in some bigger field, one can actually extend it to a derivation $\partial_N\colon N \to N$ on N. This property fails badly for endomorphisms, consider e. g. the map $\mathbb{F}_p(X) \to \mathbb{F}_p\left(X^{1/p}\right)$ defined by $X \mapsto X^{1/p}$. Thus, the assumptions in Proposition 2.34 cannot be omitted.

2.2.2. Extending along purely inseparable extensions

We know how to extend $\mathcal{B}$-operators on fields along separable extensions (see Lemma 2.29). We now want to tackle the orthogonal case of purely inseparable extensions. Let us start with an easy case which is nonetheless still immensely useful.

Lemma 2.36. *Assume that* $\mathrm{Fr}(\ker \pi_{\mathcal{B}}) = 0$. *Then, for any* $\mathcal{B}$*-field* (K, ∂) *and any* $a \in K^{\partial}$ *there is a* $\mathcal{B}$*-fields structure on* $K\left(a^{1/p}\right)$.

Proof. Assume that $a \notin K^p$ and extend ∂ to a morphism $\partial_1\colon K[X] \to \mathcal{B}\left(K\left(a^{1/p}\right)\right)$ by setting $\partial_1(X) = \iota\left(a^{1/p}\right)$. By Lemma 2.32 we have

$$\partial_1(X^p - a) = \partial_1(X)^p - \partial(a) = \iota(a) - \iota(a) = 0$$

thus $X^p - a \in \ker \partial_1$, hence ∂_1 can be lifted to a morphism $\partial_2\colon K[X]/(X^p - a) \to \mathcal{B}\left(K\left(a^{1/p}\right)\right)$. But $K[X]/(X^p - a) \cong K\left(a^{1/p}\right)$, so we are done. □

When the equality $\mathrm{Fr}(\ker \pi_{\mathcal{B}}) = 0$ does not hold, we need to do a lot more work to get a result like Lemma 2.36. First, we will introduce certain k-algebra schemes related to $\mathcal{B}$ which will allow us to speak about compositions of $\mathcal{B}$-operators.

Definition 2.37. For a coordinate k-algebra scheme $\mathcal{B}$ and a natural number $n \in \omega$ we define $\mathcal{B}^{(n)}$ as the n-fold composition of functors $\mathcal{B} \circ \ldots \circ \mathcal{B}\colon \mathrm{Alg}_k \to \mathrm{Alg}_k$, where $\mathcal{B}^{(0)} = \mathbb{S}_k$. This is naturally a coordinate k-algebra scheme $\mathcal{B}$ and for $m > n$ we have a natural projection $\mathcal{B}^{(m)} \to \mathcal{B}^{(n)}$ defined using π. This projections are morphisms of coordinate k-algebra schemes in the sense that they respect the morphisms $\pi_n\colon \mathcal{B}^{(n)} \to \mathbb{S}_k$. We define $\mathcal{B}^{(\omega)}$ as the inverse limit of the inverse system $\left(\mathcal{B}^{(m)} \to \mathcal{B}^{(n)}\right)_{m>n}$. We also denote by $\pi_n^\omega \colon \mathcal{B}^{(\omega)} \to \mathcal{B}^{(n)}$ the induced projection.

Remark 2.38. Inverse limit in Definition 2.37 is computed in the category of functors. The resulting functor $\mathcal{B}^{(\omega)}$ is an affine scheme represented by the polynomial ring in ω-many variables, as expected since each $\mathcal{B}^{(n)}$ is represented by a polynomial ring and an inverse limit of representable functors is represented by the direct limit of their representing objects. Inverse limits of functors are computed "objectwise" and we do not really need that $\mathcal{B}^{(\omega)}$ is a functor, we really need only its values on specific k-algebras R. We have that $\mathcal{B}^{(\omega)}(R)$ is the inverse limit of the system of rings $\ldots \to B^{(2)}(R) \to B^{(1)}(R) \to R$.

Definition 2.39. Assume (K, ∂) is a $\mathcal{B}$-ring and $n \in \omega$. We define $\partial^n \colon K \to \mathcal{B}^{(n)}(K)$ as the composition $\partial \circ \mathcal{B}(\partial) \circ \ldots \circ \mathcal{B}^{(n-1)}(\partial)$. It is easy to check that this system of morphisms is a cone for the system $\left(\mathcal{B}^{(m)} \to \mathcal{B}^{(n)}\right)_{m>n}$, thus it gives rise to a morphism $\partial_n^\omega \colon K \to \mathcal{B}^{(\omega)}(K)$

Remark 2.40. For every $n \in \omega$, the pair (K, ∂^n) is a $\mathcal{B}^{(n)}$-ring. If $\partial = (\mathrm{id}, \partial_1, \ldots, \partial_e)$, then the components of ∂^n are simply $\partial_{i_1} \circ \ldots \circ \partial_{i_m}$ where $m \leqslant n$ and $1 \leqslant i_1, \ldots, i_m \leqslant e$.

The following is a direct generalization of [**36**, Proposition 7.1].

Lemma 2.41. *Let K be a $\mathcal{B}$-field and $a \in K$. The following are equivalent:*

(1) *There is a $\mathcal{B}$-field extension of K containing $a^{1/p}$.*
(2) *For every $n \in \omega$ we have $\partial^n(a) \in \mathcal{B}^{(n)}\left(K^{\mathrm{alg}}\right)^p$.*

Proof. (1) $\Longrightarrow$ (2) Let $K \subseteq L$ be a $\mathcal{B}$-extensions such that $a \in L^p$. Clearly for every $n \in \omega$ we have $\partial^n(a) \in \mathcal{B}^{(n)}(L)^p$. The equality $b^p = \partial^n(a)$ is system of polynomial equations over K, where the unknowns are the components of b. Since this system has a solution in some extension of K, it has already a solution in K^{alg}. Thus $\partial^n(a) \in \mathcal{B}^{(n)}\left(K^{\mathrm{alg}}\right)^p$.

(2) $\Longrightarrow$ (1) For any $n \in \omega$ there is some tuple $b_n \in \mathcal{B}^{(n)}\left(K^{\mathrm{alg}}\right)$ such that $b_n^p = \partial^n(a)$, but the problem is that a priori the tuples b_n do not have to be coherent in the sense that they extend one another. We will first fix that, at the cost of replacing K^{alg} with some bigger field. Let Ω be a saturated algebraically closed field containing K. Let $(x_0, x_1, \ldots)$ an ω-tuple of variables. The equality

$$\mathrm{Fr}_{\mathcal{B}^{(\omega)}}(x_0, x_1 \ldots) = \partial^\omega(a)$$

defines a partial type over countably many parameters from K. The condition (2) says precisely that this partial type is finitely satisfiable, thus by saturation it is realized in Ω, say by b. Set $b_n = \partial_n^\omega(b)$. This sequence satisfies $b_n^p = \partial^n(a)$ and moreover is coherent. Note that $b_0 = a^{1/p}$. Now, since the tuples b_n are coherent and $b_n^p = \partial^n(a)$, there is a well-defined $\mathcal{B}$-operator $\partial\colon K(b) \to \mathcal{B}(K(b))$ extending $\partial\colon K \to \mathcal{B}(K)$ and satisfying $\partial(b_n) = b_{n+1}$ for $n \in \omega$ (see the proof of [**36**, Proposition 7.1] for more elaboration on that point in the case of B-operators; everything there applies to $\mathcal{B}$-operators in the same manner). The $\mathcal{B}$-field $(K(b), \partial)$ is the desired extension of K. $\square$

2.2.3. Coproducts of $\mathcal{B}$-rings

We also need the existence of coproducts in the (naturally defined) category of $\mathcal{B}$-rings. In fact, it is enough for our purposes to know that the tensor product of two $\mathcal{B}$-rings extensions over a $\mathcal{B}$-field has a natural $\mathcal{B}$-ring structure (see Corollary 2.43), but this assumption does not simplify the (already immediate) proof, so we present the stronger statement.

Proposition 2.42. *Let K, R, S be $\mathcal{B}$-rings and $K \to R, K \to S$ be $\mathcal{B}$-morphisms. Then the tensor product $R \otimes_K S$ carries a unique $\mathcal{B}$-operator such that the natural maps $R \to R \otimes_K S, S \to R \otimes_K S$ are $\mathcal{B}$-morphisms. Moreover $R \otimes_K S$ with this $\mathcal{B}$-operator is the coproduct of $K \to R$ and $K \to S$ in the category of $\mathcal{B}$-rings.*

Proof. We have the following solid commutative diagram:

$$\begin{tikzcd}
 & R \arrow[r] \arrow[dr] & \mathcal{B}(R) \arrow[dr] & \\
K \arrow[ur] \arrow[dr] & & R \otimes_K S \arrow[r, dashed] & \mathcal{B}(R \otimes_K S) \\
 & S \arrow[r] \arrow[ur] & \mathcal{B}(S) \arrow[ur] &
\end{tikzcd}$$

Since the tensor product is the coproduct in category of algebras, this diagram extends uniquely to a dashed diagram. This proves uniqueness of the desired $\mathcal{B}$-operator, once we prove that it is really a $\mathcal{B}$-operator. For this, note that by the functoriality of π we may enlarge the above diagram to the following one:

$$\begin{tikzcd}
 & R \arrow[r] \arrow[dr] & \mathcal{B}(R) \arrow[r] \arrow[dr] & R \arrow[dr] & \\
K \arrow[ur] \arrow[dr] & & R \otimes_K S \arrow[r] & \mathcal{B}(R \otimes_K S) \arrow[r] & R \otimes_K S \\
 & S \arrow[r] \arrow[ur] & \mathcal{B}(S) \arrow[r] \arrow[ur] & S \arrow[ur] &
\end{tikzcd}$$

35:11190

and by taking compositions we arrive at:

$$\begin{tikzcd}
R \arrow[dr] \arrow[drrr] & & & \\
& R \otimes_K S \arrow[r] & \mathcal{B}\left(R \otimes_K S\right) \arrow[r] & R \otimes_K S \\
S \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}$$

Since $R \to \mathcal{B}\left(R\right),\ S \to \mathcal{B}\left(S\right)$ were $\mathcal{B}$-operators, the added arrows $R \to R \otimes_K S,\ S \to R \otimes_K S$ are the canonical maps. Thus, by the universal property of the tensor product of algebras, we get that the composition of the middle arrows in this diagram is the identity, thus $R \otimes_K S \to \mathcal{B}\left(R \otimes_K S\right)$ is a $\mathcal{B}$-operator. It is also easy to show that this yields a coproduct in the category of $\mathcal{B}$-rings. $\square$

Corollary 2.43. *If K is a $\mathcal{B}$-field and $K \subseteq R, K \subseteq S$ are $\mathcal{B}$-ring extensions, then there is a unique $\mathcal{B}$-operator on the ring $R \otimes_K S$ such that the natural maps $R \to R \otimes_K S, S \to R \otimes_K S$ are $\mathcal{B}$-morphisms.*

2.3. Prolongations

The goal of this Section is to construct for every variety V over a $\mathcal{B}$-field (K, ∂) its *prolongation* $\tau^{\partial} V$. The corresponding construction for $\mathcal{D}$-rings (i. e. coordinate k-algebra schemes of characteristic zero in our terminology) was done by Moosa and Scanlon in [**34**]. We will give two motivations explaining what this object should be, one very down-to-earth (which would suffice for our intended purposes) and one more abstract but also giving more insight than the previous one.

Let us work in a big algebraically closed field Ω. Suppose $V \subseteq \Omega^n$ is a variety (or even just an algebraic set) over a $\mathcal{B}$-field (K, ∂). We would like be able to speak about point of the form $\partial\left(a\right) \in \Omega^{ne}$ for $a \in V\left(K\right)$,. This would allow us to nicely code ∂-equations (i. e. equations involving ∂ and its iterations) using varieties, as any ∂-equation (for now involving no iterations of ∂) in a tuple of variables X can be phrased as "$\partial\left(X\right) \in W$" for an appropriate algebraic set W.[4] Thus it would be nice to have a K-variety (or just an algebraic set) $\tau^{\partial} V$ such that for any $\mathcal{B}$-extension $K \subseteq L$ and any $a \in V\left(L\right)$ we have $\partial\left(a\right) \in \tau^{\partial} V\left(L\right)$, and moreover $\tau^{\partial} V$ is the smallest algebraic set with this property.

[4]Concretely, if the equation is $F\left(X, \partial_1\left(X\right), \partial_2\left(X\right), \dots\right) = 0$ for a polynomial $F\left(X_0, X_1, X_2, \dots\right)$, then W is simply the zero set of F.

Let us go to the more abstract functorial motivation. In algebraic and differential geometry one frequently interprets tangent vectors to an object X as derivations on certain ring of functions R associated to X. Tangent vectors to a nice geometric object also form a nice geometric object, namely the tangent bundle. Since X and R are dual to each other, we might as well look at this process from the point of view of R and see there is some sort of "moduli space" of derivations on R, i. e. a geometric space whose points correspond to derivations on R. This "moduli space" (or *prolongation*) can be constructed for an arbitrary ring R. Therefore, extensions of derivations on R correspond to points in some geometric space. Let us apply these ideas also for $\mathcal{B}$-operators. We fix a coordinate k-algebra scheme $\mathcal{B}$ with $(\mathcal{B},+) = \mathbb{G}_{\mathrm{a}}^{e}$. In our situation, we are interested in $\mathcal{B}$-operators, i.e elements[5] of $\mathrm{Hom}_k\left(R, \mathcal{B}\left(S\right)\right)$ for k-algebras R and S. Denote by $\tau\left(R\right)$ the coordinate ring of our desired prolongation X associated to R. We want S-rational points of X to correspond to $\mathcal{B}$-operators of the form $\partial\colon R \to \mathcal{B}\left(S\right)$ is some natural manner. Thus we want a bijection

$$X\left(S\right) = \mathrm{Hom}_k\left(\tau\left(R\right), S\right) \longleftrightarrow \mathrm{Hom}_k\left(R, \mathcal{B}\left(S\right)\right)$$

which is "natural" in S. But this precisely means that we seek for a left-adjoint of the functor $\mathcal{B}$!

Our aim here is thus to construct a left-adjoint functors. Actually, we will construct such an adjoint for some specializations $\mathcal{B}^{\partial}$ of $\mathcal{B}$ relative to a fixed $\mathcal{B}$-ring (K, ∂) - in this process points of prolongations will correspond to some $\mathcal{B}$-operators extending ∂ on K. This will allow us to formulate our geometric axioms for existentially closed $\mathcal{B}$-fields in Chapter 3, as existential closedness will translate into the existence of sufficiently many point in sufficiently nice prolongations[6].

Let us fix a $\mathcal{B}$-ring (K, ∂). We denote by Alg_K the category of K-algebras. From now on, unless mentioned otherwise, Hom refers to the category Alg_K.

2.3.1. Motivation: twisted tangent bundle

Let us look first at a concrete instance of the above ideas. The motivating example is the *twisted tangent bundle* in differential algebraic geometry. Let us fix a differential field (K, ∂) and let V be an affine variety over K. There is a naturally defined (algebraic) tangent bundle

[5]Note that any morphism $\partial\colon R \to \mathcal{B}\left(S\right)$ is a $\mathcal{B}$-operators of $f := \pi_S \circ \partial$. Geometrically, the choice of $f\colon R \to S$ will correspond to taking a base-point in our "moduli space", and the corresponding $\mathcal{B}$-operators will be rational points of the fiber over that point.

[6]...which is the idea behind any geometric axiomatization of a theory of fields with operators.

TV of V. Its defining property is that for any K-algebra R we have a bijection (natural in R)

$$TV\left(R\right)\longleftrightarrow V\left(R\left[\varepsilon\right]\right) \tag{2.1}$$

where $R\left[\varepsilon\right]:=R\left[X\right]/\left(X^{2}\right)$ is the ring of dual numbers. The construction of TV is straightforward and goes as follows. Assume that V is an affine subvariety of $\mathbb{A}_{K}^{n}$ with vanishing ideal $I\subseteq K\left[\bar{X}\right]$. Then TV is the affine subvariety of $\mathbb{A}_{K}^{2n}$ given by the ideal $\left(I,D\left(I\right)\right)\subseteq K\left[\bar{X},\bar{X}'\right]$, where $D\colon K\left[\bar{X}\right]\to K\left[\bar{X},\bar{X}'\right]$ is the unique derivation vanishing on K and satisfying $D\left(\bar{X}\right)=\bar{X}'$. It is easy to check that TV verifies (2.1).

The above construction works actually for arbitrary affine schemes. Given an affine K-scheme $V=\operatorname{Spec}A$, we can present A as a quotient $K\left[\bar{X}\right]/I$ (where $\bar{X}$ is a possibly infinite tuple of variables) and construct $\tau A:=K\left[\bar{X},\bar{X}'\right]/\left(I,D\left(I\right)\right)$ in the same way as above. Setting $TV=\operatorname{Spec}\left(\tau A\right)$ yields a scheme verifying (2.1). Using the definition of R-points (see Preliminaries) we see that (2.1) means actually

$$\operatorname{Hom}\left(\tau A,R\right)\longleftrightarrow\operatorname{Hom}\left(A,R\left[\varepsilon\right]\right) \tag{2.2}$$

i. e. the functor[7] $\tau\colon\operatorname{Alg}_{K}\longrightarrow\operatorname{Alg}_{K}$ is left-adjoint to the functor $\left(K\left[\varepsilon\right]\right)_{\otimes}:\operatorname{Alg}_{K}\longrightarrow\operatorname{Alg}_{K}$.

Another way of seeing (2.2) is that R-points of TV correspond precisely to K-derivations $K\left[V\right]\to R$, i. e. derivations vanishing on K. We would like to have an analogous scheme $T^{\partial}V$, whose R-points corresponds to derivations $\widetilde{\partial}\colon K\left[V\right]\to R$ which extend $\partial\colon K\to K$. Let us identify a derivation with the corresponding $k\left[\varepsilon\right]$-operator. An R-point of $T^{\partial}V$ should be then a ring homomorphism $\widetilde{\partial}\colon K\left[V\right]\to R\left[\varepsilon\right]$ such that the diagram

$$\begin{tikzcd}
K\left[V\right] \arrow[r, "\widetilde{\partial}"] & R\left[\varepsilon\right] \\
K \arrow[u] \arrow[r, "\partial"] & K\left[\varepsilon\right] \arrow[u]
\end{tikzcd}$$

commutes. Denote by $R^{\partial}\left[\varepsilon\right]$ the K-algebra whose underlying ring is $R\left[\varepsilon\right]$ and whose structure morphism is the composition

$$K\xrightarrow{\partial}K\left[\varepsilon\right]\longrightarrow R\left[\varepsilon\right]\ .$$

[7]We actually did not yet construct a functor, since we did not say what τ does to morphisms. The point is that we do not need to do that when constructing adjoint functors, see Remark 2.45.

Using this definition we have that an R-point of $T^{\partial}V$ should correspond to a K-algebra homomorphisms $\widetilde{\partial}\colon K \to R^{\partial}[\varepsilon]$, i. e. we should have a natural bijection

$$\operatorname{Hom}\left(K\left[T^{\partial}V\right], R\right) \longleftrightarrow \operatorname{Hom}\left(K[V], R^{\partial}[\varepsilon]\right),$$

so again at the level of K-algebras we should have a functor $\tau^{\partial}\colon \mathrm{Alg}_K \longrightarrow \mathrm{Alg}_K$ which is left-adjoint to the functor $R \mapsto R^{\partial}[\varepsilon]$. It is more or less clear how to modify the previous construction to get such τ^{∂}. We again present an algebra A as a quotient $K\left[\bar{X}\right]/I$ and set $\tau^{\partial}A := K\left[\bar{X}, \bar{X}'\right]/(I, D(I))$ where now $D\colon K\left[\bar{X}\right] \to K\left[\bar{X}, \bar{X}'\right]$ is the unique derivation extending ∂ and satisfying $D\left(\bar{X}\right) = \bar{X}'$. This yields the desired left-adjoint and we can define $T^{\partial}V$ - the *twisted tangent bundle*.

2.3.2. The general case

Our goal now is to do the above construction for general $\mathcal{B}$-operators, not only derivations. The construction is mutatis mutandis the same as the one described above, but we will give it in full details.

Remark 2.44. In the case of B-operators one can just use the classical Weil restrictions (as it was done in [**36**]). We can not use the classical Weil restriction here, since for some field extensions $k \subseteq K$, the algebra $\mathcal{B}(K)$ need not be a finite over K and then the criterion from [**6**, Section 7.6, Theorem 4] is not applicable. Still, the construction we perform "by hand" is the same as the standard construction of the Weil restrictions.

Let us fix a field extension $k \subseteq K$ and a $\mathcal{B}$-operator ∂ on K. For any K-algebra R, we set $\mathcal{B}^{\partial}(R) := \mathcal{B}(R)$ as rings and give $\mathcal{B}^{\partial}(R)$ a K-algebra structure via the composition map

$$K \xrightarrow{\partial} \mathcal{B}(K) \xrightarrow{\mathcal{B}(\rho)} \mathcal{B}(R)$$

where $\rho : K \to R$ is the structure map. It is easy to see that for any K-algebra map $f : R \to S$, the induced map $\mathcal{B}(f) : \mathcal{B}(R) \to \mathcal{B}(S)$ is a $\mathcal{B}(K)$-algebra map. Hence, it is also a K-algebra map $\mathcal{B}^{\partial}(R) \to \mathcal{B}^{\partial}(S)$ and we get a functor $\mathcal{B}^{\partial} : \mathrm{Alg}_K \longrightarrow \mathrm{Alg}_K$.

For any affine scheme V over K, we want to define its prolongation $\tau^{\partial}(V)$. The defining property of $\tau^{\partial}(V)$ is that for any K-algebra R, we should have a natural bijection:

$$\tau^{\partial}(V)(R) \longleftrightarrow V\left(\mathcal{B}^{\partial}(R)\right).$$

Again, at the level of K-algebras this means that we seek for a left-adjoint functor of the functor $\mathcal{B}^{\partial}$.

Remark 2.45. We will use the following known fact about adjoint functors (see e.g. [**29**]): a functor $R\colon \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint if and only if for any object $X \in \mathcal{C}$ there is some object $L(X) \in \mathcal{D}$ and a natural isomorphism of functors

$$\mathrm{Hom}_{\mathcal{D}}(L(X), -) \longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(X, R(-)).$$

In other words, it is enough to construct a left adjoint "objectwise". To see why this is the case, let us see how to extend the assignment $X \mapsto L(X)$ to a functor. Let $f\colon X_0 \to X_1$ be a morphism in $\mathcal{C}$. We have the following solid diagram (of natural transformations of functors)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(X_1), -) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_{\mathcal{C}}(X_1, R(-)) \arrow[d, "f_*"] \\
\mathrm{Hom}_{\mathcal{D}}(L(X_0), -) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X_0, R(-))
\end{tikzcd}$$

where f_* is induced by f. Since the horizontal arrows are isomorphisms, there is a unique dashed arrow making the above diagram commute - it is just the transport of f_* via the horizontal arrows. By Yoneda lemma, this arrow is induced by a unique morphism $L(f) : L(X_0) \to L(X_1)$ and using the uniqueness of $L(f)$ one immediately checks that the assignment $f \mapsto L(f)$ makes L into a functor, which by force is left adjoint to R.

Theorem 2.46. *The functor $\mathcal{B}^{\partial}\colon \mathrm{Alg}_K \longrightarrow \mathrm{Alg}_K$ has a left-adjoint functor.*

Proof. Using Remark 2.45 it is enough to construct for any K-algebra R a K-algebra $\tau^{\partial}R$ such that there is a natural isomorphism of funtors:

$$\mathrm{Hom}\left(R, \mathcal{B}^{\partial}(-)\right) \longleftrightarrow \mathrm{Hom}\left(\tau^{\partial}(R), -\right).$$

For a (possibly infinite) tuple of variables X, let $\bar{X} := (X_1, \ldots, X_e)$ denote a new tuple of variables (each X_i has the same length as X). Firstly, we extend ∂ to a $\mathcal{B}$-operator $\bar{\partial}$ from $K[X]$ to $K[\bar{X}]$. Since we have a natural bijection:

$$\mathcal{B}\left(K\left[\bar{X}\right]\right) \longleftrightarrow K\left[\bar{X}\right]^{\times e},$$

we can consider $\bar{X}$ as an element of $\mathcal{B}\left(K[\bar{X}]\right)$ and we define $\bar{\partial}(X) := \bar{X}$. Together with ∂ on K, this determines a K-algebra homomorphism

$$\bar{\partial} : K[X] \longrightarrow \mathcal{B}\left(K\left[\bar{X}\right]\right),$$

which is a $\mathcal{B}$-operator from $K[X]$ to $K[\bar{X}]$, so it can be identified with a tuple $(\partial_1, \ldots, \partial_e)$, where each ∂_i is a map from $K[X]$ to $K[\bar{X}]$ (see Remark 2.24).

We can assume that $R = K[X]/I$ for an appropriate X as above and an ideal I in $K[X]$. We define our adjoint functor by setting:

$$K[X]/I \mapsto K\left[\bar{X}\right]/\bar{I}, \qquad \bar{I} := \left(\partial_1(I) \cup \ldots \cup \partial_e(I)\right).$$

We will check that this construction gives the desired adjunction. That is, we need to find the following natural bijection:

$$\operatorname{Hom}\left(K[X]/I, \mathcal{B}^\partial(T)\right) \longleftrightarrow \operatorname{Hom}\left(K\left[\bar{X}\right]/\bar{I}, T\right).$$

Since we have a K-scheme identification $\mathcal{B}^\partial = \mathbb{A}^e_K$, we get that the K-algebra maps $\Psi : K[X] \to \mathcal{B}^\partial(T)$ are in a natural bijection with the K-algebra maps $\bar{\Psi} : K[\bar{X}] \to T$, where $\bar{\Psi}(X_i) = \Psi_i(X)$ (after the identification $\Psi = (\Psi_1, \ldots, \Psi_e)$, where $\Psi_i : K[X] \to T$). It is easy to see (by checking the commutativity on X) that we have the following commutative diagram:

$$\begin{tikzcd} K[X] \arrow[r, "\Psi"] \arrow[d, "\bar{\partial}"'] & \mathcal{B}^\partial(T) \\ \mathcal{B}^\partial(K\left[\bar{X}\right]), \arrow[ur, "\mathcal{B}^\partial(\bar{\Psi})"'] & \end{tikzcd}$$

which implies that for all $f \in K[X]$, we have:

$$\Psi(f) = 0 \qquad \Longleftrightarrow \qquad \forall i \;\; \bar{\Psi}\left(\partial_i(f)\right) = 0.$$

In particular, we get that:

$$I \subseteq \ker(\Psi) \qquad \Longleftrightarrow \qquad \bar{I} \subseteq \ker\left(\bar{\Psi}\right).$$

Hence, the bijection $\Psi \leftrightarrow \bar{\Psi}$ extends to our adjointness bijection. $\square$

Remark 2.47. For the above construction, we do not need all the data from the definition of a coordinate k-algebra scheme. It is enough to assume (restricting to the category of K-algebras) that $\mathcal{B}$ is a ring scheme over K whose underlying scheme is $\mathbb{A}^e_K$.

For a K-algebra R, we describe now a natural homomorphism of K-algebras

$$\pi^\partial : R \longrightarrow \tau^\partial(R),$$

whose properties will be important in the sequel. The construction of this homomorphism uses the morphism π from the coordinate k-algebra scheme data. Consider the adjointness bijection:

$$\operatorname{Hom}\left(\tau^{\partial}(R), \tau^{\partial}(R)\right) \longrightarrow \operatorname{Hom}\left(R, \mathcal{B}^{\partial}(\tau^{\partial}(R))\right), \qquad f \mapsto f^{\sharp}.$$

Then, π^{∂} is defined as the composition of the following maps:

$$R \xrightarrow{\left(\mathrm{id}_{\tau^{\partial}(R)}\right)^{\sharp}} \mathcal{B}^{\partial}(\tau^{\partial}(R)) \xrightarrow{\pi_{\tau^{\partial}(R)}} \tau^{\partial}(R).$$

Definition 2.48. Let $V = \operatorname{Spec}(R)$ be an affine K-scheme.

(1) We define the *∂-prolongation* of V as

$$\tau^{\partial} V := \operatorname{Spec}\left(\tau^{\partial} R\right).$$

(2) We have a natural morphism

$$\pi^{\partial} : \tau^{\partial} V \longrightarrow V,$$

which was described above on the level of K-algebras.

Remark 2.49. Let us point out a more down-to-earth description of the above notions. Let $V \subseteq \mathbb{A}_K^n$ be an affine K-variety defined by a (prime) ideal $I \subseteq K\left[\bar{X}\right]$. The ∂-prolongation of of V is then the algebraic set $\tau^{\partial} V \subseteq \mathbb{A}_K^{ne}$ defined by the ideal generated by the set $\partial_1(I) \cup \ldots \cup \partial_e(I)$, where $\partial_1, \ldots, \partial_e$ are as in the proof of Theorem 2.46. Since $\partial_1 = \mathrm{id}$, the projection onto the first n coordinates defines a morphism $\tau^{\partial} V \to V$ and this is precisely the morphism π^{∂} mentioned above.[8]

We would like to point out an interpretation of rational points of prolongations in terms of $\mathcal{B}$-operators, as promised in the beginning of this section. If R is a K-algebra, then (by adjointness) the set of R-rational points $\tau^{\partial} V(R)$ is in a natural bijection with the set of $\mathcal{B}$-operators from $K[V]$ to R extending ∂. Moreover, if $b \in V(R)$, then the fiber $\tau_b^{\partial} V(R)$ (considered as a subset of $\tau^{\partial} V(R)$) corresponds to $\mathcal{B}$-operators $\widetilde{\partial} : K[V] \to \mathcal{B}(R)$ such that $\pi_R \circ \widetilde{\partial} = b$, where b is considered as K-algebra homomorphism $K[V] \to R$. In particular, we get the following.

Remark 2.50. There is a natural (in V) map (not a morphism!):

$$\partial_V : V(K) \longrightarrow \tau^{\partial} V(K) = V\left(\mathcal{B}^{\partial}(K)\right)$$

[8] We are slightly lying here. If we want to pass between varieties and schemes, then the algebraic set $\tau^{\partial} V$ in the classical sense corresponds to the reduced scheme associated to $\tau^{\partial} V$ in the scheme-theoretic sense. This distinction is negligible for our purposes.

coming from composing the map $x : K[V] \to K$ with ∂ (equivalently, $\partial_V = V(\partial)$).

Lemma 2.51. *Suppose that V and W are affine K-schemes and $W \subseteq \tau^{\partial}(V)$. Then, we get a natural $\mathcal{B}$-operator*

$$\partial_V^W : K[V] \longrightarrow \mathcal{B}(K[W])$$

extending ∂.

Proof. The inclusion $W \subseteq \tau^{\partial}(V)$ gives a rational point $\partial_V^W \in \tau^{\partial}(V)(K[W])$, which corresponds to the desired $\mathcal{B}$-operator extending ∂ as it was observed above. $\square$

Let us now look at what the above considerations tell in the classical case. We fix a tower of fields $k \subseteq K \subset \Omega$ and a $\mathcal{B}$-operator ∂ on K. We assume that Ω is a big algebraically closed field. From now on, if we say "$\mathcal{B}$-operator from R to S" it means that S is an extension of R and we are considering a $\mathcal{B}$-operator of the inclusion $R \subseteq S$. We also fix the following data:

- $n > 0$, $a \in \Omega^{\times n}$ and $(a, b) \in \Omega^{\times ne}$;
- $V = \mathrm{locus}_K(a)$;
- $W = \mathrm{locus}_K(a, b)$.

In this set-up Lemma 2.51 translates into the following.

Lemma 2.52. *The following are equivalent.*

(1) *There is a $\mathcal{B}$-operator $\partial' : K[a] \to \mathcal{B}\left(K[a, b]\right)$ extending ∂ such that $\partial'(a) = (a, b)$.*
(2) *$W \subseteq \tau^{\partial}(V)$.*

2.3.3. $\mathcal{B}$-varieties

The point of this subsection is to adapt classical geometric notions from algebraic and differential-algebraic geometry to the context of $\mathcal{B}$-algebra. We fix a $\mathcal{B}$-field (K, ∂).

Definition 2.53. A **$\mathcal{B}$-variety over** (K, ∂) is a pair (V, s) consisting of an affine K-variety V together with a morphism $s\colon V \to \tau^{\partial}V$ which is a section of $\pi_V^{\partial}\colon \tau^{\partial}V \to V$, i. e. it satisfies $\pi_V^{\partial} \circ s = \mathrm{id}_V$.

Since most of the time we will work only with varieties over our fixed $\mathcal{B}$-field (K, ∂), we will often say "$\mathcal{B}$-variety" instead of "$\mathcal{B}$-variety over (K, ∂)", if no confusion arises. Also, we will say that (V, s) is absolutely irreducible (resp. separable) if the underlying K-variety V is such.

Remark 2.54. Let (V, s) be a $\mathcal{B}$-variety. Then s correspond to a K-algebra homomorphism $s^*\colon K\left[\tau^{\partial}V\right] \to K[V]$, which in turn corresponds by adjointness to a K-algebra homomorphism $\partial_s\colon K[V] \to \mathcal{B}^{\partial}(K[V])$. One easily checks that under this correspondence the condition $\pi_V^{\partial} \circ s = \mathrm{id}_V$ translates into $\pi_{K[V]} \circ \partial_s = \mathrm{id}_{K[V]}$, i. e. that ∂_s is a $\mathcal{B}$-operator on $K[V]$ extending ∂ on K. Thus a $\mathcal{B}$-variety structure on V is the same as a $\mathcal{B}$-ring (over (K, ∂)) structure on $K[V]$.

Remark 2.55. Remark 2.54 can be elegantly phrased as: the category of finitely generated (in the algebraic sense) $\mathcal{B}$-domains over (K, ∂) is dual to the naturally defined category of $\mathcal{B}$-varieties. This is completely analogous to the case of classical algebraic geometry. In the same vain we could define (affine) $\mathcal{B}$-schemes *et cetera*. We do not do any of this, since we really only use $\mathcal{B}$-varieties as a simple way of encoding certain $\mathcal{B}$-algebras over K. We point out however that e. g. difference schemes where fundamental in Hrushovski's work on the elementary theory of the Frobenius automorphism (see **[20]**).

The reason we need $\mathcal{B}$-varieties is because they can code "systems of ∂-equation". This is analogous to algebraic geometry where a variety codes a system of polynomial equations. In the classical setting a K-rational point of a variety V is the same as a solution in K of the system coded by V. The $\mathcal{B}$-algebraic counterpart of this notion is the following.

Definition 2.56. Let (V, s) be a $\mathcal{B}$-variety and let L be a $\mathcal{B}$-extension of K. An **L-rational $\mathcal{B}$-point** (V, s) is a point $a \in V(L)$ such that $s(a) = \partial(a)$. We denote the set of all L-rational $\mathcal{B}$-points by $(V, s)^{\sharp}(L)$.

Elaborating on Remark 2.54 yields the following useful lemma.

Lemma 2.57. *Let (V, s) be a $\mathcal{B}$-variety and let a be a generic point of V over K. Under the identification $K(V) = K(a)$ we have that $a \in (V, s)^{\sharp}(K(V))$.*

Proof. Let I be the ideal of V, write $K[V] = K[X]/I$ and identify a with $X + I$. By Theorem 2.46 $K\left[\tau^{\partial}V\right] = K\left[\bar{X}\right]/\bar{I}$ where $\bar{X} = (X_1, \ldots, X_e)$ and $\bar{I} = (\partial_1(I) \cup \ldots \cup \partial_e(I)) \trianglelefteq K\left[\bar{X}\right]$. The adjoint of s^*, i. e. ∂_s, is by definition the map $\partial_s\colon K\left[\bar{X}\right]/\bar{I} \to \mathcal{B}(K[X]/I)$ induced by sending X to the tuple $(s^*(X_1), \ldots, s^*(X_e))$, thus $\partial_s(X + I) = s^*(X_i) + \bar{I} = s(X + I)$, i. e. $X + I$ is a $\mathcal{B}$-point of (V, s). $\square$

2.4. Iterative $\mathcal{B}$-fields

In this section we will first review various types of iterative operators appearing in the literature and how they generalize to $\mathcal{B}$-operators, see Subsection 2.4.1. In Subsection 2.4.2

we generalize some of the results of Section 2.2 to the case of iterative $\mathcal{B}$-operators. Most importantly, we prove a variant of Proposition 2.34 for a wide class of iterative operators.

2.4.1. Examples

The most natural example of iterative operators are group actions on fields, which are of course extremely important in mathematics. Also, model-theoretic considerations of group actions on fields are very interesting and fruitful (most notably, the consideration of $\mathbb{Z}$-actions, i. e. the case of the theory ACFA).

For a fixed finite group G, the theory of G-fields (i. e. fields with an action of G by field automorphisms) is companionable, as proved by Hoffmann-Kowalski in [**16**]. The datum of a G-field K, i. e. the sequence of automorphisms $(\sigma_g\colon K \to K | g \in G)$, forms a k^G-operator, where k^G is equipped with a convolution product (alternatively, k^G is the bialgebra dual to the group ring $k[G]$). The iterativity condition (i. e. the statements $\sigma_{gh} = \sigma_g \circ \sigma_h$ for $g, h \in G$) can be expressed using the Hopf algebra structure of k^G, as we will see in the next paragraph.

More generally, we can consider actions of *finite group schemes*. Given a finite (affine) group scheme $\mathfrak{g}$ over k whose corresponding Hopf algebra is H, we can consider $\mathfrak{g}$-fields, i. e. fields K with an action of $\mathfrak{g}$ on $\operatorname{Spec} K$. This is the same as equipping K with an H-operator $\partial\colon K \to K \otimes_k H$ (where $\pi\colon H \to k$ is the counit of H) such that the following diagram commutes

$$\begin{tikzcd}
R \arrow[r, "\partial"] \arrow[d, "\partial"'] & R \otimes_k H \arrow[d, "\partial \otimes \mathrm{id}"] \\
R \otimes_k H \arrow[r, "\mathrm{id} \otimes \mu"'] & R \otimes_k H \otimes_k H
\end{tikzcd}$$

where $\mu\colon H \to H \otimes_k H$ is the comultiplication map. Thus, finite group schemes provide a very natural way of describing iterativity of Moosa-Scanlon operators.

Model theory of $\mathfrak{g}$-fields was analysed by Hoffmann and Kowalski in [**17**], where they proved that the theory of $\mathfrak{g}$-fields has a model companion, which is moreover simple. Thus, the basic model-theoretic properties of the "right" notion of iterativity of B-operators are settled.

It seems now natural to ask if we can continue the pursuit of iterativity with $\mathcal{B}$-operators. We claim that it is indeed possible by means of *comonads*. These are certain category-theoretical objects (appearing also in computer science, e. g. in functional programming, see [**48**]), which in our context can be seen as "generalized Hopf algebras" (see Remark 2.59).

Definition 2.58. A **comonad** on a category $\mathcal{C}$ is a comonoid object in the category of endofunctors on $\mathcal{C}$. More concretely, a comonad on $\mathcal{C}$ consists of a functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}$ together with two natural transformations:

(1) the *counit* $\varepsilon\colon F \longrightarrow \mathrm{id}_{\mathcal{C}}$,
(2) the *comultiplication* $\mu\colon F \longrightarrow F^2$ (here F^2 is the composition of F with itself),

such that the following diagrams commute:

$$\begin{tikzcd} F \arrow[r, "\mu"] \arrow[d, "\mu"'] & F^2 \arrow[d, "F\mu"] \\ F^2 \arrow[r, "\mu F"'] & F^3 \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\mu"] \arrow[d, "\mu"'] \arrow[dr, "\mathrm{id}"] & F^2 \arrow[d, "F\varepsilon"] \\ F^2 \arrow[r, "\varepsilon F"'] & F \end{tikzcd}$$

Remark 2.59. We are only interested in the case where $\mathcal{C} = \mathrm{Alg}_k$ is the category of k-algebras and $F = \mathcal{B}$ is a coordinate k-algebra scheme. In this case comonads can be seen as generalizations of Hopf algebras: if B is a k-algebra, then a comonoid structure on the functor $- \otimes_k B$ is the same as a Hopf algebra structure on B (the first diagram above expresses then the coassociativity of comultiplication and the second expresses the counit condition). Thus, comonads on Alg_k might be seen of as "nonlinear" Hopf algebras and are a natural candidate for a way of describing iterativity. Also, since our functors $\mathcal{B}$ are representable (by polynomial rings), a comonad amount to adding some structure to the representing object. This additional structure is the structure of a *plethory*, see [**5**].

Definition 2.60. Let $\mathcal{B}$ be a coordinate k-algebra scheme and give it the structure of a comonad with comultiplication μ whose counit is the morphism $\pi\colon \mathcal{B} \to \mathbb{S}_k$. We say that a $\mathcal{B}$-operator $\partial\colon R \to \mathcal{B}(R)$ is **μ-iterative** if the diagram

$$\begin{tikzcd} R \arrow[r, "\partial"] \arrow[d, "\partial"'] & \mathcal{B}(R) \arrow[d, "\mathcal{B}(\partial)"] \\ \mathcal{B}(R) \arrow[r, "\mu_R"'] & \mathcal{B}^2(R) \end{tikzcd}$$

commutes.

Example 2.61. If we take $\mathcal{B} = H_{\otimes}$ where B is a Hopf algebra and μ is induced by the comultiplication, then a field with a μ-iterative B-operator is the same thing as a $\mathfrak{g}$-field, where $\mathfrak{g} = \operatorname{Spec} H$.

Remark 2.62. As affine schemes $\mathcal{B}$ and $\mathcal{B}^2$ are just affine spaces over k of dimension e and e^2 respectively. Thus μ correspond to e^2 polynomials $\mu_{i,j}\left(\bar{X}\right) \in k\left[\bar{X}\right]$ in e variables. A $\mathcal{B}$-operator $(\partial_1, \dots, \partial_e) = \partial\colon R \to \mathcal{B}(R)$ is μ-iterative if and only if for every $i, j \leqslant e$ the equality

$$(\partial_i \circ \partial_j)(x) = \mu_{i,j}(\partial_1(x), \dots, \partial_e(x))$$

holds for every $x \in R$. In particular, finitely generated μ-iterative field extension are finitely generated as pure field extensions.

In a different direction, we can take a coordinate k-algebra scheme $\mathcal{B}$ and consider fields K with n distinct $\mathcal{B}$-operators $\partial_1, \dots \partial_n\colon K \to \mathcal{B}(K)$ and demand that they all (or just some of them) commute, i. e. $\partial_i \circ \partial_j = \partial_j \circ \partial_i$. Note that both sides of this equality are morphisms $K \to \mathcal{B}^{(2)}(K)$, so it makes sense to state this equality.

In yet another direction, we can consider fields with a $\mathcal{B}$-field (K, ∂) together with an action of a finite group G via $\mathcal{B}$-automorphisms (i. e. the action of G commutes with ∂).

2.4.2. Some algebraic properties of $\mathcal{B}_\varphi$-fields

Motivated by Section 2.4.1, we give the following definition.

Definition 2.63. An **iterativity condition** is a universal $\mathcal{L}_{\mathcal{B}}$-sentence φ of the form form $(\forall x)\,\theta(x)$ where $\theta(x)$ is a quantifier-free $\mathcal{L}_{\mathcal{B}}$-formula in one free variable x, such that the following holds:

(1) for any $\mathcal{B}$-field K generated as a field by a set A we have that $K \models \varphi$ if and only if $K \models \theta(a)$ for any $a \in A$,

(2) for any field K, the "zero $\mathcal{B}$-field" (K, ι_K) satisfies φ.

Example 2.64. If $(\mathcal{B}, \mu)$ is a comonad, then the sentence describing μ-iterative $\mathcal{B}$-fields is $\varphi =$ "$(\forall x)\, \theta(x)$" where $\theta(x)$ is the conjunction of the equalities appearing in Remark 2.62. One immediately checks that φ is an iterativity condition.

Definition 2.65. Let (K, ∂) be a $\mathcal{B}$-field and let (V, s) be a $\mathcal{B}$-variety over K. We say that K is a **$\mathcal{B}_\varphi$-field** it satisfies φ. We say that a $\mathcal{B}$-variety (V, s) over a $\mathcal{B}_\varphi$-field (K, ∂) is a **$\mathcal{B}_\varphi$-variety** if the corresponding $\mathcal{B}$-field $K\,(V)$ is a $\mathcal{B}_\varphi$-field.

Since all examples from Section 2.4.1 are given in a functorial manner, it is easy to check that they all give rise to iterativity conditions in the above sense.

Remark 2.66. Being a $\mathcal{B}_\varphi$-variety is definable condition, by which mean the following. Fix a $\mathcal{B}_\varphi$-field K and any natural numbers n, k, d. Let us consider all K-algebraic sets $V \subseteq \Omega^n$ together with a morphism $s\colon V \to \tau^\partial V$ such that the ideal $I_K\,(V)$ is generated by at most k polynomials of degree at most d and that the morphism s is given by polynomials of degree at most d. Having fixed that, we can code all the polynomials mentioned in the previous sentence using tuples $\bar{c}$ of of fixed length of elements of K. This length depends only on n, k, d. Thus we can code V via a tuple of elements of K (in a highly non-unique manner). Then, there is some $\mathcal{L}_\mathcal{B}$-formula $\psi\,(\bar{x})$ depending only on n, k, d such that (V, s) is a $\mathcal{B}_\varphi$-variety if and only if $K \models \psi\,(\bar{c})$. To see that recall the classical results of van den Dries (see [**47**]) on bounds for polynomial ideals. Namely, for any $n, k, d \in \omega$ there is a number N such that for any field K and any polynomials $h, f_1, \ldots, f_k \in K\,[X_1, \ldots, X_n]$ of degree at most d **if** there are some $g_1 \ldots, g_k \in K\,[X_1, \ldots, X_n]$ such that

$$h = g_1 f_1 + \ldots + g_k f_k,$$

then there are such $g_1, \ldots, g_k$ of degree at most N. Using this, one can write down a formula $\chi\,(\bar{x})$ such that $K \models \chi\,(\bar{c})$ if and only if V is K-irreducible, the image of s is a subset of $\tau^\partial V$ and that (V, s) is a $\mathcal{B}$-variety. Since φ is universal can be checked on generators (see item (1) in Definition 2.63), there is also a formula $\psi\,(\bar{x})$ asserting that moreover (V, s) is a $\mathcal{B}_\varphi$-variety.

We want to extend Proposition 2.34 to the case of $\mathcal{B}_\varphi$-fields. Obviously, some restrictions are needed (e. g. see Remark 2.35), so let us give some motivating intuitions what sort of operators satisfy a variant of Proposition 2.34.

(1) Automorphisms fail that miserably in general (see Remark 2.35).

(2) On the other hand actions of finite group are fine. The algebraic property behind that is the following: if a G-field K is generated as a G-field by a tuple a, then K is generated as a pure field by $G \cdot a$. In the same manner action of finite groups schemes or more generally actions of comonads are fine (see also Remark 2.62).

(3) $\mathcal{B}$-operators for $\mathcal{B}$ satisfying $\operatorname{Fr}(\ker \pi_{\mathcal{B}}) = 0$ are fine by Proposition 2.34 and so are $\mathcal{B}$-operators for local $\mathcal{B}$ of characteristic zero.

(4) Combining (2) and (3) should also be fine, because of the strong finiteness property of (2).

Motivated by the above, we introduce the following useful class of pairs $(\mathcal{B}, \varphi)$.

Definition 2.67. We say that the pair $(\mathcal{B}, \varphi)$ is **nice** if one of the following holds:

(1) There is a comultiplication μ on $\mathcal{B}$ such that $(\mathcal{B}, \mu)$ is a comonad (with the counit being $\pi \colon \mathcal{B} \to \mathbb{S}_k$) and φ is as in Example 2.64,

(2) $\operatorname{char} k = 0$ and $\mathcal{B}$ is local and φ is trivial (i. e. holds always).

(3) $\operatorname{char} k = p > 0$ and there is coordinate k-algebra scheme $\mathcal{B}_l$, a finite group G and a natural number $n \in \omega$ such that $\mathcal{B}_\varphi$-fields are fields K with n (possibly commuting) $\mathcal{B}$-operators $\partial_1, \ldots, \partial_n$ and an G-action on K commuting with $\partial_1, \ldots, \partial_n$.

Proposition 2.68. *Assume that $(\mathcal{B}, \varphi)$ is nice. Let $(K, \partial_K) \subseteq (L, \partial_L)$ be $\mathcal{B}_\varphi$-fields, let $a \in L$ be a finite tuple and set $K_1 = K(\partial_L a)$. Then there is a $\mathcal{B}$-operator $\partial \colon K_1 \to \mathcal{B}(K_1)$ such that $\partial(a) = \partial_L(a)$, $\partial|_K = \partial_K$ and (K_1, ∂) is a $\mathcal{B}_\varphi$-field.*

Proof. If we are in item (1) of Definition 2.67, then there is nothing to do, since by Remark 2.62 we have that K_1 is a $\mathcal{B}$-subfield of L.

Assume we are in item (2) of Definition 2.67, then we apply Lemma 2.29: the extension $K(a) \subseteq K_1$ is separable (since we are in characteristic zero), hence étale, thus the $\mathcal{B}$-operator $\partial \colon K(a) \to \mathcal{B}(K_1)$ extends to a $\mathcal{B}$-operator on K_1 and since φ is trivial, the resulting $\mathcal{B}$-field K_1 is a $\mathcal{B}_\varphi$-field.

Finally, assume that we are in item (3) of Definition 2.67. We write ∂^l for the $\mathcal{B}_l^{\times n}$-operator $(\partial_1, \ldots, \partial_n)$. As in the case of comonads, K_1 is preserved under the action of G and we will not alter this action. We will now define ∂^l on K_1 We will repeat the proof Proposition 2.34 with slightly more care. By Lemma 2.33 applied to the tower $K(a) \subseteq K_1 \subseteq L$ and $\partial_\Omega = \partial_L^l$ we get that there is a field K_0 intermediate between $K(a)$ and K_1 such that $\partial_L^l[K_0] \subseteq \mathcal{B}_l(K_1)$. Define $\partial_0 \colon K_0 \to \mathcal{B}_l(K_1)$ as the restriction of ∂_L^l to K_0. Let

$b = (b_1, \ldots, b_n)$ be a minimal subtuple of a such that $K_0(b) = K_1$ (since a is a finite tuple and $K_0(a) = K_1$ such b exists). By minimality $b_1 \notin K_1$ so we may proceed as in the proof of Proposition 2.34 and construct a (necessarily unique) $\mathcal{B}_l$-operator $\partial_1 \colon K_0(b_1) \to \mathcal{B}(K_1)$ such that $\partial_1(b_1) = \iota_{\mathcal{B}_l}(b_1)$.

Again by minimality we have that $b_{k+1} \notin K_1(b_1, \ldots, b_k)$ for any $k < n$, thus we may repeat the above construction to arrive at a $\mathcal{B}_l$-operator ∂^l on $K_1 = K_0(b_1, \ldots, b_n)$ such that ∂^l extends ∂_0 and $\partial^l(b_k) = \iota_{\mathcal{B}_l}(b_k)$ for $k = 1, \ldots, n$. In particular $\partial^l(a) = \partial_L(a)$, $\partial^l|_K = \partial_K$. Since we sent all the generators to zero, it is easily to check that the resulting ∂^l is preserved under the action of G and still satisfies the commutativity condition. ☐

Let $(K, \partial) \subseteq (L, \partial)$ be $\mathcal{B}_\varphi$-fields and let $a \in L$ be a finite tuple such that $L = K(a)$. In general there is no well-defined "$\mathcal{B}$-locus of a over K", as $K[a]$ is not necessarily closed under ∂. The following lemma says however, that (for some $\mathcal{B}$) we can construct the $\mathcal{B}$-locus at the cost of extending the tuple a.

Lemma 2.69. *Assume that either $\mathcal{B}$ is local or $(\mathcal{B}, \varphi)$ is nice. Let $(K, \partial) \subseteq (L, \partial)$ be $\mathcal{B}_\varphi$-fields and let $a \in L$ be a finite tuple such that $L = K(a)$. Then a can be extended to a finite tuple b so that ∂ restricts to a $\mathcal{B}$-operator on $K[b]$.*

Proof. Assume $\mathcal{B}$ is local. Note that we have a $\mathcal{B}$-operator $\partial \colon K[a] \to \mathcal{B}(K[\partial a])$ and since the extension $K[a] \subseteq K(\partial a)$ is étale (because it is a localization) this $\mathcal{B}$-operator extends uniquely to a $\mathcal{B}$-operator ∂' on $K[a] \subseteq K(\partial a)$ by Lemma 2.29. But by uniqueness we must have $\partial' = \partial$ so the tuple $b := \partial(a)$ works.

If $\mathcal{B}$ is a comonad, then pick $f \in K[a]$ such that $\partial(a) \in \mathcal{B}(K[a, 1/f])$. By Remark 2.62 $\partial(1/f) \in \mathcal{B}(K[a, 1/f])$, so we can take $b := (a, 1/f)$.

Finally, if we are in the case (3) of Definition 2.67, then we can combine the proof of both cases above (since finite group actions are governed by comonads). ☐

Amalgamation also transfers easily to the case of $\mathcal{B}_\varphi$-fields.

Proposition 2.70. *Assume $\mathcal{B}$ is local and φ is an iterativity condition. Then class of $\mathcal{B}_\varphi$-fields has the amalgamation property in the language $\mathcal{L}_{\mathcal{B}}^{\lambda}$ - in other words, any two separable extensions of a $\mathcal{B}_\varphi$-field can be amalgamated into a separable extension.*

Proof. We work inside some large algebraically closed field Ω. Let $L \subseteq M, N$ be separable extensions of $\mathcal{B}_\varphi$-fields. We will first find an amalgam of M, N over L in the class of $\mathcal{B}$-fields. By Lemma 2.29 we can replace L, M, N by their separable closures, so without loss of generality assume that L, M, N are separably closed. Since L is separably closed and the extensions $L \subseteq M, N$ are separable, they are in fact regular. Thus their tensor product $M \otimes_K N$ is a domain by Fact 1.1, so we may form its field of fractions F. By Corollary 2.31 and Corollary 2.43 there is a (unique) $\mathcal{B}$-field structure on F extending the ones on M and N, and since the extensions $M, N \subseteq F$ are separable, F is an amalgam of M, N over K considered in the language $\mathcal{L}_{\mathcal{B}}^{\lambda}$. Since M, N are $\mathcal{B}_{\varphi}$-fields, item (1) in Definition 2.63 implies that F is also a $\mathcal{B}_{\varphi}$-field. $\square$

Remark 2.71. Assume $\mathcal{B}$ is local. If $L \subseteq M, N$ are finitely generated[9] separable $\mathcal{B}$-field extensions, then one can take the amalgam of M, N over L to be finitely generated. Indeed, if K is any amalgam, then the $\mathcal{B}$-subfield of K generated by (the images of) M and N has the desired properties.

Reasoning as in the proof of Proposition 2.70, we can deduce from Lemma 2.36 the following result.

Lemma 2.72. *Assume that* $\operatorname{Fr}(\ker \pi_{\mathcal{B}}) = 0$. *Then, for any* $\mathcal{B}_{\varphi}$*-field* (K, ∂) *and any* $a \in K^{\partial}$ *there is a* $\mathcal{B}_{\varphi}$*-field structure on* $K\left(a^{1/p}\right)$. *In particular, existentially closed* $\mathcal{B}_{\varphi}$*-fields are* ***strict****, i. e.* $K^{\partial} = K^{p}$.

[9]As $\mathcal{B}$-extensions or as extensions of pure fields.

CHAPTER 3

Model theory of $\mathcal{B}_\varphi$-fields

Since the pioneering work of Robinson existentially closed models and model companions are indispensable objects in the model theory of algebraic structures. What we want to investigate in this Chapter is the class of $\mathcal{B}_\varphi$-fields which are existentially closed, possibly in some restricted class of extensions (e. g. only in regular extensions). This fit into a well-established line of research (see Remark 3.17).

In **Section 3.1** we prove a very general statement about elementarity of existential closedness of $\mathcal{B}_\varphi$-fields (Theorem 3.13). It can be roughly stated as the following: if a suitable variant of Proposition 2.29 holds, then an appropriate variant of existential closedness is an elementary property. In **Section 3.2** we study model companions of various theories of $\mathcal{B}$-fields. We classify coordinate k-algebra schemes $\mathcal{B}$ for which the theory of (free) $\mathcal{B}$-fields has a model companion. We prove some results about stability and quantifiers elimination of the theories involved. In **Section 3.3** we study pseudo algebraically closed structures in the context of $\mathcal{B}_\varphi$-fields, which we call *pseudo $\mathcal{B}_\varphi$-closed fields*. We show that they form an elementary class for suitable $\mathcal{B}$, answering a vast generalization of a question of Hoffmann and Kowalski and also simplifying some of their work (see Remark 3.48). Finally, in **Section 3.4** we use the methods of this chapter to investigate some further classes of fields.

3.1. Elementarity of some variants of existential closedness

In this Section we will prove a very general statement about elementarity of some variants of existential closedness (Theorem 3.13). This entails and generalizes many results from the literature, as we will see in the later sections.

We fix a coordinate k-algebra scheme $\mathcal{B}$ and an iterativity condition φ (see Definition 2.63). Let $\mathcal{K}$ be a class of extensions of $\mathcal{B}_\varphi$-fields.[1] The main examples to keep in mind are the class of all extensions, the class of separable extensions and the class of regular extensions (see also Example 3.2).

[1]In order to avoid confusion: by this we mean that the elements of $\mathcal{K}$ are pairs (K, L) where $K \subseteq L$ is an extension of $\mathcal{B}_\varphi$-fields.

Definition 3.1. Let K be a $\mathcal{B}_\varphi$-field. We say that a $\mathcal{B}_\varphi$-variety (V,s) over K is **of type** $\mathcal{K}$ if the $\mathcal{B}_\varphi$-extension $K \subseteq K(V)$ is in $\mathcal{K}$.

From now on we assume that $\mathcal{K}$ is **definable** in the following sense: given a $\mathcal{B}_\varphi$-variety (V,s) over a $\mathcal{B}_\varphi$-field K, the property "(V,s) is of type $\mathcal{K}$" is definable (in the same sense as in Remark 2.66).

Example 3.2. There are three natural examples of definable classes $\mathcal{K}$:

(1) $\mathcal{K}$ is the class of all $\mathcal{B}_\varphi$-extension. This class is definable directly by Remark 2.66, as $\mathcal{B}_\varphi$-varieties of type $\mathcal{K}$ are just $\mathcal{B}_\varphi$-varieties.

(2) $\mathcal{K}$ is the class of all regular extensions. Let (V,s) be a $\mathcal{B}_\varphi$-variety coded by $\bar{a}$ (in the sense of Remark 2.66). Then (V,s) is of type $\mathcal{K}$ if and only if V is absolutely irreducible, i. e. irreducible over K^{alg}. As in Remark 2.66, there is a formula $\psi(\bar{x})$ (independent of V and K) such that V is irreducible over K^{alg} if and only if $K^{\mathrm{alg}} \models \psi(\bar{a})$.[2] Since ACF eliminates quantifiers, we can assume that $\psi(\bar{x})$ is quantifier free, thus $K^{\mathrm{alg}} \models \psi(\bar{a})$ if an only if $K \models \psi(\bar{a})$. Hence $\mathcal{K}$ is definable.

(3) $\mathcal{K}$ is the class of all separable extensions. The argument is almost the same as in the previous point, but this time we have to use the theory SCF in an appropriate language.

Inspired by **[23]**, we introduce the following notion, which is our central object of interest.

Definition 3.3. We say that a $\mathcal{B}_\varphi$-field (K,∂) is **existentially closed in** $\mathcal{K}$ (or simply $\mathcal{K}$-**closed**) if for every $\mathcal{B}_\varphi$-field extension $(K,\partial) \subseteq (L,\partial)$ in $\mathcal{K}$ we have that (K,∂) is existentially closed in (L,∂) in the language $\mathcal{L}_{\mathcal{B}}$.

As we will later many interesting notions from differential algebra and beyond can be interpreted as $\mathcal{K}$-closedness for appropriate $\mathcal{K}$. The question we are interested in is the following: when is the property "being $\mathcal{K}$-closed" an elementary property? For the sake brevity, if this property is elementary we will say that $\mathcal{K}\mathcal{B}_\varphi\,\mathrm{CF}$ **exists** and denote by $\mathcal{K}\mathcal{B}_\varphi\,\mathrm{CF}$ the first order theory axiomatising the class of $\mathcal{K}$-closed $\mathcal{B}_\varphi$-fields. Otherwise we will say that $\mathcal{K}\mathcal{B}_\varphi\,\mathrm{CF}$ **does not exist**. If $\mathcal{K}$ is the class of all extensions of $\mathcal{B}_\varphi$-fields, then we will drop $\mathcal{K}$ from the notation and write $\mathcal{B}_\varphi\,\mathrm{CF}$ and if moreover φ is trivial we will omit φ and write simply $\mathcal{B}\,\mathrm{CF}$.

Example 3.4. Let us see what does being $\mathcal{K}$-closed mean for $\mathcal{K}$ as in Example 3.2.

[2]Alternatively, one could use that in ACF the Morley degree is definable.

(1) If $\mathcal{K}$ is the class of all $\mathcal{B}_\varphi$-extensions, then $\mathcal{K}\mathcal{B}_\varphi\,\mathrm{CF}$ exists if and only if the theory of $\mathcal{B}_\varphi$-fields has a model companion in the language $\mathcal{L}_\mathcal{B}$.

(2) Assume that $\mathcal{B} = (k\,[X]\,/\,(X^2))_\otimes$, φ is trivial (i. e. $\mathcal{B}_\varphi$-fields are differential fields) and $\mathcal{K}$ is the class of all separable extensions of differential fields. Then $\mathcal{K}\mathcal{B}_\varphi\,\mathrm{CF}$ exists and has a very nice axiomatization via "Wood axioms", as was proved by Ino and León Sánchez in **[23]**. In fact precisely that paper inspired the author of this thesis to investigate existential closedness in restricted classes of extensions.

(3) If $\mathcal{K}$ is the class of all regular extensions, then being $\mathcal{K}$-closed is closely related to being *pseudo algebraically closed* in the sense of **[18]**, which we will discuss in Section 3.3.

When checking existential closedness, we will often use the following standard reduction. We skip its proof.

Lemma 3.5. *Let $K \subseteq L$ be an extension of $\mathcal{B}$-fields. Then the following are equivalent:*

(1) *K is existentially closed in L in the language $\mathcal{L}_\mathcal{B}$.*

(2) *For every K-polynomials $f_0, \dots, f_n$, if $L \models (\exists \bar{x})\left(\bigwedge_{i\leqslant n} f_i\,(\partial \bar{x}) = 0\right)$ then $K \models (\exists \bar{x})\left(\bigwedge_{i\leqslant n} f_i\,(\partial \bar{x}) = 0\right)$.*

We introduce the following finiteness condition, which will allow us to give a simpler criterion for $\mathcal{K}$-closedness (see Lemma 3.9), which in turn will assure that $\mathcal{K}\mathcal{B}_\varphi\,\mathrm{CF}$ exists.

Assumption 3.6. Whenever $(K, \partial) \subseteq (L, \partial)$ is in $\mathcal{K}$ and $a \in L$ is a finite tuple, then there are finite tuple b containing ∂a and a $\mathcal{B}$-operator $\partial'\colon K\,(b) \to \mathcal{B}\,(K\,(b))$ such that ∂' extends the $\mathcal{B}$-operator $\partial|_{K(a)}\colon K\,(a) \to \mathcal{B}\,(K\,(\partial a))$ and $(K, \partial) \subseteq (K\,(b)\,, \partial')$ is in $\mathcal{K}$.

Remark 3.7. Let $\mathcal{K}$ be any of the classes in Example 3.2 and let $(\mathcal{B}, \varphi)$ be a nice pair. Immediately by Proposition 2.68 we get that Assumption 3.6 holds in this case. This gives a plethora of examples of $\mathcal{K}, \mathcal{B}, \varphi$ satisfying Assumption 3.6.

Notation 3.8. We denote by $\mathcal{K}_{\mathrm{fin}}$ the subclass of $\mathcal{K}$ consisting of those extensions $K \subseteq L$ in $\mathcal{K}$ which are finitely generated as pure fields. Note that $\mathcal{K}_{\mathrm{fin}}$ always satisfies Assumption 3.6, as one can take b to be a tuple of generators of the extension $K \subseteq L$. It is also clearly definable, provided $\mathcal{K}$ is.

The meaning of Assumption 3.6 is contained in the following result.

Lemma 3.9. *Assume $\mathcal{K}$ satisfies Assumption 3.6. Then a $\mathcal{B}_\varphi$-field (K, ∂) is $\mathcal{K}$-closed if and only if it is $\mathcal{K}_{\text{fin}}$-closed.*

Proof. Assume that (K, ∂) is $\mathcal{K}_{\text{fin}}$-closed. Let $\phi(x)$ be a quantifier-free $\mathcal{L}_\mathcal{B}(K)$-formula such that there is some $\mathcal{B}_\varphi$-field extension $(K, \partial) \subseteq (L, \partial_L)$ in $\mathcal{K}$ and some tuple $a \in L$ such that $L \models \phi(a)$. Using Lemma 3.5 we may assume that $\phi(\bar{x})$ is of the form $\theta(\partial\bar{x})$ where $\theta(\bar{y})$ is a quantifier-free $\mathcal{L}_{\text{rng}}(K)$-formula. Take $b \in L$ and ∂' as in the conclusion of Assumption 3.6. Then still $K(b) \models \theta(\partial' a)$ as $\partial_L(a) = \partial'(a)$. Since (K, ∂) is $\mathcal{K}_{\text{fin}}$-closed and $K \subseteq K(b)$ is a finitely generated extension in $\mathcal{K}$, we have that there is some tuple $c \in K$ such that $K \models \theta(\partial c)$, which finishes the proof. $\square$

For technical reasons we also introduce the following assumption, saying that "$(\mathcal{B}, \varphi)$ satisfies Lemma 2.69".

Assumption 3.10. Let $(K, \partial) \subseteq (L, \partial)$ be $\mathcal{B}_\varphi$-fields and let $a \in L$ be a finite tuple such that $L = K(a)$. Then a can be extended to a finite tuple b so that ∂ restricts to a $\mathcal{B}$-operator on $K[b]$.

Remark 3.11. Let us comment on the nature of the introduced assumptions. Both of the them are finiteness conditions. Assumption 3.6 allows us (at least partially) to reduce the study of $\mathcal{B}_\varphi$-fields to the case of $\mathcal{B}_\varphi$-fields which are finitely generated as field over some base $\mathcal{B}_\varphi$-field K. Assumption 3.10 on the other hand reduces the latter to the study of $\mathcal{B}_\varphi$-ring over K which are finitely generated as K-algebras, i. e. to the study of $\mathcal{B}_\varphi$-varieties (see Proposition 3.12).

Proposition 3.12. *Suppose Assumption 3.10 holds. Then, for a $\mathcal{B}_\varphi$-field (K, ∂) the following conditions are equivalent.*

(1) (K, ∂) *is $\mathcal{K}_{\text{fin}}$-closed.*
(2) *Every $\mathcal{B}_\varphi$-variety over K of type $\mathcal{K}$ has a Zariski-dense set of K-rational $\mathcal{B}$-point.*
(3) *Every $\mathcal{B}_\varphi$-variety over K of type $\mathcal{K}$ has a K-rational $\mathcal{B}$-point.*

Proof. (1) $\Longrightarrow$ (2) : Assume that (K, ∂) is $\mathcal{K}_{\text{fin}}$-closed, let (V, s) be a $\mathcal{B}_\varphi$-variety over K of type $\mathcal{K}$ and let f be a non-zero regular function on V. We aim to prove that (V, s) has a K-rational $\mathcal{B}$-point c such that $f(c) \neq 0$. Let $a \in V$ be a generic point of V over K and ∂ be the $\mathcal{B}$-operator on $K(a) = K(V)$ corresponding to s. Let $\phi(x)$ be a quantifier-free $\mathcal{L}_\mathcal{B}(K)$-formula expressing the property "$x \in (V, s)^\sharp$ and $f(x) \neq 0$".

55:52398

By Lemma 2.57 we have that $a \in (V,s)^{\sharp}(K(a))$ and $f(a)$, so that $K(V) \models (\exists x)\,\phi(x)$. Since the extension $K \subseteq K(V)$ is in $\mathcal{K}$ we have that $K \models (\exists x)\,\phi(x)$, i. e. there exists a K-rational $\mathcal{B}$-point of (V,s) outside from the zero set of f. Thus $(V,s)^{\sharp}(K)$ is Zariski-dense in V.

$(2) \implies (3)$: Obvious.

$(3) \implies (1)$: Assume that every $\mathcal{B}_{\varphi}$-variety over K of type $\mathcal{K}$ has a K-rational $\mathcal{B}$-point. Let $\theta(x)$ be a quantifier-free $\mathcal{L}_{\mathcal{B}}(K)$-formula such that there is some finitely generated $\mathcal{B}_{\varphi}$-field extension $(K,\partial) \subseteq (L,\partial)$ in $\mathcal{K}$ and some tuple $a \in L$ such that $L \models \theta(a)$. Our goal is to prove that we can find such a tuple already in K. By Lemma 3.5 we can assume that $\theta(x) = \phi(\partial x)$ where φ is of the form $\bigwedge_{i=0}^{n} f_i(y) = 0$ for some K-polynomials $f_0, \dots, f_n$. Since L is finitely generated we can assume that $L = K(a)$ (by possibly enlarging the tuple a and adding some dummy variables to ϕ) and using Assumption 3.10 we can furthermore assume that ∂ restricts to a $\mathcal{B}$-operator on $K[a]$. Set $V = \mathrm{locus}_K(a)$ and let $s\colon V \to \tau^{\partial}V$ be the section of π_V^{∂} corresponding to the $\mathcal{B}$-operator $\partial\colon K[a] \to \mathcal{B}(K[a])$. In particular $\partial(a) = s(a)$. By assumption, we have that (V,s) has a K-rational $\mathcal{B}$-point, say $b \in K$. Thus $\partial(b) = s(b)$. Since a satisfies the $\mathcal{L}_{\mathcal{B}}(K)$-formula $\phi(\partial(x))$ and $\partial(a) = s(a)$, we have that a satisfies the $\mathcal{L}_{\mathrm{ring}}(K)$-formula $\phi(s(x))$. Since $\phi(s(x))$ is a positive $\mathcal{L}_{\mathrm{ring}}(K)$-formula and $b \in V = \mathrm{locus}_K(a)$ we have that b satisfies $\phi(s(x))$ and thus $\phi(\partial(x))$, as desired. $\square$

Axioms for $\mathcal{K}\mathcal{B}_{\varphi}\,\mathrm{CF}$

> (K,∂) is a $\mathcal{B}_{\varphi}$-field such that every $\mathcal{K}$-variety over (K,∂) has a K-rational $\mathcal{B}$-point.

Combining Lemma 3.9, Proposition 3.12 and the "definability property" of $\mathcal{K}$ we get the following result.

Theorem 3.13. *Suppose that Assumption 3.6 and Assumption 3.10 hold. Then, a $\mathcal{B}_{\varphi}$-field (K,∂) is $\mathcal{K}$-closed if and only if it satisfies the Axioms for $\mathcal{K}\mathcal{B}_{\varphi}\,\mathrm{CF}$ written above. In particular, $\mathcal{K}\mathcal{B}_{\varphi}\,\mathrm{CF}$ exists.*

In particular, any nice pair $(\mathcal{B},\varphi)$ meets the assumptions of Theorem 3.13 by Lemma 2.69 and Remark 3.7, hence we get the following.

Theorem 3.14. *Assume $(\mathcal{B},\varphi)$ is nice and $\mathcal{K}$ is any of the classes in Example 3.2. Then, for a $\mathcal{B}_{\varphi}$-field (K,∂) the following properties are equivalent:*

(1) *(K,∂) is $\mathcal{K}$-closed.*

(2) *Every $\mathcal{B}_\varphi$-variety over (K, ∂) has a K-rational $\mathcal{B}$-point.*

Moreover, the latter property is expressible by a scheme of first-order sentences in the language $\mathcal{L}_\mathcal{B}$. In particular, $\mathcal{K}\mathcal{B}_\varphi \operatorname{CF}$ exists.

The rest of this Chapter is devoted to applying Theorem 3.14 for various $\mathcal{B}, \varphi, \mathcal{K}$.

Remark 3.15. We could well ignore any pair $(\mathcal{B}, \varphi)$ which is not nice, in particular we could refrain from introducing Assumption 3.6. We think however that it is appropriate to introduce them in order to show that our proof work under some abstract structural properties and is not dependent on the particular class of operators involved. Nice pairs are simply a wide class of natural examples to which we can apply our results. Theorem 3.13 can be applied also to prove the existence of model companions of seemingly random theories, e. g. the theory of fields of characteristic 17 together with two commuting derivations ∂_1, ∂_2 such that $\partial_1^3 = \partial_2^{37}$.

3.2. Model companions of theories of $\mathcal{B}$-fields

In Subsection 3.2.1 we will use Theorem 3.13 to prove the existence of a model companion of various theories of $\mathcal{B}_\varphi$-fields, generalizing and unifying many results from the literature. After that we focus on the case when $\mathcal{B}$ satisfies $\operatorname{Fr}(\ker \pi_\mathcal{B}) = 0$. In the latter part of Subsection 3.2.1 we give a different axiomatization of the resulting model companion, in the spirit of the Pierce-Pillay axioms for DCF_0. In Subsection 3.2.2 we prove that $\mathcal{B}_\varphi \operatorname{CF}$ eliminates quantifiers after adding λ_0 to the language, and in Subsection 3.2.3 we give an explicit description of forking independence in $\mathcal{B}_\varphi \operatorname{CF}$ and prove that $\mathcal{B}_\varphi \operatorname{CF}$ is stable. In Subsection 3.2.4 we say a few things about the cases not covered by Theorem 3.13.

3.2.1. Existence and different axiomatizations

Immediately from Theorem 3.13 and Proposition 2.68 we get the following results.

Theorem 3.16. *Assume that $(\mathcal{B}, \varphi)$ satisfies Assumption 3.6 where $\mathcal{K}$ is the class of all $\mathcal{B}_\varphi$-extensions. Then the theory of $\mathcal{B}_\varphi$-fields has a model companion. In particular, this holds for any nice pair $(\mathcal{B}, \varphi)$.*

Remark 3.17. The above theorem unifies and generalizes in one swift motion many existing results about the existence of model companions for theories of fields with operators, mostly in positive characteristic. Below is a list of some of those theories (see also the discussion Section 2.4 and the chart provided in the Introduction). The point is that any of the example below can be described using a nice pair $(\mathcal{B}, \varphi)$.[3]

(1) free "local" $\mathcal{D}$-ring structures in characteristic zero (see [**36**]),
(2) B-operators for B satisfying $\operatorname{Fr}(\ker \pi_B) = 0$ (see [**2**]; these are precisely the local B for which a model companion exists),
(3) fields with an action of a (fixed) finite group (see [**16**]) or a finite group scheme (see [**17**])
(4) ordinary differential fields with finite group actions (this was done for characteristic zero in [**19**], and for positive characteristic in [**18**, Theorem 4.36.]),
(5) partial differential fields with finite group actions in positive characteristic (this is a positive-characteristic counterpart of [**19**]).

The notion of a nice pair covers of course much more, e. g. n commuting B-operators together with a finite group action (for B satisfying $\operatorname{Fr}(\ker \pi_B) = 0$). What Theorem 3.14 does not entail however are partial differential fields in characteristic zero. In fact, the model companion in this case can not be axiomatized using $\mathcal{B}_\varphi$-varieties (see Section 3.4.3).

Let us now assume that $\operatorname{Fr}(\ker \pi_{\mathcal{B}}) = 0$ and that φ is trivial. We will now give different axioms for $\mathcal{B}\,\mathrm{CF}$, in the spirit of the Pierce-Pillay axioms for DCF_0 given in [**39**].

New Axioms for $\mathcal{B}\,\mathrm{CF}$

For every K-varieties V and W, if $W \subseteq \tau^{\partial} V$ and the projection $W \to V$ is separable, then there is some $a \in V(K)$ such that $\partial_V(a) \in W(K)$.

Theorem 3.18. *Assume $\mathcal{B}$ satisfies $\operatorname{Fr}(\ker \pi_{\mathcal{B}}) = 0$. Then a $\mathcal{B}$-field (K, ∂) is existentially closed if and only if it satisfies the New Axioms.*

Proof. ($\Longrightarrow$) Assume (K, ∂) is existentially closed and take V, W as in the axioms. Let us work inside some big algebraically closed field Ω. Since the projection $W \to V$ is separable, it is by definition dominant. Thus $W = \operatorname{locus}_K(a, b)$ and $V = \operatorname{locus}_K(a)$ for some tuples $a, b \in \Omega$. Since $W \subseteq \tau^{\partial} V$, the is a natural $\mathcal{B}$-operator $\partial \colon K[a] \to \mathcal{B}(K[a, b])$ of the inclusion $K[a] \subseteq K[a, b]$ such that $\partial(a) = (a, b)$. By Corollary 2.31 ∂ extends to a $\mathcal{B}$-operator of the field extension $K(a) \subseteq K(a, b)$. This extension is separable, so by Lemma 2.29 ∂ and the fact that separable extensions are formally smooth ∂ extends further to a $\mathcal{B}$-operator on the field $K(a, b)$. Let x be a tuple of variables of length equal to the length of a and let $\phi(x)$ be the quantifier-free $\mathcal{L}_{\mathcal{B}}(K)$-formula expressing that $\partial_V(x) \in W$. Since

[3]The examples below are precisely the reason why we crafted the definition of a nice pair the way we did.

$\phi(x)$ has a solution in $K(a,b) \supseteq K$, it has a solution in K by existential closedness. Thus K satisfies the above axioms.

($\Longleftarrow$) Assume (K,∂) satisfies the axioms. We will check that (K,∂) satisfies the assumptions of Theorem 3.14, i. e. that every $\mathcal{B}$-variety over K has a $\mathcal{B}$-point in K. Let (V,s) be a $\mathcal{B}$-variety over K and define $W := s[V] \subseteq \tau^{\partial}V \subseteq \tau^{\partial}V$. Note that W is a closed subset of $\tau^{\partial}V$, since it is equal to the set of all $b \in \tau^{\partial}V$ such that $s(\pi(b)) = b$. Moreover, the projection $W \to V$ is an isomorphism, so in particular it is separable, thus by the New Axioms there is some $a \in V(K)$ such that $\partial_V(a) \in W(K)$ and for this a we have

$$\partial_V(a) = s(\pi(\partial_V(a))) = s(a),$$

thus a is a $\mathcal{B}$-point of (V,s). $\square$

Remark 3.19. Let us point out how Theorem 3.18 relates to other results in the literature. In [**2**, Theorem 3.8] Beyarslan, Hoffmann, Kamensky and Kowalski give an axiomatization of existentially closed B-fields with B satisfying $\mathrm{Fr}_B(\ker \pi_B) = 0$. In [**13**] the author and Kowalski do the same for $\mathcal{B}$-fields with $\mathcal{B}$ satisfying $\mathrm{Fr}(\ker \pi_{\mathcal{B}}) = 0$ and by taking $\mathcal{B} = B_{\otimes}$ for B as in the previous sentence one recovers the axiomatization given in [**2**]. The axioms in [**13**, Theorem 4.5] are as follows.

> For every K-varieties V and W, if
> (1) $W \subseteq \tau^{\partial}V$,
> (2) the projection $W \to V$ is dominant,
> (3) the projection $E \to W$ is dominant,
> then there is some $a \in V(K)$ such that $\partial_V(a) \in W(K)$.

Here E is a certain algebraic subset of $\tau^{\partial}(W)$, defined as the equalizer of certain maps involving V and W. Anyway, E looks strange and unnatural, but the assumptions on V, W are *exactly* the conditions assuring that the natural $\mathcal{B}$-operator $\partial_V^W : K[V] \to \mathcal{B}^{\partial}(K[W])$ extends to a $\mathcal{B}$-operator on the field $K(W)$ (see [**13**, Proposition 4.4]).

In our New Axioms instead of introducing E we demand that the projection $W \to V$ is separable. This condition is stronger than "$E \to W$ is dominant", thus our Theorem 3.18 is stronger than [**13**, Theorem 4.5] (since we axiomatize the same theory using fewer axioms). At the same time, we achieve an axiomatization of $\mathcal{B}$CF faster and with less technicalities than in [**2**] or [**13**]. Also, Theorem 3.18 answers [**27**, Question 4], which asked for an axiomatization of this form in the case of derivations of the Frobenius map (an answer to this question was also achieved in the paper [**12**] by the author using a completely different method, see [**12**, Remark 3.13] there; see also Remark 3.51).

3.2.2. Quantifier elimination

In this subsection we will prove that for $(\mathcal{B}, \varphi)$ such that $\mathrm{Fr}\,(\ker \pi_{\mathcal{B}}) = 0$ the theory $\mathcal{B}_{\varphi}\,\mathrm{CF}$ eliminates quantifiers after adding the inverse of the Frobenius to the language. **Throughout this subsection we assume that** $\mathrm{Fr}\,(\ker \pi_{\mathcal{B}}) = 0$**.**

First we need a certain easy lemma, which is nonetheless very general and useful. Let us introduce a few local definitions. Let (K, ∂) be a field of characteristic exponent p (i. e. p is the characteristic of K if it is positive and $p = 1$ otherwise) with some *operators*, i. e. a tuple (possibly infinite) of unary functions $\partial = (\partial_i \colon K \to K)_{i \in I}$. We define the *constants* of (K, ∂) as the set of common zeroes of all ∂_i and denote it by K^{∂}. We assume that K^{∂} is a field. We also assume that every ∂_i is additive and "Fr^m-linear over the constants", i. e. there is some natural number m_i such that for any $a \in K^{\partial}$, $x \in K$ we have $\partial_i\,(ax) = a^{p^{m_i}} \partial_i\,(x)$. Note that under the assumption $\mathrm{Fr}\,(\ker \pi_{\mathcal{B}}) = 0$, $\mathcal{B}$-operators fit into this set-up and constants in the sense above are the same as the constant in the sense of $\mathcal{B}$-operators.

Let $(K, \partial) \subseteq (L, \partial')$ be an extension of fields with operators in the above sense, that is, $\partial'|_K = \partial$ and the numbers m_i mentioned above are the same for K and L. By abuse of notation, we will use the same symbol ∂ for the operators on K and on L. The following result was proved in [**12**] by the author.

Lemma 3.20. *Let $(K, \partial) \subseteq (L, \partial)$ be as above. Assume that $p > 1$, $L^p \subseteq L^{\partial}$ and that K is strict, i. e. $K^{\partial} = K^p$. Then L^{∂} and K are linearly disjoint over K^{∂}.*

Proof. Assume the conclusion is not true and take the minimal $n > 1$ such that there are some $x_1, \ldots, x_n \in L^{\partial}$ linearly dependent over K, but linearly independent over K^{∂}. By the minimality assumption, there are $a_1, \ldots, a_n \in K \setminus \{0\}$ such that:

$$a_1 x_1 + \ldots + a_n x_n = 0$$

Then for any $i \in I$:

$$0 = \partial_i \left(\frac{a_1}{a_n} x_1 + \ldots + \frac{a_{n-1}}{a_n} x_{n-1} + x_n \right) = \partial_i \left(\frac{a_1}{a_n} \right) x_1^{p^{m_i}} + \ldots + \partial_i \left(\frac{a_{n-1}}{a_n} \right) x_{n-1}^{p^{m_i}}$$

If some $\partial_i\left(\frac{a_j}{a_n}\right)$ is nonzero, then $x_1^{p^{m_i}},\ldots,x_{n-1}^{p^{m_i}} \in L^p \subseteq L^{\partial}$ are linearly dependent over K, so by the minimality assumption on m we get that $x_1^{p^{m_i}},\ldots,x_{n-1}^{p^{m_i}}$ are linearly dependent over $K^{\partial}=K^p$, hence $x_1^{p^{m_i-1}},\ldots,x_{n-1}^{p^{m_i-1}}$ are linearly dependent over K. Repeating this reasoning yields that $x_1,\ldots,x_{n-1}$ are linearly dependent over K^p, contrary to the assumption, that they are independent over $K^{\partial}=K^p$.

Therefore for any i we have $\partial_i\left(\frac{a_1}{a_n}\right)=\ldots=\partial_i\left(\frac{a_{n-1}}{a_n}\right)=0$, hence $\frac{a_1}{a_n},\ldots,\frac{a_{n-1}}{a_n}\in K^{\partial}$. By the strictness assumption we get that for some $b_1,\ldots,b_{n-1}\in K\setminus\{0\}$:

$$a_1=b_1^pa_n,\ldots,a_{n-1}=b_{n-1}^pa_n,$$

thus

$$0=a_1x_1+\ldots+a_nx_n=a_n\left(b_1^px_1+\ldots+b_{n-1}^px_{n-1}+x_n\right),$$

hence $x_1,\ldots,x_n$ are linearly dependent over $K^p=K^{\partial}$, contrary to the assumption. $\square$

From the proof it is clear that in the linear case (i. e. for every i we have $m_i=0$) we have the following

Lemma 3.21. *Let $(K,\partial)\subseteq(L,\partial)$ be an extension of fields with operators linear over the constants. Then L^{∂} and K are linearly disjoint over K^{∂}.*

The strictness assumption in Lemma 3.20 is necessary (which gives a negative answer to Question 1 in [**27**]), as shown by the example below.

Example 3.22. Take $K=\mathbb{F}_p(X,Y,\lambda,\mu)\,,L=\mathbb{F}_p\left(X^{1/p},Y^{1/p},\lambda,\mu\right)$ and define a derivation of the Frobenius map on L by setting

$$\partial\left(X^{1/p}\right)=\partial\left(Y^{1/p}\right)=0,\quad \partial(\lambda)=Y,\quad \partial(\mu)=-X.$$

We will show that L^{∂} and K are not linearly disjoint over K^{∂}. Note that

$$\partial\left(\lambda X^{1/p}+\mu Y^{1/p}\right)=X\partial(\lambda)+Y\partial(\mu)=0.$$

Thus, $X^{1/p},Y^{1/p},\lambda X^{1/p}+\mu Y^{1/p}$ are elements of L^{∂}, linearly dependent over K. However, they are independent over K^{∂}: indeed, for any $a,b,c\in K^{\partial}$, if

$$aX^{1/p}+bY^{1/p}+c\left(\lambda X^{1/p}+\mu Y^{1/p}\right)=0$$

then

$$(a+c\lambda)\,X^{1/p}+(b+c\mu)\,Y^{1/p}=0,$$

but $X^{1/p}$ and $Y^{1/p}$ are linearly independent over K, so $a + c\lambda = b + c\mu = 0$. If $c \neq 0$, then $\lambda = -\frac{a}{c} \in K^{\partial}$, which is not the case. Thus $c = 0$ and therefore $a = b = 0$, hence $X^{1/p}, Y^{1/p}, \lambda X^{1/p} + \mu Y^{1/p}$ are linearly independent over K^{∂}.

Lemma 3.20 implies that strict $\mathcal{B}$-fields are $\mathcal{B}$-differentially perfect, i. e. any $\mathcal{B}$-field extension is separable (for fields with a derivation of the Frobenius morphism this was asked in Question 2 in [**27**]).

Lemma 3.23. *Let K be a model of $\mathcal{B}_{\varphi}\mathrm{CF}$ and let K_0 be subfield of K. If K is $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$-substructure of K then the extension $K_0 \subseteq K$ is separable.*

Proof. By Remark 2.72 we have that $K^{\partial} = K^p$ an assumption we have that $K^p \cap K_0 = K_0^p$, thus $K_0^p = K_0^{\partial}$. Therefore by Lemma 3.20 we have that $K^{\partial} = K^p$ and K_0 are linearly disjoint over K_0^p, i. e. the extension $K_0 \subseteq K$ is separable. $\square$

We need the following technical lemma, which will be also useful in Section 3.3. The same sort of reduction happens in the middle of the proof of [**18**, Theorem 4.34.].

Lemma 3.24. *Let $K \subseteq L$ be an $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$-extension of strict $\mathcal{B}_{\varphi}$-fields. Then K is existentially closed in L in the language $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$ if and only if it is existentially closed in the language $\mathcal{L}_{\mathcal{B}}$.*

Proof. Assume that K is existentially closed in L in the language $\mathcal{L}_{\mathcal{B}}$ and let $\phi(\bar{x})$ be a quantifier-free $\mathcal{L}_{\mathcal{B}}^{\lambda_0}(K)$-formula realisable in L. Our goal is to prove that $\phi(\bar{x})$ is realisable in K. We will do this by "rewriting ϕ into the language $\mathcal{L}_{\mathcal{B}}$", i. e. we will find a quantifier-free $\mathcal{L}_{\mathcal{B}}(K)$-formula $\phi_0(\bar{x}')$, where $\bar{x}'$ extends $\bar{x}$, such that

(1) $\phi_0(L) \neq \emptyset$ (hence also $\phi_0(K) \neq \emptyset$),
(2) $L \models \phi_0(\bar{x}') \to \phi(\bar{x})$.

This immediately implies that $\phi(K) \neq \emptyset$, as desired.

Using the standard procedures we may assume that $\phi(\bar{x}) = \psi(\partial(\bar{x}), \lambda_0(\bar{x}))$, where $\psi(\bar{y}, \bar{z})$ is a formula in language of pure fields with parameters from K.[4] We can moreover assume that $\psi(\bar{y}, \bar{z})$ is of the form $\bigwedge_{i=1}^{n} f_i(\bar{y}, \bar{z}) = 0$ where $f_1, \ldots, f_n$ are K-polynomials. Say that $\bar{x} = (x_1, \ldots, x_m), \bar{z} = (z_1, \ldots, z_m)$. Let $\bar{a} = (a_1, \ldots a_m) \in L$ be a realization of $\phi(\bar{x})$. We partition the set $\{1, \ldots, m\}$ into three parts A, B, C so that

- A consists of those i for which $\lambda_0(a_i) \neq 0$,
- B consists of those i for which $\lambda_0(a_i) = 0$ and $a_i \neq 0$,

[4]Recall that the first coordinate of the tuple $\partial(x)$ is x.

62:94224

- C consists of those i for which $a_i = 0$.

Define $\bar{x}'$ as the concatenation of $\bar{x}$ and $\bar{z}$. Finally, define $\phi_0(\bar{x}')$ as the following formula

$$\psi(\bar{x}, \bar{z}) \wedge \bigwedge_{i \in A} (z_i^p = x_i) \wedge \bigwedge_{i \in B} \left(z_i = 0 \wedge \bigvee_{j=1}^{e-1} \partial_j(x_i) \neq 0 \right) \wedge \bigwedge_{i \in C} (x_i = 0).$$

Since L is strict, we have that for any nonzero $b \in L$ it holds that $\lambda_0(b) = 0$ if and only if $\partial(b) \neq \iota(b)$, i. e. for some j we have $\partial_j(b) \neq 0$. Thus the tuple $(\bar{a}, \lambda_0(\bar{a}))$ witnesses that $\phi_0(L) \neq \emptyset$. Moreover by the same fact we have that $L \models \phi_0(\bar{x}, \bar{z}) \to (\phi(\bar{x}) \wedge \bar{z} = \lambda_0(\bar{x}))$, hence ϕ_0 is as desired. $\square$

Immediately from the above lemma we get the following result.

Corollary 3.25. *The theory $\mathcal{B}_\varphi \mathrm{CF}$ is the model companion of the theory of strict $\mathcal{B}_\varphi$-fields in the language $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$.*

Finally, putting all pieces together we arrive at the main result of this Subsection.

Theorem 3.26. *The theory $\mathcal{B}_\varphi \mathrm{CF}$ has quantifier elimination in the language $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$.*

Proof. By Corollary 3.25 the theory $\mathcal{B}_\varphi \mathrm{CF}$ is the model companion of the theory of strict $\mathcal{B}_\varphi$-fields, in particular it is model complete as an $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$-theory. Since any model complete theory with the amalgamation property admits quantifier elimination, we are done by Proposition 2.70. $\square$

Corollary 3.27. *Assume that k is perfect. Then, the theory $\mathcal{B}_\varphi \mathrm{CF}$ is complete.*

Proof. This follows from Theorem 3.26 and the fact that (k, ι_k) is a common $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$-substructure of all models of $\mathcal{B}_\varphi \mathrm{CF}$. $\square$

3.2.3. Stability

In this subsection we prove stability of the theory $\mathcal{B}_\varphi \mathrm{CF}$ for $\mathcal{B}$ satisfying $\mathrm{Fr}(\ker \pi_{\mathcal{B}}) = 0$. The arguments are standard and follow closely the proofs of the corresponding results in [**2**] and [**27**].

Again, **throughout this subsection we assume that** $\mathrm{Fr}(\ker \pi_{\mathcal{B}}) = 0$**.** We also assume that $(\mathcal{B}, \varphi)$ is not trivial, i. e. that there is a $\mathcal{B}_\varphi$-field (K, ∂) where ∂ is not the zero operator ι_K. Otherwise $\mathcal{B}_\varphi$-fields are simply pure fields and $\mathcal{B}_\varphi \mathrm{CF} = \mathrm{ACF}_p$.

We fix a monster model $(\mathfrak{C}, \partial) \models \mathcal{B}_\varphi \mathrm{CF}$. Immediately from Lemma 2.29 and the fact that separable algebraic extensions are étale we get the following fact.

Corollary 3.28. *The field $\mathfrak{C}$ is separably closed.*

Note that since $(\mathcal{B},\varphi)$ is not trivial we have that $\mathfrak{C}\neq\mathfrak{C}^{\partial}$ and by Lemma 2.72 $\mathfrak{C}^{\partial}=\mathfrak{C}^{p}$. Thus, the imperfection degree of $\mathfrak{C}$ is nonzero. It is natural to ask what is exactly the imperfection degree of $\mathfrak{C}$. This depends heavily on φ, e. g. for actions of finite groups schemes this degree is sometimes finite (see [**17**, Proposition 5.3]) and for free B-operators it is infinite (see [**2**, Remark 4.12]). Instead of giving a full classification, we point out in Lemma 3.29 a class of examples where the imperfection degree is infinite. This class includes in particular all pairs $(\mathcal{B},\varphi)$ where φ is trivial or is a commutativity condition. In order to state Lemma 3.29, we need to impose some technical assumptions as explained below.

By Theorem 2.17 and Remark 2.26 we may assume that $\mathcal{B}=B_{(n_1,\ldots,n_e)}$ where B is a finite k-algebra and $n_1=0$. If $n_2=\ldots=n_e=0$ and φ is trivial then we are in the case of free B-operators and the imperfection degree of $\mathfrak{C}$ is infinite by [**2**, Remark 4.12]. Thus, let us assume (only for the purpose of the next lemma!) that e. g. $n_2>0$ and set $q=p^{n_2}$.[5] Now, extend ∂ to a $\mathcal{B}$-operator on field of (univariate) rational functions $\mathfrak{C}\left(X\right)$ so that $\partial_i\left(X\right)=X$ for $i=1,\ldots,e$.

Lemma 3.29. *Assume that the $\mathcal{B}$-field $\mathfrak{C}\left(X\right)$ satisfies φ. Then, the imperfection degree of $\mathfrak{C}$ is infinite.*

Proof. We will reason as in [**27**]. Assume that $\mathfrak{C}$ has finite imperfection degree and let $x_1,\ldots,x_m$ is a basis of the vector space $\mathfrak{C}$ over $\mathfrak{C}^p$. Let K be a model of $\mathcal{B}_{\varphi}\mathrm{CF}$ extending $\mathfrak{C}\left(X\right)$. Since the extension $\mathfrak{C}\subseteq K$ is elementary we have that $x_1,\ldots,x_m$ is a basis of K over K^p, thus

$$X=\sum_{i=1}^{m}\alpha_i x_i$$

for some $\alpha_1,\ldots,\alpha_m\in K^p$. Let $b_{ij}\in\mathfrak{C}^p$ for $i,j=1,\ldots,e$ be such that $\partial\left(a_i\right)=\sum_{j=1}^{m}b_{ij}x_j$ for $i=1,\ldots,e$. We have

$$\sum_{i=1}^{m}\alpha_i x_i=X=\partial_2\left(X\right)=\sum_{i=1}^{m}\alpha_i^q\partial_2\left(x_i\right)=\sum_{i,j=1}^{m}\alpha_i^q b_{ij}x_i,$$

thus if we set $f_i\left(Y_1,\ldots,Y_m\right)=Y_i x_i-\sum_{j=1}^{m}Y_i^q b_{ij}x_i$ we have that the tuple $\alpha=(\alpha_1,\ldots\alpha_m)$ is a common zero the $\mathfrak{C}$-polynomials $f_1,\ldots,f_m$. The Jacobian matrix of the tuple $(f_1,\ldots,f_m)$ is equal to the diagonal matrix with entries $x_1,\ldots,x_m$ (here we use $n_2>0$ here), thus its

[5] Of course by making such assumptions we completely ignore the case of iterative B-operators. Let us point out that the proof in [**2**, Remark 4.12] works just as fine in the case of B-operators with a commutativity restrain.

determinant is nonzero. Hence by Fact 1.2 we have that α is separably algebraic over $\mathfrak{C}$, thus $\alpha \in \mathfrak{C}$. But then also $X = \sum_{i=1}^{m} \alpha_i x_i \in \mathfrak{C}$, which is absurd. $\square$

We denote the forking independence in $\mathfrak{C}$, considered as a separably closed field, by $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{SCF}}$. Similarly for types, algebraic closure, definable closure and groups of automorphisms, e.g. we use the notation $\mathrm{acl}^{\mathrm{SCF}}$. On the other hand, $\mathrm{acl}^{\mathcal{B}}$ corresponds to the algebraic closure computed in the $\mathcal{B}$-field $(\mathfrak{C}, \partial)$. The proof of the next result is verbatim the same as the proof of [**2**, Lemma 4.14], which deals with the case of B-operators.

Lemma 3.30. *For any small subset A of $\mathfrak{C}$, we have:*

$$\mathrm{acl}^{\mathcal{B}}(A) = \mathrm{acl}^{\mathrm{SCF}}(\langle A\rangle_{\mathcal{B}}),\ \mathrm{dcl}^{\mathcal{B}}(A) = \mathrm{dcl}^{\mathrm{SCF}}(\langle A\rangle_{\mathcal{B}})$$

where $\langle A\rangle_{\mathcal{B}}$ denotes the $\mathcal{B}$-subfield of $\mathfrak{C}$ generated by A.

Proof. We will prove the claim about $\mathrm{acl}^{\mathcal{B}}$, the proof of the claim about $\mathrm{dcl}^{\mathcal{B}}$ being almost the same. We need to show that $E := \mathrm{acl}^{\mathrm{SCF}}(\langle A\rangle_{\mathcal{B}})$ is $\mathcal{B}$-algebraically closed. Assume not, and take $d \in \mathrm{acl}^{\mathcal{B}}(E) \setminus E$. Let $(K, \partial) \prec (\mathfrak{C}, \partial)$ be such that $E \subseteq K$ and such that K also contains the (finite) orbit of d under the action of $\mathrm{Aut}^{\mathcal{B}}(\mathfrak{C}/E)$. There is $f \in \mathrm{Aut}^{\mathrm{SCF}}(\mathfrak{C}/E)$ such that $f(K)$ is algebraically disjoint from K over E. If $f(d) \in K$, then $d \in E$ (since E is SCF-algebraically closed). Therefore $f(d) \notin K$. Let us denote by ∂^f the $\mathcal{B}$-operator on $f(K)$ which is the transport of $\partial\colon K \to \mathcal{B}(K)$ via f. Arguing as in Theorem 3.26, we get that $Kf(K) \cong (f(K) \otimes_E K)_0$ and that there is a unique $\mathcal{B}$-operator on this field extending ∂, ∂^f. This field of fractions can be embedded (as a $\mathcal{B}$-field) over K into $\mathfrak{C}$. Hence, we can assume that $f(K)$ is an elementary substructure of $(\mathfrak{C}, \partial)$ and f is a $\mathcal{B}$-isomorphism. Therefore, f extends to an element of $\mathrm{Aut}^{\mathcal{B}}(\mathfrak{C}/E)$. But then we get

$$f(d) \in \big(\mathrm{Aut}^{\mathcal{B}}(\mathfrak{C}/E) \cdot d\big) \setminus K,$$

which is a contradiction. $\square$

We proceed now towards a description of the forking independence in the theory $\mathcal{B}_{\varphi}\mathrm{CF}$, which corresponds exactly to the description from [**2**] and is also "the obvious candidate for forking independence". We define the following ternary relation on small subsets A, B, C of $\mathfrak{C}$:

$$A \mathop{\smile\!\!\!\!\!\mid}_{C}^{\mathcal{B}} B \qquad \iff \qquad \mathrm{acl}^{\mathcal{B}}(A) \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathcal{B}}(C)}^{\mathrm{SCF}} \mathrm{acl}^{\mathcal{B}}(B).$$

We will prove that the ternary relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{B}}$ defined above satisfies the well-known properties of forking in stable theories (see Preliminaries, Section 1.7)

(P1) (invariance) The relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{B}}$ is $\mathrm{Aut}^{\mathcal{B}}(\mathfrak{C})$-invariant.

(P2) (symmetry) For every small $A, B, C \subset \mathfrak{C}$, it follows that

$$A \mathop{\smile\!\!\!\!\!\mid}_{C}^{\mathcal{B}} B \qquad \iff \qquad B \mathop{\smile\!\!\!\!\!\mid}_{C}^{\mathcal{B}} A.$$

(P3) (monotonicity and transitivity) For all small $A \subseteq B \subseteq C \subset \mathfrak{C}$ and small $D \subset \mathfrak{C}$, it follows that

$$D \mathop{\smile\!\!\!\!\!\mid}_{A}^{\mathcal{B}} C \qquad \iff \qquad D \mathop{\smile\!\!\!\!\!\mid}_{A}^{\mathcal{B}} B \quad \text{and} \quad D \mathop{\smile\!\!\!\!\!\mid}_{B}^{\mathcal{B}} C.$$

(P4) (existence) For every finite $a \subset \mathfrak{C}$ and every small $A \subseteq B \subset \mathfrak{C}$, there exists $f \in \mathrm{Aut}^{\mathcal{B}}(\mathfrak{C})$ such that $f(a) \mathop{\smile\!\!\!\!\!\mid}_{A}^{\mathcal{B}} B$.

(P5) (local character) For every finite $a \subset \mathfrak{C}$ and every small $B \subset \mathfrak{C}$, there is $B_0 \subseteq B$ such that $|B_0| \leqslant \omega$ and $a \mathop{\smile\!\!\!\!\!\mid}_{B_0}^{\mathcal{B}} B$.

(P6) (finite character) For every small $A, B, C \subset \mathfrak{C}$, we have:

$$A \mathop{\smile\!\!\!\!\!\mid}_{C}^{\mathcal{B}} B \text{ if and only if } a \mathop{\smile\!\!\!\!\!\mid}_{C}^{\mathcal{B}} B \text{ for every finite } a \subseteq A.$$

(P7) (uniqueness over a model) Any complete type over a model is $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{B}}$-stationary.

The properties (P1), (P2), (P3) and (P6) follow easily from the definition of $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{B}}$. The property (P5) follows from the local character and the finite character of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{SCF}}$.

Lemma 3.31. *Property (P4) holds.*

Proof. Let $a \subset \mathfrak{C}$ be finite, let $A \subseteq B$ be small subsets of $\mathfrak{C}$ and set $E = \mathrm{acl}^{\mathcal{B}}(A)$. Take a small submodel $K \prec \mathfrak{C}$ containing a, B, E. As in the proof of Lemma 3.30, we see that there is some $f \in \mathrm{Aut}^{\mathrm{SCF}}(\mathfrak{C}/E)$ such that $f(K)$ is algebraically disjoint from K over E and $f(K)$ is a elementary substructure of $\mathfrak{C}$. By the definition of $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{B}}$ and Lemma 3.30 we get $f(a) \mathop{\smile\!\!\!\!\!\mid}_{A}^{\mathcal{B}} B$, as desired. $\square$

Lemma 3.32. *Any complete type over an algebraically closed set is $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{B}}$-stationary. In particular, property (P7) holds.*

Proof. Let $M \prec \mathfrak{C}$ be a small model, K a small set with $\mathrm{acl}^{\mathcal{B}}(K) = K$ and let $a, b \in \mathfrak{C}$ are such that

$$\mathrm{tp}^{\mathcal{B}}(a/K) = \mathrm{tp}^{\mathcal{B}}(b/K), \quad a \mathop{\smile\!\!\!\!\!\mid}_{K}^{\mathcal{B}} M \quad \text{and} \quad b \mathop{\smile\!\!\!\!\!\mid}_{K}^{\mathcal{B}} M.$$

We want to show that also $\mathrm{tp}^{\mathcal{B}}(a/M) = \mathrm{tp}^{\mathcal{B}}(b/M)$.

Let $f \in \mathrm{Aut}^{\mathcal{B}}(\mathfrak{C}/K)$ be such that $f(a)$. We see that f constitutes a $\mathcal{B}$-isomorphism between the $\mathcal{B}$-fields $K_a := \mathrm{dcl}^{\mathcal{B}}(Ka)$ and $K_b := \mathrm{dcl}^{\mathcal{B}}(Kb)$. Since K is algebraically closed, the field extension $K \subseteq M$ is regular, so we can do the same juggling as in the proof of Lemma 3.30 to arrive at the following: K_a and M are linearly disjoint over K (and the same for K_b), $K_a \otimes M$ and $K_b \otimes M$ are $\mathcal{B}$-domains and f lifts naturally to an isomorphism between their fractions fields, i.e to an $\mathcal{B}$-automorphism $\widetilde{f}\colon K_aM \to K_bM$ which agrees with f on K_a (in particular $\tilde{f}(a) = b$) and is the identity on M. It is now enough to extend $\widetilde{f}$ to an automorphism of $\mathfrak{C}$.

Since K, K_a, K_b and M are definably closed, all the extensions $K \subseteq K_a, K_b, M \subseteq \mathfrak{C}$ are separable, so by the linear disjointedness of K_a and M over K (and the same for K_b) we get that the extensions $K_aM, K_bM \subseteq \mathfrak{C}$ are separable, in particular they are $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$ substructures of $\mathfrak{C}$, thus by quantifier elimination (Theorem 3.26) $\widetilde{f}\colon K_aM \to K_bM$ extends to an automorphism of $\mathfrak{C}$, as desired. $\square$

Theorem 3.33. *The theory $\mathcal{B}_{\varphi}\mathrm{CF}$ is stable, not superstable and the relation $\downarrow^{\mathcal{B}}$ coincides with the forking independence.*

Proof. Stability and the description of forking is due to properties (P1)-(P7) and the results from stability theory mentioned in Preliminaries, Section 1.7. The theory $\mathcal{B}_{\varphi}\mathrm{CF}$ is not superstable because its reduct to the language of pure fields is the theory of a imperfect separably closed field, thus it is not superstable. $\square$

As it was mentioned in [**2**, Section 5.1], finer model-theoretic results (as Zilber's trichotomy) are unknown even in the "simplest" case of the theory DCF_p, so they seem to be out of reach in the context of $\mathcal{B}$-operators.

Remark 3.34. One can also prove the stability of $\mathcal{B}_{\varphi}\mathrm{CF}$ by counting types. This was done by Shelah for the theory DCF_p (see [**45**]). Using Shelah's strategy Kowalski proved an analogous results for derivations of the Frobenius morphism and his proof might be easier to read (see [**27**]). In order to employ this strategy for $\mathcal{B}_{\varphi}\mathrm{CF}$ we have to show that for any tuple a the type $\mathrm{tp}^{\mathcal{B}}(a/K)$ can be naturally identified with the type $\mathrm{tp}^{\mathrm{SCF}}(\partial^{<\omega}(a)/K)$. This is more or less clear - by quantifier elimination (Theorem 3.26) both of those types describe the isomorphism type (over K) of the $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$-substructure of $\mathfrak{C}$ generated by $K(a)$. Having done this and using the stability of SCF one immediately gets that over any set K there are at most $|K|^{\aleph_0}$ complete types.

3.2.4. (Non-)existence: remaining cases

Theorem 3.16 shows that the theory of $\mathcal{B}$-fields has a model companion provided that $\mathrm{Fr}\,(\ker \pi_{\mathcal{B}}) = 0$. In this Subsection we want to say something about the existence of a model companion for other $\mathcal{B}$. For simplicity of the exposition, we will talk only about free $\mathcal{B}$-fields, i. e. φ is trivial in this Subsection. **We also assume that k is algebraically closed.** This assumption is harmless and does not affect the results in this Subsection - using standard base-change arguments one can deduce from this case the general case. This is done in details in [**13**, Section 4.].

Let us review what is known in the case of B-operators, i. e. when $\mathcal{B} = B_{\otimes}$ for a finite k-algebra B (see Preliminaries). By the structure theorem for Artin algebras over fields we get that B is isomorphic to a product $B_1 \times \ldots \times B_n$ where each B_i is either local or a field. In [**2**] a full classification of when the aforementioned model companion exists is given. This happens precisely when one the following holds:

(1) B is separable, thus (since k is algebraically closed) $B = k^n$ or
(2) B is local (thus $n = 1$) and $\mathrm{Fr}\,(\ker \pi_B) = 0$.

Let us go back to an arbitrary coordinate k-algebra scheme $\mathcal{B}$ and set $B = \mathcal{B}\,(k)$. From what we did until now we can easily deduce the following "transfer principle".

Proposition 3.35. *If the theory of $\mathcal{B}\,(k)$-fields has a model companion, then so does the theory of $\mathcal{B}$-fields.*

Proof. Assume that item (1) holds. By Theorem 2.17 and Example 2.5 we get that in this case $\mathcal{B} = \mathbb{S}_k^e$, thus $\mathcal{B}$-fields are simply fields of characteristic p with $e-1$ automorphisms and this theory is known to have a model companion (see the proof of [**2**, Corollary 3.9]). Assume now that B satisfies item (2). As in Remark 2.28 we get that $\mathrm{Fr}\,(\ker \pi_{\mathcal{B}}) = 0$, so by Theorem 3.14 the theory of $\mathcal{B}$-fields has a model companion. $\square$

It is tempting to ask whether the implication in Proposition 3.35 can be reversed. Again, write $B := \mathcal{B}\,(k)$ as a product $B_1 \times \ldots \times B_n$ where each B_i is either local or a field and assume that the theory of B-fields has no model companion. By the above discussion this means that either

(1) B is neither local nor separable or
(2) B is local but $\mathrm{Nil}\,B \neq \ker \pi_B$.

We will prove that in the first case theory of $\mathcal{B}$-fields has no model companion (see Proposition 3.37). In the second case we will only obtain some partial results (see Proposition 3.38). In both cases we will employ a strategy similar to the one used in [**36**] and [**2**]. Our main tool is the following.

Proposition 3.36. *Assume that for every natural number n there is a $\mathcal{B}$-field (K_n, ∂_n) and an element a_n such that $\partial_n^n(a_n) \in \mathcal{B}^{(n)}\left(K_n^{alg}\right)^p$, but $\partial_n^{n+1}(a_n) \notin \mathcal{B}^{(n+1)}\left(K_n^{alg}\right)^p$. Then the theory of $\mathcal{B}$-fields does not have a model companion.*

Proof. Assume that the theory of $\mathcal{B}$-fields has a model companion. Extend each K_n to an existentially closed $\mathcal{B}$-field $\widetilde{K}_n$. Clearly still $\partial_n^n(a_n) \in \mathcal{B}^n\left(\widetilde{K}_n^{alg}\right)^p$ and $\partial^{(n+1)}(a_n) \notin \mathcal{B}^{(n+1)}\left(\widetilde{K}_n^{alg}\right)^p$. Take a free ultrafilter $\mathcal{U}$ on the set of natural number and consider the ultraproduct $K = \prod_{n\in\omega} \widetilde{K}_n/\mathcal{U}$. Since we assumed that the theory of $\mathcal{B}$-fields has a model companion (i. e. the class of existentially closed $\mathcal{B}$-fields is elementary), by Łoś's Theorem we have that K is existentially closed. Let $a = (a_n)_{n\in\omega}/\mathcal{U} \in K$. Note that by the choice of the elements a_n we have $a \in \mathcal{B}^\omega\left(K^{alg}\right)^p$, thus by Lemma 2.41 there exists a $\mathcal{B}$-extension of K containing $a^{1/p}$, thus by existential closedness of K we have $a \in K^p$. Thus for $\mathcal{U}$-almost all $n \in \omega$ we have $a_n^p \in \widetilde{K}_n^p$, hence also $\partial^\omega(a_n) \in \mathcal{B}^\omega\left(\widetilde{K}_n^{alg}\right)^p$, a contradiction. ☐

Proposition 3.37. *Suppose that B is neither local nor separable. Then, the theory $\mathcal{B}\mathrm{CF}$ does not exist.*

Proof. We can represent $(\mathcal{B}, +)$ as:

$$(\mathcal{B}, +) = \mathbb{G}_{\mathrm{a}}^m \times \mathbb{G}_{\mathrm{a}}^{e-m},$$

where $\mathbb{G}_{\mathrm{a}}^m$ corresponds to separable part of B. By assumption we get that $1 < m < e$. Then, we clearly have:

$$\mathbb{G}_{\mathrm{a}}^m \subseteq \mathrm{Im}\left(\mathrm{Fr}_{\mathcal{B}}\right).$$

For simplicity, we assume that $n = 2$, $e = 3$, and $\pi(x, y, z) = x$. Then, we set the following for each i, n such that $0 < i < n$:

$$K_n := \mathbb{F}_p^{\mathrm{alg}}(X_1, \ldots, X_n), \qquad \partial(X_n) = (X_n, X_{n+1}, 0), \qquad \partial(X_n) = (X_n, 0, 1).$$

It is easy to check that K_n satisfy the assumptions of Proposition 3.36 for $a_n := X_1$. ☐

Using the result above, we can assume that we are in the Case 2 situation, that is: B is local and $\mathrm{Nil}(B) \neq \ker(\mathrm{Fr}_B)$. By Theorem 2.17, $\mathcal{B}$ is the transport of $B_{(1,n_2,\ldots,n_e)}$ using the transport by $(\mathrm{id}, \mathrm{Fr}^{n_2}, \ldots, \mathrm{Fr}^{n_e})$ for some $n_2, \ldots, n_e \in \omega$. If $n_2 = \ldots = n_e = n$, then $B_{(0,n_2,\ldots,n_e)} = B^{\mathrm{Fr}^n}$ and one can prove the nonexistence of $\mathcal{B}\,\mathrm{CF}$ almost exactly as in the case of $B\,\mathrm{CF}$ (see [**36**, Proposition 7.2]). Thus we get the following.

Proposition 3.38. *Suppose that there is $n \in \omega$ such that $\mathcal{B} \cong B^{\mathrm{Fr}^n}$ Then, the theory $\mathcal{B}\,\mathrm{CF}$ does not exist.*

By the last result, we are left with Case 2, where moreover not all the n_i's are equal to each other. However, even the simplest example of such a situation is hard to tackle as we will see below. Let us take $B = k[X]/(X^3)$ with basis $v_1 = 1+(X^3)$, $v_2 = X+(X^3)$, $v_3 = X^3+(X^3)$ and consider $\mathcal{B} = B_{(0,1,2)}$. Then, we have the following formulas for $\mathrm{Fr}_{\mathcal{B}}$ and $\mathrm{Fr}_{\mathcal{B}^{(2)}}$:

$$(x,y,z)^p = \left(x^p, 0, y^{p^2}\right),$$

$$((x_1,x_2,x_3),(y_1,y_2,y_3),(z_1,z_2,z_3))^p = \left(\left(x_1^p, 0, x_2^{p^2}\right), (0,0,0), \left(y_1^{p^2}, 0, 0\right)\right).$$

The next lemma shows that in this case, it is impossible to use Proposition 3.36. Actually, the desired construction of a non-satisfiable (in any existentially closed $\mathcal{B}$-field) and finitely satisfiable partial type already fails after the second step!

Lemma 3.39. *Let (K,∂) be a $\mathcal{B}$-field ($\mathcal{B}$ is as above) and $x \in K$ be such that:*

$$\partial(x) \in \left(\mathcal{B}\left(K^{\mathrm{alg}}\right)\right)^p, \quad \partial^{(2)}(x) \in \left(\mathcal{B}^{(2)}\left(K^{\mathrm{alg}}\right)\right)^p.$$

Then, for any $i > 0$ we have that:

$$\partial^{(i)}(x) \in \left(\mathcal{B}^{(i)}\left(K^{\mathrm{alg}}\right)\right)^p.$$

Proof. Let $\partial(x) = (x,y,z)$. Then we have:

$$\partial^{(2)}(x) = (\partial(x), \partial(y), \partial(z)).$$

By our assumption on $\partial(x)$, we get that $y = 0$. So, we have:

$$\partial^{(2)}(x) = ((x,0,z),(0,0,0),\partial(z)).$$

By our assumption on $\partial^{(2)}(x)$, we get that $\partial(z) = (z,0,0)$. Hence $z \in K^{\partial}$, which finishes the proof. □

3.3. PAC structures in the context of $\mathcal{B}$-fields

Pseudo algebraically closed (PAC) fields were introduced by Ax in his seminal work [**1**] on pseudofinite fields. Later on Hrushovski in [**22**] considered PAC structures in a more general model-theoretic context, more precisely he considered *pseudo algebraically substructures* of strongly minimal structures. In [**40**] Pillay and Polkowska considered PAC substructure in the general context of stable theories and so did recently Hoffmann and Kowalski in [**18**].[6] We will base on the latter work in this section. Let T is be a stable theory and let $\mathfrak{C}$ be a monster model of T.

Definition 3.40 (Definition 2.3. in [**18**])**.** We say that a small substructure $F \subset \mathfrak{C}$ is **pseudo algebraically closed** (or T-PAC) if and every stationary type over F is finitely satisfiable in F.

For $T = \mathrm{ACF}$ it is an easy exercise to prove that pseudo algebraically closed substructures are precisely perfect pseudo algebraically closed fields. Moreover, in ACF stationary types correspond to absolutely irreducible varieties, so Definition 3.40 generalizes the "geometric" definition of being pseudo algebraically closed, i. e. "PAC = absolutely irreducible varieties have points". The point is that one can view this notion also as a variant of existential closedness, as explained below in Fact 3.43.

Definition 3.41 (Definition 2.3. in [**18**])**.** We say that an extension of small substructures $F \subseteq K$ is **regular** if $\mathrm{dcl}\,(K) \cap \mathrm{acl}\,(F) = F$.

Note that in particular, if the extension $F \subseteq K$ is regular then F is definably closed.

Example 3.42. If $L = \mathcal{L}^{\lambda}_{\mathrm{rng}}$ and $T = \mathrm{SCF}_{p,e}$ (where e is finite or not), then a regular extension in the above sense is the same as regular extension in the field theoretic sense (see Fact 4.17 and Fact 4.20 in [**18**]).

Fact 3.43 (Remark 2.17 in [**18**])**.** *Assume the following conditions hold:*

(1) *T has quantifier elimination.*
(2) *T codes finite tuples (i.e. eliminates finite imaginaries).*
(3) *T is stable and types over algebraically closed sets are stationary.*

*Then, a small subset F is T-PAC if and only if $F = \mathrm{dcl}(F)$ and F is **regularly closed**, i. e. existentially closed in every regular extension.*

[6] There is a lot more work on PAC structures and people we should mention, see the introduction to [**18**].

Specializing to $T = \mathrm{ACF}$ in Fact 3.43 we recover the well-known fact that PAC fields are exactly the fields which are existentially closed in any (field-theoretically) regular extension.

Remark 3.44. Any theory of a field (with possibly some additional structure) codes finite tuples - given a tuple $\bar{a} = (a_1, \ldots, a_n)$ the tuple $(s_1(\bar{a}), \ldots, s_n(\bar{a}))$ is a code for $\bar{a}$, where $s_1, \ldots, s_n$ are the elementary symmetric polynomials in n variables.

Let us apply the above notions in the case of $\mathcal{B}_\varphi$-fields. Fix a coordinate k-scheme $\mathcal{B}$ satisfying $\mathrm{Fr}(\ker \pi_{\mathcal{B}}) = 0$ and an iterativity condition φ. From now on let T be the theory $\mathcal{B}_\varphi\,\mathrm{CF}$ considered in the language $\mathcal{L}_{\mathcal{B}}^{\lambda_0}$ and let $\mathfrak{C}$ be a monster model of T. Note that T meets the assumptions of Fact 3.43: The theory T is stable by Theorem 3.33, stationary of types over algebraically closed sets is the content of Lemma 3.32, by the choice of the language it eliminates quantifiers thanks to Theorem 3.26 and Remark 3.44 implies T codes finite tuples.

We aim to prove that $\mathcal{B}_\varphi\,\mathrm{CF}$-PAC is an elementary property. Fact 3.43 tells us that $\mathcal{B}_\varphi\,\mathrm{CF}$-PAC is an instance of $\mathcal{K}$-closedness for an appropriate $\mathcal{K}$, so this fits perfectly into the set-up of Section 3.1. The next few result technical lemmas will show this even more concretely, namely we will prove the following (see Lemma 3.46): a $\mathcal{B}_\varphi$-field K is $\mathcal{B}_\varphi\,\mathrm{CF}$-PAC if and only if K is strict and $\mathcal{K}_{\mathrm{reg}}$-closed where $\mathcal{K}_{\mathrm{reg}}$ denotes the class of all $\mathcal{B}_\varphi$-field extensions which are regular as extensions of pure fields.

Until the end of this Section, all $\mathcal{B}_\varphi$-field considered are small subsets of $\mathfrak{C}$.

Lemma 3.45. *An extension of small $\mathcal{B}_\varphi$-fields $F \subseteq K$ is regular in the sense of Definition 3.41 if and only if $F \subseteq K$ is regular as a pure field extension.*

Proof. By Lemma 3.30 we have

$$\mathrm{dcl}^{\mathcal{B}}(K) \cap \mathrm{acl}^{\mathcal{B}}(F) = \mathrm{dcl}^{\mathrm{SCF}}(K) \cap \mathrm{acl}^{\mathrm{SCF}}(F),$$

thus we are done by Corollary 3.28 and Example 3.42. $\square$

Lemma 3.46. *Let K be a $\mathcal{B}_\varphi$-field. Then K is $\mathcal{B}_\varphi\,\mathrm{CF}$-PAC if and only if K is strict and $\mathcal{K}_{\mathrm{reg}}$-closed.*

Proof. By Fact 3.43 K is $\mathcal{B}_\varphi\,\mathrm{CF}$-PAC if and only if $K = \mathrm{dcl}(K)$ and regularly closed. Thus the result follows immediately from combining Lemma 3.30, Lemma 3.24 and Lemma 3.45. $\square$

Finally, from Theorem 3.14, Example 3.2 and Lemma 3.46 we immediately get the following result.

Theorem 3.47. *For a $\mathcal{B}_\varphi$-field (K, ∂) the following are equivalent:*

(1) *(K, ∂) is $\mathcal{B}_\varphi\,\mathrm{CF}$-$\mathrm{PAC}$.*
(2) *K is strict and every absolutely irreducible $\mathcal{B}_\varphi$-variety over (K, ∂) has a K-rational $\mathcal{B}$-point.*

In particular, being $\mathcal{B}_\varphi\,\mathrm{CF} - \mathrm{PAC}$ is an elementary property.

Remark 3.48. Theorem 3.47 applies in particular to the theory $\mathrm{DCF}_{p,m}$ for $p > 0$, which answers a question of Hoffmann-Kowalski in [**18**] (see Remark 4.45 there). The case $m = 0$ was done by Hoffmann-Kowalski in [**18**], but their axiomatization of $\mathrm{DCF}_{p,0} - \mathrm{PAC}$ is a bit complicated (see the paragraph above [**18**, Theorem 4.34]). The source of this complication is twofold:

- There is a certain equalizer variety E involved (see also Remark 3.19).
- The axioms use a certain notion of "admissible tuples". It is not at all obvious that this notion is expressible in first-order logic and in fact the proof that it is uses a slight enhancement of a theorem by Tamagawa (see [**11**, Proposition 11.4.1]).

Our axioms are much more transparent and also the proof that they work is much easier. Additionally, Theorem 3.47 goes well beyond the case of $\mathrm{DCF}_{p,m}$

Remark 3.49. $\mathcal{K}_{\mathrm{reg}}$-closed $\mathcal{B}_\varphi$-fields are also interesting for other classes of $\mathcal{B}$ than considered in this Section. For example, let G be a finitely generated group and let $(\mathcal{B}, \varphi)$ be the pair describing G-fields. Then, $\mathcal{K}_{\mathrm{reg}}$-closed $\mathcal{B}_\varphi$-fields are (by definition) the same as *pseudo existentially closed* G-fields (see [**3**]). They implicitly appear in Hrushovski's proof that the theory of fields with two commuting automorphisms does not have a model companion, since what is really proved there is the much stronger statement that there is no $\aleph_0$-saturated pseudo existentially closed $(\mathbb{Z} \times \mathbb{Z})$-field. We send the reader to [**3**, Section 6.2] for more details. In any case, using Theorem 3.14 we get that for a finite group G the class of pseudo existentially closed G-fields is elementary.

3.4. Miscellaneous examples

In this Section we comment on some further applications of the techniques used in this Chapter.

3.4.1. Separably $\mathcal{B}_\varphi$-closed fields

In [**23**] Ino and León Sánchez considered separably differentially closed field, i. e. differential field which are closed in separable differential field extensions (here separable is in the sense of pure field). Among other things, they prove that this class of field is elementary and give a very nice axiomatization of its theory SDCF_p, similar to the axiomatization of DCF_0 by Blum (see [**4**]) or the axiomatization of DCF_0 by Wood (see [**50**]). They also give a geometric axioms for SDCF_p.

This clearly fits into our context. Fix some coordinate k-algebra scheme $\mathcal{B}$ and an iterativity condition φ. Let $\mathcal{K}_{\mathrm{sep}}$ be the class of all separable (in the field-theoretic sense) extensions of $\mathcal{B}_\varphi$-fields. Let us say that a $\mathcal{B}_\varphi$-field (K,∂) is **separably $\mathcal{B}_\varphi$-closed** if it is $\mathcal{K}_{\mathrm{sep}}$-closed. Immediately from Theorem 3.14 we get the following.

Corollary 3.50. *Assume that $(\mathcal{B},\varphi)$ is nice. Then, for a $\mathcal{B}_\varphi$-field (K,∂) the following are equivalent:*

(1) *(K,∂) is separably $\mathcal{B}_\varphi$-closed.*
(2) *Every separable $\mathcal{B}_\varphi$-variety over (K,∂) has a K-rational $\mathcal{B}_\varphi$-point.*

In particular, there is an $\mathcal{L}_{\mathcal{B}}$-theory $\mathrm{S}\,\mathcal{B}_\varphi\,\mathrm{CF}$ whose models are precisely separably $\mathcal{B}_\varphi$-closed $\mathcal{B}_\varphi$-fields.

Remark 3.51. In particular setting $\mathcal{B} = k\,[\varepsilon]_\otimes$ where $k\,[\varepsilon] = k\,[X]\,/\,(X^2)$ we get a new axiomatization of the theory SDCF_p considered in [**23**]. There are two axiomatizations given in [**23**]. One of them roughly corresponds to our New Axioms for $\mathcal{B}\,\mathrm{CF}$ given in Section 3.2.1. The second one is similar to the axiomatization of DCF_0 given by Blum or the axiomatization of DCF_p (where $p > 0$ is prime) given by Wood, and it speaks about the solvability of certain differential equations in one variable. An analogous results for derivations of the Frobenius map was proven by the author in [**12**, Theorem 3.12].

3.4.2. Largeness

Recall that a field K is called **large** if every K-variety defined with a smooth K-rational point has a Zariski-dense set of K-rational points. In particular, the class of large fields includes pseudo algebraically closed fields (hence also separably and algebraically closed fields) and real closed fields. In general, large fields are in some sense the widest class of "tame fields" and they have many remarkable properties (e. g. the regular inverse Galois problem is solvable over large fields). Moreover being large is an elementary property in the language of rings. We send the reader to [**41**] for a survey on large fields.

In [**42**] León Sánchez and Tressl introduce a differential counterpart of large fields and among many other thing proved that the class of differentially large fields is elementary. We want to generalize this result to the case of $\mathcal{B}_\varphi$-fields. Let $\mathcal{K}_{\mathrm{ec}}$ be the class of all $\mathcal{B}_\varphi$-extensions $K \subseteq L$ such that K is existentially closed in L as a pure field.

Definition 3.52. Let (K, ∂) be a $\mathcal{B}_\varphi$-field. We say that K is **$\mathcal{B}_\varphi$-large** if it is large as a pure field and $\mathcal{K}_{\mathrm{ec}}$-closed.

Specializing to $(\mathcal{B}, \varphi)$ describing partial differential fields, we recover the definition given in [**42**].

We aim to show that for a nice pair $(\mathcal{B}, \varphi)$ being $\mathcal{B}_\varphi$-large is an elementary property. For partial differential fields in characteristic zero this was done in [**42**] and for ordinary differential fields of positive characteristic in [**43**].

The following lemma is well-known, see e.g [**41**, Fact 2.3].

Lemma 3.53. *Let K be a field and V a K-variety. Then, the following are equivalent:*

(1) *K is existentially closed in $K(V)$.*
(2) *$V(K)$ is Zariski-dense in V.*

Corollary 3.54. *Assume that $(\mathcal{B}, \varphi)$ is nice. Then, for a $\mathcal{B}_\varphi$-field (K, ∂) the following are equivalent:*

(1) *(K, ∂) is $\mathcal{B}_\varphi$-large.*
(2) *K is large and every $\mathcal{B}_\varphi$-variety over (K, ∂) with a smooth K-rational point has a K-rational $\mathcal{B}_\varphi$-point.*

Moreover, being $\mathcal{B}_\varphi$-large is an elementary property.

Proof. (1) $\Longrightarrow$ (2) Assume (1) and let (V, s) be a $\mathcal{B}_\varphi$-variety such that V has a smooth K-rational point. Since K is large, V has in fact a Zariski-dense set of such points, thus by Lemma 3.53 we have that K is existentially closed (as a pure field) in $K(V)$. Since K is $\mathcal{K}_{\mathrm{ec}}$-closed, K is existentially closed in $K(V)$ as a $\mathcal{B}_\varphi$-field. By Lemma 2.57 there is a $K(V)$-rational $\mathcal{B}_\varphi$-point of (V, s), thus by existential closedness there is already a K-rational one.

(2) $\Longrightarrow$ (1) Assume (2). Since $(\mathcal{B}, \varphi)$ is nice, by Lemma 3.9 and Proposition 2.68 we have to check only that K is $(\mathcal{K}_{\mathrm{ec}})_{\mathrm{fin}}$-closed. But this follows from (2) using Lemma 3.53 as in the previous implication.

The moreover claim follows since being large is an elementary property and "V has smooth K-rational point" is a definable condition in the sense of Remark 2.66. $\square$

Remark 3.55. The main thing we want to point out is that as a special case of Corollary 3.54 the class of fields with several commuting derivations in positive characteristic, which are large in the above sense, is an elementary class. This results does not appear neither in [**42**] nor in [**43**].

3.4.3. $\mathrm{DCF}^{\mathrm{fin}}_{0,m}$

Let us note a peculiar example which pops out naturally from our work. For simplicity of the exposition, let us work with fields of characteristic zero with $m \geq 2$ commuting derivations, which we will also call **partial differential fields** (suppressing m from the nomenclature). Denote by $\mathrm{DF}_{0,m}$ the theory of such fields in the language $\mathcal{L}_D = \mathcal{L}_{\mathrm{rng}} \cup \{\partial_1, \ldots, \partial_m\}$. McGrail proved in [**33**] that $\mathrm{DF}_{0,m}$ has a model companion $\mathrm{DCF}_{0,m}$. Also, by Theorem 3.13 we get that there is a theory $\mathrm{DCF}^{\mathrm{fin}}_{0,m}$ whose models are exactly those $K \models \mathrm{DF}_{0,m}$ which are existentially closed in any $L \models \mathrm{DF}_{0,m}$ such that $K \subseteq L$ is finitely generated as a pure field extension. There is a natural question whether $\mathrm{DCF}_{0,m} = \mathrm{DCF}^{\mathrm{fin}}_{0,m}$. Note that by Theorem 3.14 this is true for $m = 1$ or for p instead of 0.

Unfortunately the answer is no. Consider the field $K = \mathbb{Q}(x_1, \ldots x_m)$ with the obvious derivations. By [**24**, Theorem 2] the (consistent) equation

$$\partial_1(f) = \left(1 - \frac{x_1}{x_2}\right) \partial_2(f) + 1$$

has no solution f such that $K\langle f \rangle$ has finite transcendence degree over K, where $K\langle f \rangle$ is the partial differential field generated over K by f.

Using this one can construct a model M of $\mathrm{DCF}^{\mathrm{fin}}_{0,m}$ which is not a model of $\mathrm{DCF}_{0,m}$. We will invoke Fraïssé theory for this purpose, though one can construct M directly, similarly to how one construct an existentially closed model of an inductive theory, although some care is needed.

Let us recall some classical material about Fraïssé theory. We refer to Fraïssé's original paper [**10**].

Definition 3.56. Let $\mathcal{L}$ be a countable language. We say that a countable $\mathcal{L}$-structure M is **ultrahomogeneous** if any isomorphism between finitely generated substructures of M extends to an automorphism of M.

Definition 3.57. Let M be an $\mathcal{L}$-structure. The **age of** M is the class $\operatorname{age}(M)$ of all finitely generated $\mathcal{L}$-structures which embed into M. Equivalently, it is (the closure under isomorphic images of) the class of all finitely generated $\mathcal{L}$-substructures of M.

Definition 3.58. Let $\mathcal{C}$ be a class of finitely generated $\mathcal{L}$-structures. If

(1) $\mathcal{C}$ is closed under isomorphisms and finitely generated substructures,
(2) $\mathcal{C}$ has only countably many members up to isomorphism,
(3) $\mathcal{C}$ has the joint embedding property,
(4) $\mathcal{C}$ has the amalgamation property,

the we say that $\mathcal{C}$ is a **Fraïssé class**.

The following is the celebrated Fraïssé theorem.

Fact 3.59. *Let $\mathcal{C}$ be a Fraïssé class. Then there is a unique (up to isomorphism) countable ultrahomogeneous structure M whose age is equal $\mathcal{C}$ called the **Fraïssé limit** of $\mathcal{C}$.*

For our purposes, we work in the language $\mathcal{L}_{D,\mathrm{inv}} = \mathcal{L}_D \cup \{\mathrm{inv}\}$ where "inv" is a unary function symbol. Any partial differential field K is naturally an $\mathcal{L}_{D,\mathrm{inv}}$-structure, where we interpret inv as follows:

$$\mathrm{inv}^K(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{otherwise} \end{cases}.$$

Of course, any extension of partial differential fields is also an $\mathcal{L}_{D,\mathrm{inv}}$-extension. We define $\mathcal{C}$ as the class of all partial differential fields which are finitely generated (as pure fields over $\mathbb{Q}$).

Lemma 3.60. *The class $\mathcal{C}$ is a Fraïssé class.*

Proof. Clearly $\mathcal{C}$ is closed under finitely generated substructures and one easily sees that $\mathcal{C}$ has only countably many objects up to isomorphism. By Proposition 2.70 and Remark 2.71 the class $\mathcal{C}$ has the amalgamation property and since $\mathcal{C}$ has an initial object (namely $\mathbb{Q}$ with m trivial derivations) it has also the joint embedding property. $\square$

Denote the Fraïssé limit of $\mathcal{C}$ by M. Note that M is a field, since any of its finitely generated substructure is one.

Proposition 3.61. *M is a model of* $\mathrm{DCF}_{0,m}^{\mathrm{fin}}$ *which is not a model of* $\mathrm{DCF}_{0,m}$.

Proof. We will first prove the latter claim. Since the finitely generated substructures of M are (up to isomorphism) precisely all models of $\mathrm{DF}_{0,m}$, we see that the equation

$$\partial_1(f) = \left(1 - \frac{x_1}{x_2}\right)\partial_2(f) + 1$$

has no solution in M (see the beginning of this section). On the other hand, this equation has a solution in some partial differential field N, hence (by taking an amalgam of M and N) also in an extension of M. Thus M is not existentially closed as a partial differential field, hence M is not a model of $\mathrm{DCF}_{0,m}$.

As for the former claim, let $M \subset N$ be an extension of partial differential fields which is finitely generated as an extension of pure fields. We want to prove that M is existentially closed in N in the language $\mathcal{L}_D$. Let $\phi(\bar{x}, \bar{y})$ be a quantifier-free $\mathcal{L}_D$-formula and let $a \in N$ and $b \in M$ be tuples such that $N \models \phi(a, b)$. Let N_0 be the partial differential field generated by a and b. Then $N_0 \in \mathcal{C}$, thus there is an embedding $f\colon N_0 \to M$. Set $b' = f(b)$. Then, f restricts to an isomorphism between the substructures of M generated by b and b', hence by ultrahomogeneity there is an automorphism σ of M such that $\sigma(b) = b'$. Since $N_0 \models \phi(a, b)$ and f is an embedding we have that $M \models \phi(f(a), b')$. Applying σ^{-1} yields $M \models \phi((\sigma^{-1} \circ f)(a), b)$, thus $\phi(\bar{x}, b)$ is satisfiable in M. Therefore M is existentially closed in N and thus M is a model of $\mathrm{DCF}_{0,m}^{\mathrm{fin}}$. $\square$

In more fancy terms, models of $\mathrm{DCF}_{0,m}^{\mathrm{fin}}$ are differential fields K (inside some monster model of $\mathrm{DCF}_{0,m}$) with the property that every finite dimensional type over K is finitely satisfiable in K.

References

[1] James Ax. The elementary theory of finite fields. *Annals of Mathematics*, 88(2):239–271, 1968.

[2] Özlem Beyarslan, Daniel Hoffmann, Moshe Kamensky, and Piotr Kowalski. Model theory of fields with free operators in positive characteristic. *Transactions of the American Mathematical Society*, 372(8):5991–6016, 2019.

[3] Özlem Beyarslan and Piotr Kowalski. Model theory of Galois actions of torsion abelian groups. *Journal of the Institute of Mathematics of Jussieu*, 22(6):2943–2985, 2023.

[4] Lenore Blum. Differentially closed fields: a model-theoretic tour. In *Contributions to Algebra*, pages 37–61. Academic Press, 1977.

[5] James Borger and Ben Wieland. Plethystic algebra. *Advances in Mathematics*, 194(2):246–283, 2005.

[6] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models*. A Series of Modern Surveys in Mathematics Series. Springer, 1990.

[7] A. Buium. Arithmetic analogues of derivations. *Journal of Algebra*, 198:290–299, 1997.

[8] Guy Casale, James Freitag, and Joel Nagloo. Ax-Lindemann-Weierstrass with derivatives and the genus 0 Fuchsian groups. *Annals of Mathematics*, 192(3):721 – 765, 2020.

[9] Zoé Chatzidakis and Ehud Hrushovski. Difference fields and descent in algebraic dynamics. I. *Journal of the Institute of Mathematics of Jussieu*, 7(4):653–686, 2008.

[10] Roland Fraïssé. Sur quelques classifications des systèmes de relations. *Publications scientifiques de l'Université d'Alger, Série A*, 1:35–182, 1954.

[11] Michael D. Fried and Moshe Jarden. *Field arithmetic*. Springer, 2008.

[12] Jakub Gogolok. Model theory of derivations of the Frobenius map revisited. *The Journal of Symbolic Logic*, 88(3):1213–1229, 2023.

[13] Jakub Gogolok and Piotr Kowalski. Operators coming from ring schemes. *Journal of the London Mathematical Society*, 106(3):1725–1758, 2022.

[14] Marvin J. Greenberg. Algebraic rings. *Transactions of the American Mathematical Society*, 111(3):472–481, 1964.

[15] Wilfrid Hodges. *A Shorter Model Theory*. Cambridge University Press, USA, 1997.

[16] Daniel M. Hoffmann and Piotr Kowalski. Existentially closed fields with finite group actions. *Journal of Mathematical Logic*, 18(01):1850003, 2018.

[17] Daniel Max Hoffmann and Piotr Kowalski. Model theory of fields with finite group scheme actions. *The Journal of Symbolic Logic*, 88(4):1443–1468, 2023.

[18] Daniel Max Hoffmann and Piotr Kowalski. PAC structures as invariants of finite group actions. *The Journal of Symbolic Logic*, 2023. (appeared online).

[19] Daniel Max Hoffmann and Omar León Sánchez. Model theory of differential fields with finite group actions. *Journal of Mathematical Logic*, 22(01):2250002, 2022.

[20] Ehud Hrushovski. The Elementary Theory of the Frobenius Automorphisms. Preprint (24 July 2012), available on `http://www.ma.huji.ac.il/~ehud/FROB.pdf`.

[21] Ehud Hrushovski. The Manin–Mumford conjecture and the model theory of difference fields. *Annals of Pure and Applied Logic*, 112(1):43–115, 2001.

[22] Ehud Hrushovski. Pseudo-finite fields and related structures. *Quad. Mat.*, vol. 11, 2002.

[23] Kai Ino and Omar León Sánchez. Separably differentially closed fields. *arXiv e-prints*, page arXiv:2302.11319, February 2023.

[24] J. Johnson, G.M. Reinhart, and L.A. Rubel. Some counterexamples to separation of variables. *Journal of Differential Equations*, 121(1):42–66, 1995.

[25] Moshe Kamensky. Tannakian formalism over fields with operators. *International Mathematics Research Notices*, 24:5571–5622, 2013.

[26] Byunghan Kim. *Simplicity Theory*. Oxford University Press, 2013.

[27] Piotr Kowalski. Derivations of the Frobenius map. *Journal of Symbolic Logic*, 70(1):99–110, 2005.

[28] Piotr Kowalski. Jet operators on fields. *Journal of Algebra*, 289:312–319, 2005.

[29] Mac Saunders Lane. *Categories for the working mathematician*. Springer, 2010.

[30] Serge Lang. *Abelian varieties*. Dover Publications, Inc., 2019.

[31] Q. Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford Graduate Texts in Mathematics. Oxford University Press, 2006.

[32] H. Matsumura. *Commutative ring theory*. Cambridge University Press, 1987.

[33] Tracey McGrail. The model theory of differential fields with finitely many commuting derivations. *The Journal of Symbolic Logic*, 65(2):885–913, 2000.

[34] Rahim Moosa and Thomas Scanlon. Jet and prolongation spaces. *Journal of the Inst. of Math. Jussieu*, 9(2):391–430, 2010.

[35] Rahim Moosa and Thomas Scanlon. Generalized Hasse-Schmidt varieties and their jet spaces. *Proc. Lond. Math. Soc.*, 103(2):197–234, 2011.

[36] Rahim Moosa and Thomas Scanlon. Model theory of fields with free operators in characteristic zero. *Journal of Mathematical Logic*, 14(02):1450009, 2014.

[37] D. Mumford and G.M. Bergman. *Lectures on Curves on an Algebraic Surface*. Annals of Mathematics Studies. Princeton University Press, 1966.

[38] Oystein Ore. Theory of non-commutative polynomials. *Annals of Mathematics*, 34:480–508, 1933.

[39] David Pierce and Anand Pillay. A note on the axioms for differentially closed fields of characteristic zero. *Journal of Algebra*, 204:108–115, 1998.

[40] Anand Pillay and Dominika Polkowska. On PAC and Bounded Substructures of a Stable Structure. *The Journal of Symbolic Logic*, 71(2):460–472, 2006.

[41] Florian Pop. *Little survey on large fields - old and new*, page 432–463. EMS Press, September 2014.

[42] Omar León Sánchez and Marcus Tressl. Differentially large fields. *arXiv e-prints*, page arXiv:2005.00888, July 2020.

[43] Omar León Sánchez and Marcus Tressl. On ordinary differentially large fields. *arXiv e-prints*, page arXiv:2307.12977, July 2023.

[44] Stephen S. Shatz. Galois theory. In *Category Theory, Homology Theory and their Applications I*, volume 86 of *Lecture Notes in Mathematics*, pages 146–158. Springer-Verlag, Berlin, Heidelberg, New York, 1969.

[45] Saharon Shelah. Differentially closed fields. *Israel Journal of Mathematics*, 16(3):314–328, September 1973.

[46] T.A. Springer. *Linear Algebraic Groups*. Progress in mathematics 9. Birkhäuser, 1981.

[47] Laurentius Petrus Dignus van den Dries. *Model theory of fields: Decidability and bounds for polynomial ideals*. PhD thesis, Rijksuniversiteit Utrecht, 1978.

[48] Philip Wadler. Monads for functional programming. In *Program Design Calculi*. Springer Berlin Heidelberg, 1993.

[49] William C. Waterhouse. *Introduction to Affine Group Schemes*. Springer-Verlag, 1979.

[50] Carol Wood. The model theory of differential fields revisited. *Israel Journal of Mathematics*, 25:331–352, 1976.