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Koncetracja i stabilność rozwiązań równań agregacji-dyfuzji

Rozprawa doktorska

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To all those who have been kind to me.

Contents

Streszczenie

Ta rozprawa doktorska koncentruje się na równaniach agregacji-dyfuzji (ADE), które modelują rozkład gęstości obiektów (cząstek, etc.) w czasie, pod wpływem nielokalnych interakcji oraz nieliniowej dyfuzji. Równania te opisują różne zjawiska fizyczne i biologiczne, m.in.: przyciąganie grawitacyjne, chemotaksję oraz zachowanie tłumu. Interpretowane są jako ciągły opis interakcji między cząstkami, gdzie każda cząstka ma przyporządkowane położenie i pęd. Rozprawa ta dzieli się na trzy części.

W pierwszej części rozwijana jest teoria w przestrzeni L^p dla równań ADE z jądrem (opisującym interakcje) w postaci funkcji potęgowej. Rozwiązania tego zagadnienia są globalne w czasie oraz ograniczone, ale wykazują koncentrację dla dostatecznie małej dyfuzji tzn. można zaobserwować istotną akumulację części rozwiązania w małym otoczeniu punktu zero. Może to być interpretowane jako jakościowy opis osobliwości powstającej przy braku dyfuzji. Główne narzędzia zastosowane w tej części opierają się na oszacowaniach *a priori*, metodzie momentów oraz uśrednianiu rozwiązania w czasie poprzez całkowanie.

Druga część rozprawy dotyczy badania istnienia rozwiązań stacjonarnych równań ADE, z ogólniejszym jądrem potęgowym, metodą punktu stałego. Wyprowadzono jawne wzory na niektóre z tych rozwiązań, które bywają cenne w kontekście weryfikacji ogólnych metod numerycznych. Co więcej, udało się wyprowadzić jawny wzór na rozwiązanie zmiennego znaku w jednym wymiarze, posiłkując się wynikami dotyczącymi równania Burgersa.

Tematem ostatniej części jest paraboliczno-eliptyczny układ równań różniczkowych, rozważany w całej przestrzeni, znany również jako model Kellera-Segela, który modeluje rozkład gęstości komórek oraz ich interakcje poprzez stężenie chemoatraktantu. W tym problemie funkcje stałe są rozwiązaniami stacjonarnymi. Udało się rozwinąć teorię lokalnych w czasie rozwiązań w jednorodnie lokalnych przestrzeniach Lebesgue'a, wraz z opisem dynamiki rozwiązań dla dużych wartości czasu. Niektóre stałe rozwiązania stacjonarne są stabilne. Z drugiej strony, powyżej pewnej wartości krytycznej, stałe stany stacjonarne wykazują niestabilność.

Abstract

This doctoral dissertation focuses on aggregation-diffusion equations (ADE), which model the behavior of particle density under non-local interactions and nonlinear diffusion. These equations describe various physical and biological phenomena, including, e.g., gravitational attraction, chemotaxis and swarming. Modeling through ADE captures long-range attraction and short-range repulsion, serving as a continuum description of particle interactions derived from ordinary differential equations. This dissertation is divided into three parts.

In the first part, an L^p-theory is developed for an ADE with a power-law interaction kernel. Global-in-time and globally bounded solutions exhibit a concentration phenomenon for small diffusion coefficients, resulting in significant mass accumulation in small neighborhoods around the origin. This can be viewed as a qualitative description of the singularity that forms in the absence of diffusion. The main calculations are based on *a priori* and moment estimates, and time averaging through integration. Properties of the solutions for large times are also described.

The second part is devoted to discussing the application of fixed-point methods to establish the existence of steady states for certain ADE with a more general power-law kernel. Moreover, explicit formulas for some stationary solutions are derived, which are valuable for verifying general numerical methods. Finally, the existence of a sign-changing solution in one dimension is demonstrated, utilizing results from the theory of the Burgers' equation.

The last part analyzes a minimal parabolic-elliptic Keller-Segel system modeling cell density and interactions through chemoattractant concentration. In this system, constant functions serve as steady states. A framework is developed for local-in-time solutions in the uniformly local Lebesgue spaces, alongside an analysis of the long-term dynamics of these solutions. Certain constant stationary solutions are stable, indicating that small perturbations can lead to global-in-time convergence. Conversely, beyond a critical parameter value, these constant steady states exhibit instability.

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Notation

General notation

i) $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^+ = \{1, 2, 3, ...\}$ - natural numbers.

ii) $\mathbb{R}^d$ - d-dimensional Euclidean space, endowed with the norm $|x| = \sqrt{x_1^2 + ... + x_d^2}$.

iii) B_R - ball in $\mathbb{R}^d$ centered at $x_0 = 0$, with radius $R > 0$. We write $B_R(x_0)$ when $x_0 \neq 0$.

iv) σ_d - area of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, $\sigma_d = 2\pi^{d/2}/\Gamma(d/2)$.

v) ∂^α - higher order partial derivative, where α - multi-index.

vi) C - generic constant (sometimes indexed), which may vary from line to line. We write $C = C(\alpha, \beta, \gamma, ...)$ when we want to emphasize the dependence of C on such parameters.

Function spaces

i) $L^p(\mathbb{R}^d)$ - Lebesgue space with a standard norm $\|\cdot\|_p$.

ii) $L^1\big(\mathbb{R}^d, w\,\mathrm{d}x\big)$ - the w-weighted L^1 space, where $w : \mathbb{R}^d \to [a, +\infty)$, $a > 0$, is a measurable function, endowed with the norm

$$\|v\|_{L^1(\mathbb{R}^d, w\,\mathrm{d}x)} = \int_{\mathbb{R}^d} |v(x)| w(x)\,\mathrm{d}x.$$

iii) $C^n(\mathbb{R}^d)$ - Banach space of n-times continuously differentiable functions on $\mathbb{R}^d$.

iv) $C_c^\infty(\mathbb{R}^d)$ - space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

v) Any other norm in a Banach space Y is denoted by $\|\cdot\|_Y$.

Additional notation and definitions

i) For two integrable functions f, g, we define their convolution as

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y.$$

ii) M - mass of the function $v \in L^1(\mathbb{R}^d)$, defined by

$$M = \int_{\mathbb{R}^d} v(x)\,\mathrm{d}x.$$

iii) Heat semigroup is given by the formula

$$\left(e^{t\Delta} f\right)(x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\,\mathrm{d}y.$$

iv) Function f is radial (radially symmetric) if and only if it is invariant under all rotations leaving the origin fixed, i.e., $f(Sx) = f(x)$ for all $S \in SO(d)$, where $SO(d)$ is the special orthogonal group of $\mathbb{R}^d$ (orthogonal matrices satisfying $\det S = 1$).

Chapter 1

Introduction

1.1 Aggregation-diffusion equations (ADE)

In this doctoral dissertation, we consider a class of partial differential equations, known as *aggregation-diffusion equations*, in the following form,

$$u_t - \Delta u^m = \nabla \cdot (u \nabla K * u), \quad t > 0, \; x \in \mathbb{R}^d, \tag{1.1}$$

where $d \geq 1$ and $m \geq 1$. Function $u = u(t, x)$ represents the density of particles, which are subject to the non-local interactions, described by the convolution with a symmetric interaction potential $K : \mathbb{R}^d \to \mathbb{R}$. Term Δu^m is the nonlinear diffusion, which expresses dissipation of the particles. Throughout this work, we will refer to these equations as ADE, and we mention that they can sometimes be found in the literature under other names, e.g., drift-diffusion equations.

These models are used to describe behavior of the particles and their pairwise interactions in many life phenomena (e.g., physical and biological), both on the microscopic and macroscopic level, such as astrophysics (mean-field models of gravitationally attracted particles [36, 37]), chemotaxis (movement of microorganisms caused by chemical stimuli [57, 81]), angiogenesis, herding of animal populations, motion of human crowds or bacteria orientation [31]. This behavior is largely driven by long-range attractive forces, due to, e.g., chemical or social interactions, and short range repulsion, mainly due to dissipation. In some cases, equations of the form (1.1) arise from the modeling particles interacting with each other, defined by the system of ODEs, as a continuum description of their behavior, when number of particles becomes large [27, 28].

In this chapter, we provide a literature review on the following topics. In Chapter 2, we describe behavior of solutions to equation (1.1) with a small diffusion coefficient. Chapter 3 is devoted to the special solutions (e.g., steady states) to problem (1.1) and Chapter 4 provides an analysis of the solutions to a chemotaxis model in the uniformly local L^p spaces. Existing results in these topics are covered in the following sections, respectively, and references therein. For a general description of ADE, we refer the reader to the book [29], and for the introduction to techniques used in their analysis, see [45].

1.2 Approximation of blow-up

In Chapter 2, we develop a systematic L^p-theory on the following ADE,

$$u_t - \varepsilon\Delta u = \nabla \cdot (u\nabla W_k * u), \quad t > 0,\ x \in \mathbb{R}^d, \tag{1.2}$$

where the effect of particle interaction is captured through the convolution with the so-called power-law interaction kernel, denoted as W_k, defined by

$$W_k(x) = \frac{|x|^k}{k}, \tag{1.3}$$

where $k \in (0,1)$. We introduce the diffusion coefficient $\varepsilon > 0$, where we are particularly interested in the behavior of solutions when ε is sufficiently small.

The main result reported in Chapter 2 states, that solutions to problem (1.2) are global-in-time and bounded in L^p-norm, but they exhibit a concentration phenomenon for a small diffusion coefficient. Namely, under suitable assumptions on the initial condition, small neighborhoods of the origin carry an uniform portion of the total mass of the corresponding solution. We can interpret this result as a qualitative description of the singularity formed in equation (1.2) without diffusion (more widely known as aggregation equation, see below), when blow-up occurs, see Remark 2.14. This result is an extension of the idea described in paper [13], where equation (1.2) with $|x|$ was studied.

For a better understanding of our result, we present an overview of the results for equation (1.2) with fixed ε and $k > -d$. For $k < 0$, solutions either exist globally or blow up in finite time, depending on the mass and concentration of the initial condition. The same behavior is present in the critical case $k = 0$, where power-law kernel is defined as $W_k(x) = \log|x|$. Then equation (1.2) exhibits global-in-time existence/blow-up dichotomy only in terms of the mass of the solution, where the essential example of such equation is the classical Keller–Segel chemotaxis model in $\mathbb{R}^2$, described in Section 1.4. For a general description of ADE with such behavior, we refer the reader to articles [16, 18, 55, 62] and book [29], with references therein.

In the case $k = 1$, we refer mainly to the recent paper [13] and references therein, where authors consider equation (1.2) with interaction kernels, which are radially symmetric and behave like $|x|$ only locally around the origin. For such kernels, they consider solutions which are globally bounded in time and prove that they exhibit mass concentration phenomenon around the origin when ε is small.

For $k > 1$, solutions to problem (1.2) are global-in-time [35], and as far we are concerned, there are no meaningful results regarding description of the behavior of solutions when diffusivity is small. We expect that results obtained in this dissertation can be extended to range $k \in (1, 2)$ (see Remark 2.21) and kernels behaving like $|x|^k$ only locally around the origin.

On the other hand, for the aggregation equations, i.e., ADE without dissipation,

$$u_t = \nabla \cdot (u \nabla K * u), \quad t > 0, \ x \in \mathbb{R}^d, \tag{1.4}$$

we refer to the series of works by Bertozzi et al. [8, 9, 10, 11]. In general, regularity of kernel is essential for the global-in-time existence. If K is at least a C^2-function, which covers the case $k \geq 2$, then there is no finite time blow-up. Nevertheless, for some initial conditions one can expect blow-up in infinite time [8]. For $2 - d \leq k < 2$ it is known, that solution cease to exist in finite time at least for bounded, compactly supported initial data [11]. We note, that solution in the moment of blow-up is a Dirac delta function (see, e.g., [9, 25]), and this is consistent with our result (see Remark 2.14).

Proof of concentration is based on the technique described in the work [13], but these results where in general inspired by the methodology arising from considerations in [17, 21, 54]. In these results, one can observe techniques considering moment and *a priori* estimates, along with averaging the solution in time through integration.

In the end of Chapter 2, we consider potential convergence of solutions with fixed $\varepsilon > 0$ to steady state when $t \to +\infty$, by analysing suitable moment of the solution. For the references on the existence of stationary solutions, we refer to Section 1.3. Here we recall only paper [20], in which authors proved convergence to a steady state when $k \geq 2$, studying the dissipation of the Wasserstein distance between the solution and the steady state. We note that convexity of the kernel was crucial in their analysis.

Chapter 2 is constructed as follows. In Sections 2.1-2.3 we construct local and global-in-time solutions to problem (1.2), Section 2.4 is devoted to the concentration phenomenon for small ε and in Section 2.5 we describe behavior of solutions for large $t > 0$.

1.3 Special solutions to ADE

Chapter 3 is dedicated to examining an alternative method for demonstrating the existence of steady states for the problem

$$u_t - \Delta u^m = \nabla \cdot (u \nabla W_k * u), \quad t > 0, \ x \in \mathbb{R}^d, \tag{1.5}$$

where W_k defined in (1.3) with $k > 0$, as well as deriving explicit formula for some of these solutions. Furthermore, we discuss possible existence of self-similar sign-changing solutions to this problem in one-dimension, based on the solution to the Burgers' equation.

Existence of steady states to problem (1.5) is well-known for a wide range of parameters $d \geq 1$, $k > -d$ and $m \geq 1$ [4, 30, 34, 35, 26, 67, 66], where differences in reasoning depend significantly on either the parameter m or the properties of the kernel W_k (e.g., convexity, singularity). As to the properties of these solutions, they are non-negative, radially decreasing and compactly supported, if $m > 1$. Uniqueness for fixed mass is known only in several cases [23, 58, 65].

The main idea behind these results comes from the observation, that stationary solutions to problem (1.5) are equivalent to the minimizers of the following free energy functional

$$E[u] = \frac{1}{m-1} \int_{\mathbb{R}^d} u^m \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d} u(W_k * u) \,\mathrm{d}x,$$

where for $m = 1$ we replace u^m with $u \log u$. This functional can be interpreted as a gradient flow with respect to the Wasserstein metric on the space of probability measures with finite second moment. Comparison of the scaling of the two terms in the integral, depending on m and k, suggests considering three distinct regimes for the behavior of the solutions, separated by the critical diffusion exponent $m_c = 1 - k/d$. Existence of minimizers is considered only when $m > m_c$ in the so-called *diffusion dominated regime*, where existence of global-in-time solutions is well-known [5, 6, 18, 75, 82]. One would expect convergence of these solutions to the steady states, but this is known only in a few cases [20, 33, 41].

In Section 3.2, we consider a stationary version of equation (1.5),

$$\Delta u^m + \nabla \cdot (u \nabla (W_k * u)) = 0, \tag{1.6}$$

which, under certain assumptions, can be transformed into a fixed-point equation, being a novel approach compared to the previously applied method. In the particular case $m = 1$ and $k = 2$, we prove that either Banach or Schauder fixed-point theorem cannot be applied in this setting, but the possible use of other methods to solve this problem remains of interest to us.

During our study on this topic, we were able to derive explicit formulae in several cases. Moreover, equation (1.6), with some additional assumptions on u, is equivalent to the nonlinear Fredholm integral equation of the second kind, for which direct methods of solving are well known (see, e.g., [78]). Explicit solutions can be particularly useful for verifying the accuracy of numerical methods used in solving problem (1.5) (see, e.g., [24]). In the Appendix A, we attach the code for numerical approximation of some of these steady states.

At the end of Chapter 3, we derive a sign-changing solution, which follows from the equivalence of the viscous Burgers' equation and problem (1.5) with parameters $d = m = k = 1$. Sign-changing solutions and their properties have been studied in certain problems (see, e.g., [53]). The equations we consider primarily model physical phenomena, where the assumption of non-negativity is relevant, thus the existence of partially negative solutions is often deemed non-physical. Nevertheless, they can still be of interest from a mathematical perspective, e.g., in the study of the stability of steady states, where zero mass function can be considered as perturbation [56].

Considering problem (1.5) with $m = 1$, we suspect that one can generalize the sign-changing solution obtained in Section 3.3 to full range $k > 0$ at least for $d = 1$, analogously as steady states for this problem display a continuous dependence on the parameter k. For this reason, it can be crucial to understand the properties of this solution, i.e., scaling and derivation from Burgers' equation, along with the Hopf-Cole transform [39, 49].

1.4 Chemotaxis model in the uniformly local L^p spaces

In this section, we present a literature review from the paper [42] concerning a chemotaxis model in the uniformly local L^p spaces, of which the author of this dissertation is a co-author. In Chapter 4 one can find an overview of the results obtained in this paper and detailed description of the author's contribution to this publication (see Section 4.2).

There are several mathematical works on the chemotaxis model introduced by Keller and Segel [57]. Here, we refer the reader only to the monographs [12, 81] and the reviews [7, 48, 51] for a discussion of those mathematical results as well as for additional references.

We consider the following minimal parabolic-elliptic Keller-Segel system

$$u_t - \Delta u + \nabla \cdot (u \nabla \psi) = 0, \quad -\Delta \psi + \psi = u, \quad t > 0, \ x \in \mathbb{R}^d, \tag{1.7}$$

where $u = u(t,x)$ denotes the density of cells and $\psi = \psi(t,x)$ is a concentration of the chemoattractant – a substance that is responsible for the attraction of cells. In these equations all constant parameters are equal to one for simplicity of the exposition. System (1.7) was already studied in the whole space e.g., in the papers [14, 15, 40, 54, 55, 60, 61, 72], where several results either on a blow-up or on a large time behavior of solutions have been obtained.

For each constant $A \in \mathbb{R}$, the couple $(u, \psi) = (A, A)$ is a stationary solution to system (1.7) and, since the domain is unbounded, it does not belong to any Lebesgue L^p space with $p \in [1, \infty)$. Thus, in Theorem 4.2, we develop a mathematical theory concerning local-in-time solutions to the initial value problem for system (1.7) in the uniformly local Lebesgue spaces $L^p_{\text{uloc}}(\mathbb{R}^d)$.

Then, we consider a constant stationary solution $(u, \psi) = (A, A)$ with $A \in [0, 1)$ and we show in Theorem 4.4 that a small L^p-perturbation of such an initial datum gives a global-in-time solution which converges toward (A, A) as $t \to \infty$. On the other hand, we prove in Theorem 4.5 that the constant solution is unstable in the Lyapunov sense if $A > 1$.

A stability of constant solutions to chemotaxis models has been already studied in bounded domains. For example, the paper [46] describes dynamics near an unstable constant solution to the classical parabolic-parabolic Keller-Segel model in a bounded domain and obtained results are interpreted as an early pattern formation.

Another work [80] is devoted to the system

$$u_t - \Delta u + \nabla \cdot (u \nabla \psi) = 0, \quad -\Delta \psi + \mu = u, \quad \mu = \frac{1}{|B_R|} \int_{B_R} u \, \mathrm{d}x,$$

in the ball of radius $R > 0$ with Neumann boundary condition. Here, constants are also stationary solutions and it is shown that there exists a critical number m_c such that at mass levels above m_c the constant steady states are extremely unstable and blow-up can occur. On the other hand, for $m < m_c$, there exist infinitely many radial solutions with a mass equal to m.

Since our publication [42] on this topic, four publications citing it have been released. Worth mentioning is the work [76], where more regular solutions to problem (1.7) have been constructed in a subspace of space $L^p_{\text{uloc}}(\mathbb{R}^d)$.

Chapter 2

Approximation of blow-up

2.1 Statement of the problem

The goal of this chapter is to study existence and properties of solutions to the following initial value problem

$$\begin{cases} u_t - \varepsilon \Delta u = \nabla \cdot (u \nabla W_k * u), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

where $d \geq 1$, $\varepsilon > 0$ is a constant diffusion coefficient and W_k is the power-law interaction kernel defined by

$$W_k(x) = \frac{|x|^k}{k}, \quad x \in \mathbb{R}^d, \quad k \in (0,1).$$

The initial condition satisfies $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $p > 1$.

2.2 Local-in-time solutions

In this section, for the sake of clarity, we fix coefficient $\varepsilon = 1$, thus we consider the following simplified problem

$$\begin{cases} u_t - \Delta u = \nabla \cdot (u \nabla W_k * u), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.2}$$

Notice that if function u is a solution to problem (2.2) then function $u_\varepsilon(t,x) = \varepsilon u(\varepsilon t, x)$ is a solution to problem (2.1).

We begin with a theorem on the local-in-time *mild* solutions to problem (2.2), i.e., solutions to the integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t \nabla e^{(t-s)\Delta} \cdot (u(s) \nabla W_k * u(s)) \, \mathrm{d}s. \tag{2.3}$$

Notice that heat semigroup commutes with the divergence operator $\nabla\cdot$, which can be justified by applying the Fourier transform. This relation is usually not valid in a bounded domain.

Proposition 2.1. *Let $d \geq 1$ and $k \in (0,1)$. For every $p \in [2d/(2d+k-1), +\infty]$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ there exists $T > 0$ and a unique mild solution $u \in C\big([0,T], L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\big)$ of equation (2.3). Moreover, this solution has the following properties.*

(i) The solution u is non-negative for a non-negative initial datum u_0.

(ii) The solution u is radially symmetric for a radial initial datum u_0.

This proposition is proved by the contraction mapping theorem (see Proposition 2.5) applied to an integral equation (2.3) and requires some results which we are going to gather and prove below. Moreover, solution obtained by this proposition is in fact sufficiently regular to be a *classical* solution to problem (2.2), see Remark 2.6.

We begin by recalling well-known properties of the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ acting on the L^p-spaces.

Lemma 2.2 (Heat semigroup estimates)**.** *Let $d \geq 1$, $1 \leq q \leq p \leq +\infty$, $v \in L^q(\mathbb{R}^d)$ and $C = C(d,p,q)$, then*

$$\|e^{t\Delta} v\|_p \leq C t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|v\|_q \quad \text{and} \quad \|\nabla e^{t\Delta} v\|_p \leq C t^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \|v\|_q$$

for all $t > 0$.

Next we recall some properties of the convolution with a singular function ∇W_k.

Lemma 2.3 (See e.g., [50, Theorem 4.5.3])**.** *Let $d \geq 1$, $k \in (0,1)$ and $1 < p < q < +\infty$. Then there exists a number $C_{k,p} > 0$ such that*

$$\|\nabla W_k * v\|_q \leq C_{k,p} \|v\|_p, \quad \text{where} \quad \frac{1}{p} + \frac{1-k}{d} = 1 + \frac{1}{q}$$

for all $v \in L^p(\mathbb{R}^d)$.

Lemma 2.4 (See e.g., [50, Lemma 4.5.4])**.** *Let $d \geq 1$ and $k \in (0,1)$. Then there exists a number $C_k > 0$ such that*

$$\|\nabla W_k * v\|_\infty \leq C_k \|v\|_1^{1-\frac{1-k}{d}} \|v\|_\infty^{\frac{1-k}{d}}$$

for all $v \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

We obtain a solution to integral equation (2.3) from the Banach fixed-point theorem formulated in the following way.

Proposition 2.5 (See e.g., [63, Theorem 13.2]). *Let E be a Banach space and let $B = B(\cdot,\cdot) : E \times E \to E$ be a bounded bilinear form with an estimate*

$$\|B(u,v)\|_E \le C_B \|u\|_E \|v\|_E$$

for some $C_B > 0$ independent of $u, v \in E$. Assume that $\delta \in (0, 1/(4C_B))$. If $\|y_0\|_E \le \delta$, then equation

$$u = y_0 + B(u,u)$$

has a solution satisfying $\|u\|_E \le 2\delta$. This solution is unique in the set $\{u \in E : \|u\|_E \le 2\delta\}$ and stable in the following sense: if $y_1, y_2 \in E$ satisfy $\|y_1\|_E \le \delta$ and $\|y_2\|_E \le \delta$, then for the corresponding solutions $u_1, u_2 \in E$ we have

$$\|u_1 - u_2\|_E \le C \|y_1 - y_2\|_E,$$

where $C > 0$ is independent of u_1 and u_2.

It is easy to see that within these assumptions, $F(u) = y_0 + B(u,u)$ is a contraction operator on the set $\{u \in X : \|u\|_X \le 2\delta\}$.

Proof of Proposition 2.1. For $T > 0$ we introduce $X = C\big([0,T], L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\big)$ which is a Banach space with the norm $\|u\|_X = \sup_{t\in[0,T]}(\|u(t)\|_1 + \|u(t)\|_p)$. In order to apply Proposition 2.5, it suffices to estimate the bilinear form

$$B(u,v)(t) = \int_0^t \nabla e^{(t-s)\Delta} \cdot (u(s)\nabla W_k * v(s))\, \mathrm{d}s.$$

Let $u, v \in X$. Using the heat semigroup estimates from Lemma 2.2, Hölder inequality and Lemma 2.3 we have

$$\begin{aligned} \|B(u,v)(t)\|_1 &\le \int_0^t \|\nabla e^{(t-s)\Delta} \cdot (u(s)\nabla W_k * v(s))\|_1\, \mathrm{d}s \\ &\le C \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\nabla W_k * v(s)\|_1\, \mathrm{d}s \\ &\le C \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{q_1} \|\nabla W_k * v(s)\|_{q_2}\, \mathrm{d}s \\ &\le C \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{q_1} \|v(s)\|_{q_3}\, \mathrm{d}s, \end{aligned} \tag{2.4}$$

where $1/q_1 + 1/q_2 = 1$ with $q_1, q_2 \in [1, +\infty]$ and

$$\frac{1}{q_3} + \frac{1-k}{d} = 1 + \frac{1}{q_2}, \quad 1 < q_3 < q_2 < +\infty. \tag{2.5}$$

Moreover, by the standard interpolation of norms,

$$\|u(s)\|_{q_1} \le C\|u(s)\|_1^{\alpha}\|u(s)\|_p^{1-\alpha} \le C\left(\|u(s)\|_1 + \|u(s)\|_p\right),$$

and analogously for $\|v(s)\|_{q_3}$, we have conditions

$$\frac{1}{q_1} = \alpha + \frac{1}{p}(1-\alpha),\ 0 \le \alpha \le 1 \quad \text{and} \quad \frac{1}{q_3} = \beta + \frac{1}{p}(1-\beta),\ 0 \le \beta \le 1. \tag{2.6}$$

Combining all the identities we obtain a function

$$p(\alpha, \beta) = \frac{d(2-\alpha-\beta)}{d(2-\alpha-\beta)+k-1}$$

satisfying $p(\alpha, \beta) > 0$ for $\alpha+\beta < 2-(1-k)/d$. Moreover, it is strictly increasing in this domain up to $+\infty$ for both arguments and attains its minimum $p_1 = 2d/(2d+k-1)$ at $\alpha = \beta = 0$. Thus, for any $p^* \in [p_1, +\infty)$ there exists a solution (α, β) to equation $p(\alpha, \beta) = p^*$ such that

$$\alpha + \beta = 2 - (1-k)p^*/(d(p^*-1)). \tag{2.7}$$

Exponents q_i, $i \in \{1, 2, 3\}$, corresponding to p^* satisfy all the assumptions if

$$\beta > 1 - \frac{(1-k)p^*}{d(p^*-1)}, \tag{2.8}$$

where for such β we can always find α satisfying equation (2.7).

Putting $p = +\infty$ in conditions (2.6) one can choose adequate exponents q_i satisfying all the assumptions, as long as $\beta > 1 - (1-k)/d$. Taking supremum of both sides of inequality (2.4) and integrating we obtain

$$\sup_{t\in[0,T]} \|B(u,v)(t)\|_1 \le CT^{\frac{1}{2}}\|u\|_X\|v\|_X,$$

where $C > 0$ is a constant independent of T.

Now we estimate p-norm of the bilinear form with $p < +\infty$. In the case $d = 1$ we have

$$\|B(u,v)(t)\|_p \le C\int_0^t (t-s)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}\|u(s)W_k' * v(s)\|_1\,\mathrm{d}s \tag{2.9}$$

and estimates for the nonlinear term are as previously. Integral in time is convergent for all $p > 0$, therefore we have

$$\sup_{t\in[0,T]} \|B(u,v)(t)\|_p \le CT^{\frac{1}{2p}} \|u\|_X \|v\|_X.$$

We assume $d \ge 2$ and proceed with a more general approach,

$$\|B(u,v)(t)\|_p \le C \int_0^t (t-s)^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \|u(s)\|_{q_1} \|v(s)\|_{q_3}\, \mathrm{d}s, \tag{2.10}$$

where

$$1 \le q \le p \le +\infty, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \quad \text{with} \quad q_1, q_2 \in [1, +\infty],$$

along with condition (2.5). By interpolation of norms we obtain the same relation as described in expression (2.6) and integral in time is convergent when $q > pd/(p+d)$. Notice that taking $q = 1$ leads to convergence only for $p < d/(d-1)$ thus for $p \in [p_1, d/(d-1))$ we have

$$\sup_{t\in[0,T]} \|B(u,v)(t)\|_p \le CT^{-\frac{d}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}} \|u\|_X \|v\|_X.$$

Now assume $q > 1$. Once again, combining all the identities, we obtain the following function

$$p(\alpha,\beta;q) = \frac{dq(2-\alpha-\beta)}{dq(1-\alpha-\beta)+d+q(k-1)},$$

where $q > 1$ is a parameter. We consider function p on the non-empty set

$$D = \left\{(\alpha,\beta) \in [0,1]^2 : \alpha+\beta < 1 + 1/q - (1-k)/d\right\}.$$

Then for all $(\alpha,\beta) \in D$ we have $p(\alpha,\beta;q) > 0$, p is strictly increasing function for each variable, it is unbounded from above and has infimum corresponding to $p(0,0,1) = p_1$. Thus, we conclude that $p(\alpha,\beta;q) \in (p_1, +\infty)$.

Let $p^* > p_1$, $q_{\max} = p^*$ and $q_{\min} = \max\{1, p^*d/(p^*+d)\}$. If $p^* \ge p_2$, where $p_2 = d/(d+k-1)$, then p^* satisfies inequality $p(0,0;q_{\max}) \le p^*$. Thus, for all $q \in (q_{\min}, q_{\max})$ equation $p(\alpha,\beta;q) = p^*$ has a solution $(\alpha,\beta) \in D$.

For $p^* \in (p_1, p_2)$ we have $p(0,0;q_{\max}) > p^*$. There exists p^* and $q \in (q_{\min}, q_{\max})$ such that equation $p(\alpha,\beta;q) = p^*$ has no solutions. Let $q_m \in \mathbb{R}$ be a solution to equation $p(0,0,q_m) = p^*$. Then for all $q \in (q_{\min}, q_m)$ equation $p(\alpha,\beta;q) = p^*$ has a solution $(\alpha,\beta) \in D$.

Fix $p^* > p_1$, then coefficients α, β satisfy the following equation

$$\alpha + \beta = \frac{(k-1)\,p^*}{d\,(p^*-1)} + \frac{p^* q + p^* - 2q}{(p^*-1)\,q}$$

and exponents q_i are well-defined if condition (2.8) is satisfied. Intersection of this conditions is non-empty for some $q \in (q_{\min}, q_{\max})$, which concludes the argument. After taking supremum and integration in inequality (2.10), we obtain

$$\sup_{t\in[0,T]} \|B(u,v)(t)\|_p \le CT^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{1}{2}} \|u\|_X \|v\|_X.$$

When $p = +\infty$, we proceed analogously using Lemma 2.4,

$$\|B(u,v)(t)\|_\infty \le C \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_\infty \|v(s)\|_1^{1-\frac{1-k}{d}} \|v(s)\|_\infty^{\frac{1-k}{d}} \, \mathrm{d}s, \tag{2.11}$$

where by standard interpolation and integration we obtain

$$\sup_{t\in[0,T]} \|B(u,v)(t)\|_\infty \le CT^{\frac{1}{2}} \|u\|_X \|v\|_X.$$

By Proposition 2.5 we obtain a local-in-time solution to the integral equation (2.3) in the space X for a sufficiently small $T > 0$. Uniqueness of the solution can be shown in a standard way by estimating the difference

$$\|u_1(t) - u_2(t)\| = \|u_1(t) - u_2(t)\|_1 + \|u_1(t) - u_2(t)\|_p$$

for two solutions u_1, u_2 with the same initial condition u_0. For all $p \in [2d/(2d+k-1), +\infty]$, using estimates (2.4) and (2.9)-(2.11), we obtain inequality

$$\|u_1(t) - u_2(t)\| \le C \int_0^t (t-s)^{-\gamma} \|u_1(s) - u_2(s)\| \, \mathrm{d}s,$$

where $C = C(\|u_1\|_X, \|u_2\|_X) > 0$ is independent of $t \in (0,1]$ and $\gamma \in [0,1)$. Notice that this is a Volterra type inequality (see, e.g., [38, Lemma 1.2.9, p.19]), thus we can conclude that $\|u_1(t) - u_2(t)\| \le 0$ for all $t \in (0,1]$ and in consequence, uniqueness of the solution in the space X for sufficiently small $T > 0$.

We show non-negativity of the solution by a standard approach. For a non-negative initial datum $u_{0,n} \in C_c^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}^+$, one can obtain a non-negative sufficiently regular solution u_n (see, e.g., [44, 64]). By the density, we approximate a non-negative initial condition $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with a sequence $\{u_{0,n}\}_{n\in\mathbb{N}^+}$. Using Proposition 2.5, we conclude that corresponding solution u is also approximated by a sequence $\{u_n\}_{n\in\mathbb{N}^+}$, thus non-negative.

Rotational invariance of the Laplace operator is well-known. Let $v(x) = u(Sx)$, where $S \in SO(d)$. We note that $W_k * v(x) = W_k * u(Sx)$, thus we conclude that nonlinear term $\nabla \cdot (u \nabla W_k * u)$ is also rotation invariant. By the uniqueness, we obtain that solution is radially symmetric if the initial condition is so. □

Remark 2.6. Let $X = C\big([0,T], L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\big)$ with $p \in [d/(d+k-1), +\infty)$. If $u \in X$, then $u \nabla W_k * u \in X$ and this is sufficient for a *mild* solution u to problem (2.3), to be a *classical* solution, i.e.,

$$u \in X \cap C\big((0,T], W^{2,p}(\mathbb{R}^d)\big) \cap C^1\big((0,T], L^p(\mathbb{R}^d)\big),$$

where $W^{2,p}(\mathbb{R}^d)$ is the Sobolev space. For more details on this standard result see, e.g., [32, 68].

2.3 Global-in-time solutions

In this section, we continue with a result on the existence of the global-in-time solutions to problem (2.1) with $\varepsilon > 0$. Local-in-time solution u_ε to this problem is obtained from Proposition 2.1 and suitable substitution, and is well-defined for all $t \in [0,T]$ with some $T > 0$.

Theorem 2.7. *Let $d \geq 1$, $k \in (0,1)$ and $p \in [p_k, +\infty]$, where*

$$p_k = \begin{cases} \max\left\{2, \frac{1}{k}\right\}, & d = 1, \\ 2, & d \geq 2, \end{cases} \tag{2.12}$$

and let $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ be a non-negative initial condition. Then, for each $T > 0$, there exists a unique, non-negative, global-in-time solution $u_\varepsilon \in C\big([0,T], L^1(\mathbb{R}) \cap L^p(\mathbb{R})\big)$ to problem (2.1), with $\varepsilon > 0$ and initial condition u_0. Moreover, for every $p \in [p_k, +\infty)$, $\sup_{t \geq 0} \|u_\varepsilon(t)\|_p < +\infty$.

Proof of this theorem is based on Proposition 2.1 and standard *a priori* estimates, see Lemma 2.9. Similar reasoning can be found in paper [18]. Before that, we show some estimates for the solutions to problem (2.1).

Lemma 2.8 (Mass conservation property)**.** *Let $d \geq 1$, $k \in (0,1)$ and $p \in [2d/(2d+k-1), +\infty]$. Let u be a local-in-time solution to problem (2.2) corresponding to the initial condition $u_0 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Then for all $t \in [0,T]$,*

$$\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x, \tag{2.13}$$

where $T > 0$ follows from Proposition 2.1.

Proof. Using integral formulation (2.3) of the solution and properties of the heat semigroup we have

$$\int_{\mathbb{R}^d} u(t)\,\mathrm{d}x = \int_{\mathbb{R}^d} e^{t\Delta}u_0\,\mathrm{d}x + \int_{\mathbb{R}^d}\int_0^t \nabla e^{(t-s)\Delta}\cdot\left(u(s)\nabla W_k * u(s)\right)\,\mathrm{d}s\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0\,\mathrm{d}x,$$

where space-time integral is bounded by estimate (2.4) and in consequence equal to zero by Fubini's theorem. □

Notice that for a non-negative initial condition, mass conservation property can be written as $\|u(t)\|_1 = \|u_0\|_1 = M$ where

$$M = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x.$$

Lemma 2.9. *Let $d \geq 1$, $k \in (0,1)$ and $p \geq p_k$, where p_k is defined in (2.12). Let u_ε be a non-negative, local-in-time solution to problem (2.1) with $\varepsilon > 0$ and initial condition $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. Then for all $t \in [0,T]$,*

$$\|u_\varepsilon(t)\|_p \leq \max\left\{\|u_0\|_p, C(k,p)M^{1+\frac{1}{\eta}}\varepsilon^{-\frac{1}{\eta}}\right\}, \tag{2.14}$$

where $C(k,p) > 0$, $\eta = kp/(d(p-1))$ and $T > 0$ follows from Proposition 2.1.

We note that $p_k > 2d/(2d+k-1)$ for all $d \geq 1$ and $k \in (0,1)$ thus for any $p \geq p_k$ there exists a local-in-time solution, according to Proposition 2.1. Moreover, recalling Remark 2.6, $p_k \geq d/(d+k-1)$. Thus, integration by parts and time differentiation of integrals involving u in the following results are fully justified. In the proof, we simplify the notation from u_ε to u for the sake of clarity.

Proof. Assume $p \geq 2$. By the energy method, integrating equation (2.1) by parts, using Hölder inequality and Lemma 2.3, we obtain

$$\begin{aligned}
\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p &= -\varepsilon\int_{\mathbb{R}^d}|\nabla u|^2 u^{p-2}\,\mathrm{d}x - \int_{\mathbb{R}^d} u^{p-1}\nabla u\cdot(\nabla W_k * u)\,\mathrm{d}x \\
&= -\varepsilon\frac{4}{p^2}\|\nabla u^{p/2}\|_2^2 - \frac{2}{p}\int_{\mathbb{R}^d} u^{p/2}\nabla u^{p/2}\cdot(\nabla W_k * u)\,\mathrm{d}x \\
&\leq -\varepsilon\frac{4}{p^2}\|\nabla u^{p/2}\|_2^2 + C_{k,p}\frac{2}{p}\|u^{p/2}\|_s\|\nabla u^{p/2}\|_2\|u\|_q \\
&\leq \frac{4}{p^2}\|\nabla u^{p/2}\|_2\left(-\varepsilon\|\nabla u^{p/2}\|_2 + C_{k,p}\frac{p}{2}\|u^{p/2}\|_s\|u\|_q\right),
\end{aligned} \tag{2.15}$$

where

$$\frac{1}{s} + \frac{1}{q} + \frac{1-k}{d} = \frac{3}{2}. \tag{2.16}$$

We estimate each term in inequality (2.15). From the Gagliardo-Nirenberg inequality and mass conservation property (2.13) we have

$$\begin{aligned}\|u^{p/2}\|_s &\le C\|u^{p/2}\|_2^{\alpha}\|u^{p/2}\|_1^{1-\alpha} = C\|u\|_p^{\alpha p/2}\|u\|_{\frac{p}{2}}^{(1-\alpha)p/2}\\ &\le C\|u\|_p^{(\alpha+\beta-\alpha\beta)p/2}M^{(1-\alpha)(1-\beta)p/2}\end{aligned} \tag{2.17}$$

where

$$\frac{1}{s} = 1-\frac{1}{2}\alpha \quad \text{and} \quad \frac{2}{p} = 1-\left(1-\frac{1}{p}\right)\beta, \quad \alpha,\beta\in[0,1]. \tag{2.18}$$

In addition

$$\|u\|_q \le C\|u\|_p^{\gamma}M^{1-\gamma}, \quad \text{where} \quad \frac{1}{q} = 1-\left(1-\frac{1}{p}\right)\gamma, \quad \gamma\in[0,1]. \tag{2.19}$$

For the derivative we have

$$\begin{aligned}\|u\|_p^{p/2} = \|u^{p/2}\|_2 &\le C\|\nabla u^{p/2}\|_2^{d/(d+2)}\|u^{p/2}\|_1^{2/(d+2)}\\ &\le C\|\nabla u^{p/2}\|_2^{d/(d+2)}\|u\|_p^{\beta p/(d+2)}M^{(1-\beta)p/(d+2)}\end{aligned}$$

and therefore

$$\|\nabla u^{p/2}\|_2 \ge CM^{(\beta-1)p/d}\|u\|_p^{(d/2+1-\beta)p/d}. \tag{2.20}$$

Using estimates (2.17), (2.19) and (2.20) we get

$$\begin{aligned}\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p \le &\frac{4}{p^2}\|\nabla u^{p/2}\|_2 \times \Big(-\varepsilon C_1 M^{(\beta-1)p/d}\|u\|_p^{(d/2+1-\beta)p/d}\\ &+ C_2M^{(1-\alpha)(1-\beta)p/2+1-\gamma}\|u\|_p^{(\alpha+\beta-\alpha\beta)p/2+\gamma}\Big)\\ =&C\|\nabla u^{p/2}\|_2M^{(1-\alpha)(1-\beta)p/2+1-\gamma}\|u\|_p^{(\alpha+\beta-\alpha\beta)p/2+\gamma}\\ &\times\big(-\varepsilon C(k,p)M^{-\eta-1}\|u\|_p^{\eta}+1\big),\end{aligned} \tag{2.21}$$

where $\eta = (d/2+1-\beta)p/d-(\alpha+\beta-\alpha\beta)p/2-\gamma = kp/(d(p-1))$. We substitute equalities (2.18) and (2.19) into equation (2.16), and using conditions $\alpha,\gamma\le 1$ we obtain

$$\frac{1}{2}+\frac{1-k}{d} = \frac{1}{2}\alpha+\left(1-\frac{1}{p}\right)\gamma \le \frac{1}{2}+\left(1-\frac{1}{p}\right),$$

thus in consequence $p\ge d/(d+k-1)$. Notice that this restriction is significant only when $d=1$.

Let us show that inequality (2.21) implies the estimate

$$\|u(t)\|_p \le \max\left\{\|u_0\|_p, C(k,p)M^{1+\frac{1}{\eta}}\varepsilon^{-\frac{1}{\eta}}\right\} = U$$

for all $t > 0$. The following reasoning is analogous to the one from the paper [13, Lemma 4.1] and we present it for the sake of completeness. For $\delta > 0$ consider the set

$$A_\delta = \{t \geq 0 : \|u(t)\|_p \leq U + \delta\}.$$

Obviously, $0 \in A_\delta$ and the time continuity of u in $L^p(\mathbb{R})$ ensures that

$$\tau_\delta = \sup\{t \geq 0 : [0,t] \subset A_\delta\} \in (0, +\infty].$$

Assume now for contradiction that $\tau_\delta < +\infty$. On the one hand, the definition of τ_δ implies that

$$\|u(\tau_\delta)\|_p^p = (U+\delta)^p \geq \|u(t)\|_p^p \quad \text{for all } t \in (0, \tau_\delta), \tag{2.22}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\tau_\delta)\|_p^p \geq 0. \tag{2.23}$$

On the other hand, from inequalities (2.21) and (2.22), since $\eta > 0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\tau_\delta)\|_p^p &\leq C(k,p,M)\|u(\tau_\delta)\|_p^\zeta \|\nabla u^{p/2}(\tau_\delta)\|_2 \left(-\|u(\tau_\delta)\|_p^\eta U^{-\eta} + 1\right)\\ &= C(k,p,M)\|u(\tau_\delta)\|_p^\zeta \|\nabla u^{p/2}(\tau_\delta)\|_2 \left(-\left(1+\frac{\delta}{U}\right)^\eta + 1\right) < 0,\end{aligned}$$

where $\zeta = \zeta(k,p) \geq 0$ and this contradicts (2.23). Consequently, $\tau_\delta = +\infty$ and $A_\delta = [0, +\infty)$ for all $\delta > 0$. Letting $\delta \to 0$ we complete the proof. □

Remark 2.10. For fixed $\bar{p} \geq p_k$, such that $u_0 \in L^1(\mathbb{R}^d) \cap L^{\bar{p}}(\mathbb{R}^d)$, one can interpolate L^p-norms of u_ε for all $p \in (1, \bar{p})$ using Littlewood's inequality. Indeed, by estimate (2.14),

$$\begin{aligned}\|u_\varepsilon(t)\|_p &\leq M^{\frac{\bar{p}-p}{\bar{p}p-p}} \|u_\varepsilon(t)\|_{\bar{p}}^{\frac{\bar{p}p-\bar{p}}{\bar{p}p-p}}\\ &\leq M^{\frac{\bar{p}-p}{\bar{p}p-p}} \left(\max\left\{\|u_0\|_{\bar{p}}, C(k,\bar{p})M^{1+\frac{1}{\bar{\eta}}}\varepsilon^{-\frac{1}{\bar{\eta}}}\right\}\right)^{\frac{\bar{p}p-\bar{p}}{\bar{p}p-p}}\\ &\leq \max\left\{M^{\frac{\bar{p}-p}{\bar{p}p-p}}\|u_0\|_{\bar{p}}^{\frac{\bar{p}p-\bar{p}}{\bar{p}p-p}}, C(k,\bar{p})M^{1+\frac{d(p-1)}{kp}}\varepsilon^{-\frac{d(p-1)}{kp}}\right\}\\ &\leq \max\left\{M, \|u_0\|_{\bar{p}}, C(k,\bar{p})M^{1+\frac{d(p-1)}{kp}}\varepsilon^{-\frac{d(p-1)}{kp}}\right\}\end{aligned}$$

where $\bar{\eta}$ corresponds to $\bar{p}$. Notice that exponent for ε is consistent with the general result.

Proof of Theorem 2.7. Let $p < +\infty$. By the standard continuation argument, using estimates from the proof of Proposition 2.1, mass conservation property (2.13) and Lemma 2.9, we conclude that for fixed $\varepsilon > 0$, $u_\varepsilon \in C\big([0,+\infty), L^1(\mathbb{R}) \cap L^p(\mathbb{R})\big)$.

For $p = +\infty$, $u_\varepsilon \in C\big([0,+\infty), L^1(\mathbb{R}) \cap L^q(\mathbb{R})\big)$ for every $q \in [p_k, +\infty)$, by the same reasoning. By analogous calculations as in the proof of Proposition 2.1,

$$\|u_\varepsilon(t)\|_\infty \le C_1 t^{-\frac{d}{2q}} \|u_0\|_q + C_2 t^\alpha \sup_{s\in(0,t)} \left(\|u_\varepsilon(s)\|_1 + \|u_\varepsilon(s)\|_q\right)^2,$$

for some $\alpha > 0$, where numbers $C_1, C_2 > 0$ are independent of $t > 0$. We conclude that $u_\varepsilon \in C\big([0,T], L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\big)$ for every $T > 0$. □

Remark 2.11. In the proof of Theorem 2.7, we do not infer that $\|u_\varepsilon(t)\|_\infty$ is uniformly bounded in time. To improve behavior of constant $C(k,p)$ for the limit case $p = +\infty$, one would consider applying the Moser–Alikakos method of estimating the L^p-norms with $p = 2n$ recursively (see, e.g., [1] and [52, Lemma 3.1]). We note that our results for $p = +\infty$ are sufficient for the later part of this work.

2.4 Concentration around the origin

Aim of this section is to study behavior at the origin of the family of radial solutions to problem (2.1) when $\varepsilon > 0$ is small. Throughout this section we assume that the radial initial condition u_0 satisfies

$$u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \quad u_0(x) \ge 0,\ x \in \mathbb{R}^d \quad \text{and} \quad M = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x > 0, \tag{2.24}$$

and our main goal is to prove the following theorem.

Theorem 2.12 (Concentration of mass at the origin)**.** *Let $d \ge 1$ and $k \in (0,1)$. Let u_ε be a radial, non-negative, global-in-time solution to problem (2.1) with $\varepsilon > 0$ and initial condition u_0 satisfying assumptions (2.24). Moreover, suppose that there exists $\lambda > 0$ such that*

$$\int_{\mathbb{R}^d} \left(\min\{|x|, \lambda\}\right)^{2-k} u_0(x)\,\mathrm{d}x < \mu_k M \lambda^{2-k}, \tag{2.25}$$

where $\mu_k > 0$ is a constant dependent only on k, see definition (2.43), below. Then for some explicit numbers $T_* > 0$, $C_* > 0$, $\varepsilon_* > 0$ and $\nu > 0$, dependent only on d, k, u_0 and λ, the following inequality holds

$$\int_0^{T_*} \int_{B_{(\nu\varepsilon)^{1/k}}} u_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \ge C_* \tag{2.26}$$

for all $\varepsilon \in (0, \varepsilon_*)$.

We note that in this theorem, for any $\varepsilon > 0$, u_ε is a solution obtained for the same initial condition u_0. Moreover, one can derive analogous estimates for the L^p-norms of the solutions on ε-small balls.

Corollary 2.13. *Let $p \in [1, +\infty]$. Under the assumptions of Theorem 2.12 and using the same notation, solution u_ε to problem (2.1) satisfies*

$$\int_0^{T_*} \left(\int_{B_{(\nu\varepsilon)^{1/k}}} u_\varepsilon(t,x)^p \,\mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}t \geq C_{**}(p) \varepsilon^{-\frac{d(p-1)}{kp}} \tag{2.27}$$

for $p < +\infty$ and

$$\int_0^{T_*} \sup_{x \in B_{(\nu\varepsilon)^{1/k}}} |u_\varepsilon(t,x)| \,\mathrm{d}t \geq C_{**}(p) \varepsilon^{-\frac{d}{k}},$$

when $p = +\infty$. Here, number $C_{**}(p) > 0$ depends on the same parameters as number C_* in Theorem 2.12, as well as on p.

Theorem 2.12, along with Corollary 2.13, indicate that even if the interactions described by the kernel W_k do not lead to a formation of singularities for the solution u_ε with fixed $\varepsilon > 0$, neither in finite nor in the infinite time (cf. Theorem 2.7), it is possible to describe a concentration phenomena of solutions on ε-small balls with $\varepsilon > 0$ sufficiently small.

Remark 2.14. Inequality (2.26) in a sense resembles property of a sequence approximating the Dirac's delta function. Indeed, for a positive $v \in L^1(B_1)$ such that $\|v\|_{L^1(B_1)} = M$, we define $v_\varepsilon(x) = \varepsilon^{-d/k} v(x/\varepsilon^{d/k})$ and note that

$$\int_{B_{\varepsilon^{1/k}}} v_\varepsilon(x) \,\mathrm{d}x = M$$

for every $\varepsilon > 0$. Moreover, $v_\varepsilon \to M\delta_0$ weakly in the sense of measures as $\varepsilon \to 0$.

Remark 2.15. The order of growth $\varepsilon^{-d(p-1)/kp}$ of the L^p-norms stated in inequality (2.27) is the same as in the upper estimate (2.14) with small $\varepsilon > 0$, thus we conclude that it is optimal. This is a genuinely nonlinear effect, since estimates of the L^p-norms for solutions of the heat equation $w_t = \varepsilon \Delta w$ are different. Indeed, it follows from the explicit form of solutions, via the convolution with the Gauss–Weierstrass kernel, that

$$\|w(t)\|_p \asymp (\varepsilon t)^{-\frac{d(p-1)}{2p}} \|w(0)\|_1,$$

where symbol $\asymp$ denotes Hardy's asymptotic notation.

Remark 2.16. If initial condition u_0 has a finite $(2-k)$-moment, namely

$$\int_{\mathbb{R}^d} |x|^{2-k} u_0(x)\, \mathrm{d}x < +\infty,$$

then there always exists $\lambda > 0$ such that condition (2.25) is fulfilled.

We describe the concentration phenomenon of radial solution u to problem (2.1) at the origin by considering the quantity

$$\mathcal{D}_\lambda(u)(t) = (2-k)(d-k) \int_{B_{3\lambda/2}} \frac{u(t,x)}{|x|^k}\, \mathrm{d}x, \tag{2.28}$$

where scaling parameter $\lambda > 0$ is such that condition (2.25) is satisfied. The radius of the ball follows from the definition of the cutoff function defined below in (2.29). The following proposition states that, after time averaging, $\mathcal{D}_\lambda(u_\varepsilon)$ grows at least as ε^{-1} when $\varepsilon \to 0$ and it plays a crucial role in our results.

Proposition 2.17. *Let $d \geq 1$ and $k \in (0,1)$. Let u_ε be a radial, non-negative, global-in-time solution to problem (2.1) with $\varepsilon > 0$. Let initial value u_0 satisfy assumptions (2.24), as well as condition (2.25) for some $\lambda > 0$. Then, there exist numbers $\omega_k > 0$, $L_\lambda > 0$ and $T_\lambda > 0$, depending only on k, M and λ (see definitions (2.42) and (2.45), below), such that*

$$\int_0^{T_\lambda} \mathcal{D}_\lambda(u_\varepsilon)(t) e^{-\omega_k M t/\lambda^{2-k}}\, \mathrm{d}t \geq \frac{L_\lambda \lambda^{2-k}}{\varepsilon}.$$

Remark 2.18. Results obtained in Theorem 2.12 and Proposition 2.17 can be generalized for radial interaction kernels which behave like $|x|^k$, $k \in (0,1)$, only around the origin, in the same manner as it is done in the paper [13], where interaction kernels behave locally like $|x|$.

During the proofs of the aforementioned results, we simplify the notation from u_ε to u for the sake of clarity.

Proof. Let $\varphi \in C^1([0,+\infty))$ be a function satisfying

$$\varphi(s) = \begin{cases} s & \text{for } 0 \leq s \leq \left(\frac{1}{2}\right)^{2-k}, \\ 1 & \text{for } s \geq \left(\frac{3}{2}\right)^{2-k}, \end{cases} \tag{2.29}$$

along with $0 \leq \varphi(s) \leq \min\{s, 1\}$, $0 \leq \varphi'(s) \leq 1$ and $\varphi''(s) \leq 0$. Note that φ'' is not necessarily continuous, but it is bounded. An example piecewise function

$$f(s) = \begin{cases} s, & 0 \leq s \leq \frac{1}{2}, \\ -\frac{1}{2}\left(s - \frac{3}{2}\right)^2 + 1, & \frac{1}{2} \leq s \leq \frac{3}{2}, \\ 1, & s \geq \frac{3}{2}, \end{cases}$$

meets all the requirements. Another idea is to take function φ resulting from the following convolution, $\varphi = \psi * \rho(\cdot/\delta)$, where $\psi(s) = \min(s,1)$, ρ is a mollifier and $\delta > 0$ is small enough.

For each $\lambda > 0$, we set

$$\varphi_\lambda(s) = \varphi\left(\left(\frac{s}{\lambda}\right)^{2-k}\right)$$

and we introduce the "truncated moment of order k",

$$I_\lambda(t) = \int_{\mathbb{R}^d} \varphi_\lambda(|x|)u(t,x)\,\mathrm{d}x \quad \text{for all } t \geq 0, \tag{2.30}$$

in contrast to the moment with a function $|x|$ as weight, considered in e.g., [17]. Notice that

$$I_\lambda(t) \leq M \quad \text{for all } t \geq 0, \tag{2.31}$$

by mass conservation property (2.13) and properties of function φ. Our goal is to derive a differential inequality (2.41) for I_λ, see below. Thus, we multiply equation (2.1) by $\varphi_\lambda(|x|)$, and integrate the resulting identity with respect to $x \in \mathbb{R}^d$.

Let us first show that term corresponding to diffusion in equation (2.1) satisfies the following inequality,

$$\int_{\mathbb{R}^d} \varphi_\lambda(|x|)\Delta u(t,x)\,\mathrm{d}x \leq \frac{\mathcal{D}_\lambda(u)(t)}{\lambda^{2-k}} \quad \text{for all } t \geq 0. \tag{2.32}$$

Indeed, we integrate by parts and use properties of the function φ, as well as positivity and radial symmetry of u, to get

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi_\lambda(|x|)\Delta u(t,x)\,\mathrm{d}x =& -\frac{2-k}{\lambda^{2-k}}\int_{\mathbb{R}^d}\varphi'\left(\frac{|x|^{2-k}}{\lambda^{2-k}}\right)\frac{x}{|x|^k}\cdot\nabla u(t,x)\,\mathrm{d}x \\
=& -\sigma_d\frac{2-k}{\lambda^{2-k}}\int_0^{+\infty}\varphi'\left(\frac{r^{2-k}}{\lambda^{2-k}}\right)r^{d-k}u_r(t,r)\,\mathrm{d}r \\
=& \int_0^{+\infty}\left(\sigma_d\frac{(2-k)^2}{\lambda^{4-2k}}\varphi''\left(\frac{r^{2-k}}{\lambda^{2-k}}\right)r^{d+1-2k}\right. \\
&\left. +\sigma_d\frac{(2-k)(d-k)}{\lambda^{2-k}}\varphi'\left(\frac{r^{2-k}}{\lambda^{2-k}}\right)r^{d-k-1}\right)u(t,r)\,\mathrm{d}r \\
&-\sigma_d\frac{(2-k)}{\lambda^{2-k}}\left[\varphi'\left(\frac{r^{2-k}}{\lambda^{2-k}}\right)r^{d-k}u(t,r)\right]_{r=0}^{r=+\infty} \\
\leq& \sigma_d\frac{(2-k)(d-k)}{\lambda^{2-k}}\int_0^{3\lambda/2}u(t,r)r^{d-k-1}\,\mathrm{d}r = \frac{\mathcal{D}_\lambda(u)(t)}{\lambda^{2-k}},
\end{aligned} \tag{2.33}$$

where we abuse the notation of function u to emphasize its radiality.

Next, we estimate the truncated moment of the nonlinear term

$$J_\lambda(t) = \int_{\mathbb{R}^d} \varphi_\lambda(|x|)\nabla \cdot (u(t,x)\nabla W_k * u(t,x)) \, \mathrm{d}x.$$

Integrating by parts and using properties of functions W_k and φ, as well as the symmetrization argument, we obtain

$$\begin{aligned} J_\lambda(t) &= -\frac{2-k}{\lambda^{2-k}} \int_{\mathbb{R}^d} u(t,x)\varphi'\left(\frac{|x|^{2-k}}{\lambda^{2-k}}\right) \frac{x}{|x|^k} \cdot \nabla W_k * u(t,x) \, \mathrm{d}x \\ &= -\frac{k(2-k)}{\lambda^{2-k}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(t,x)u(t,y)\varphi'\left(\frac{|x|^{2-k}}{\lambda^{2-k}}\right) \frac{x}{|x|^k} \cdot \frac{x-y}{|x-y|^{2-k}} \, \mathrm{d}x \, \mathrm{d}y \\ &= -\frac{k(2-k)}{2\lambda^{2-k}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(t,x)u(t,y)\Phi_\lambda(x,y) \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{2.34}$$

We define a function

$$\Phi_\lambda(x,y) = \left(\varphi'\left(\frac{|x|^{2-k}}{\lambda^{2-k}}\right)\frac{x}{|x|^k} - \varphi'\left(\frac{|y|^{2-k}}{\lambda^{2-k}}\right)\frac{y}{|y|^k}\right) \cdot \frac{x-y}{|x-y|^{2-k}} \tag{2.35}$$

for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus Z$, where

$$Z = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = 0 \vee y = 0 \vee x = y\right\}.$$

For $(x,y) \in B_{\lambda/2} \times B_{\lambda/2} \setminus Z$, we observe that

$$\Phi_\lambda(x,y) = \Phi(x,y) = \left(\frac{x}{|x|^k} - \frac{y}{|y|^k}\right) \cdot \frac{x-y}{|x-y|^{2-k}} \tag{2.36}$$

and we introduce the following lemma.

Lemma 2.19. *Let $d \geq 1$, $k \in (0,1)$ and let function Φ be defined in formula (2.36). Then the following inequality holds,*

$$\Phi(x,y) + \Phi(x,-y) \geq 2^k,$$

for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = 0 \vee y = 0 \vee x - y = 0 \vee x + y = 0\right\}$.

Proof. We denote $g(x,y) = \Phi(x,y) + \Phi(x,-y)$. Let $d = 1$ and by the following properties of function g,

$$g(x,y) = g(x,-y) = g(-x,y) = g(y,x),$$

which follows from the symmetry of function Φ, we assume w.l.o.g. that $x, y \geq 0$ and $x > y$. By substitution $x = ty$, $t > 1$, we obtain a function $f(t)$, which is strictly increasing and satisfies $\lim_{t\to 1^+} f(t) = 2^k$.

For $d \geq 2$, we notice that $g(\lambda x, \lambda y) = g(x, y)$ for every $\lambda > 0$. Thus, we can assume that $|x| \leq |y|$ and $|x| = 1$. Moreover, for fixed x, let $S \in SO(d)$ be a rotation such that $x = Sw$, where w is a unit vector of the first axis. Then g is rotation invariant in the following sense,

$$g(x, y) = g(Sw, y) = g(w, S^{-1}y),$$

therefore we consider only function $g(w, y)$ for $y \in \mathbb{R}^d$. In fact, by the definition of w, function g can be described only by two variables y_1 and $|y|$ in the following way,

$$g(w, y) = g(y_1, |y|) = \frac{\left(1 - y_1 - y_1|y|^{-k} + |y|^{2-k}\right)}{\left(1 - 2y_1 + |y|^2\right)^{1-k/2}} + \frac{\left(1 + y_1 + y_1|y|^{-k} + |y|^{2-k}\right)}{\left(1 + 2y_1 + |y|^2\right)^{1-k/2}},$$

where $|y| \geq |y_1|$ and $(|y_1|, |y|) \neq (1, 1)$.

We note that g is a continuous function inside of the domain, $g(-y_1, |y|) = g(y_1, |y|)$ and $\lim_{|y|\to+\infty} g(y_1, |y|) = 2$. Lowest value of the function g can be deduced from the behavior on the boundary, thus we substitute $y_1 = |y| = t \geq 0$ to obtain a function $f(t)$. This function satisfies $\lim_{t\to 1} f(t) = 2^k$, which happens to be the biggest possible lower bound. □

Now we introduce the quantity

$$J_{\lambda,1}(t) = -\frac{k(2-k)}{2\lambda^{2-k}} \int_{B_{\lambda/2}} \int_{B_{\lambda/2}} u(t, x)u(t, y)\Phi_\lambda(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

which can be estimated from above using formula (2.36), symmetrization argument and Lemma 2.19,

$$\begin{aligned} J_{\lambda,1}(t) &= -\frac{k(2-k)}{2\lambda^{2-k}} \int_{B_{\lambda/2}} \int_{B_{\lambda/2}} u(t, x)u(t, y)\Phi(x, y)\, \mathrm{d}x\, \mathrm{d}y, \\ &= -\frac{k(2-k)}{4\lambda^{2-k}} \int_{B_{\lambda/2}} \int_{B_{\lambda/2}} u(t, x)u(t, y)\left(\Phi(x, y) + \Phi(x, -y)\right) \mathrm{d}x\, \mathrm{d}y, \\ &\leq -\frac{k(2-k)}{2^{2-k}\lambda^{2-k}} \int_{B_{\lambda/2}} \int_{B_{\lambda/2}} u(t, x)u(t, y)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

By the mass conservation property (2.13), the inclusion

$$\left(\mathbb{R}^d \times \mathbb{R}^d\right) \setminus \left(B_{\lambda/2} \times B_{\lambda/2}\right) \subset \left(\mathbb{R}^d \times \left(\mathbb{R}^d \setminus B_{\lambda/2}\right)\right) \cup \left(\left(\mathbb{R}^d \setminus B_{\lambda/2}\right) \times \mathbb{R}^d\right), \tag{2.37}$$

symmetry of u and inequality

$$\frac{1}{2^{2-k}} \leq \varphi_\lambda(|x|) \quad \text{for all } x \in \mathbb{R}^d \setminus B_{\lambda/2}, \tag{2.38}$$

we conclude, that

$$
\begin{aligned}
J_{\lambda,1}(t) &\leq -\frac{k(2-k)}{2^{2-k}\lambda^{2-k}}M^2 + \frac{k(2-k)}{2^{2-k}\lambda^{2-k}}\int_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus(B_{\lambda/2}\times B_{\lambda/2})} u(t,x)u(t,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq -\frac{k(2-k)}{2^{2-k}\lambda^{2-k}}M^2 + \frac{k(2-k)}{2^{1-k}\lambda^{2-k}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_{\lambda/2}} u(t,x)u(t,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= -\frac{k(2-k)}{2^{2-k}\lambda^{2-k}}M^2 + \frac{k(2-k)}{2^{1-k}\lambda^{2-k}}M2^{2-k}\int_{\mathbb{R}^d\setminus B_{\lambda/2}} \frac{u(t,x)}{2^{2-k}}\,\mathrm{d}x \\
&\leq -\frac{k(2-k)}{2^{2-k}\lambda^{2-k}}M^2 + \frac{2k(2-k)}{\lambda^{2-k}}MI_\lambda(t).
\end{aligned}
\tag{2.39}
$$

In the next step we show estimate for $|J_\lambda(t) - J_{\lambda,1}(t)|$ from above, therefore we prove the following estimate for function Φ_λ.

Lemma 2.20. *Let $d \geq 1$, $k \in (0,1)$ and let function Φ_λ be defined in formula (2.35) for all $\lambda > 0$. Then there exists a constant $\psi_k > 0$, independent of λ, such that*

$$|\Phi_\lambda(x,y)| \leq \psi_k$$

for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus Z$.

Proof. We begin by noting that $\Phi_\lambda(x,y) = \Phi_1(x/\lambda, y/\lambda)$, which is just rescaling of the variables, thus we consider only function $|\Phi_1(x,y)|$. By the properties of function φ and reverse triangle inequality, we have

$$\lim_{|x|\to+\infty} |\Phi_1(x,y)| = \lim_{|y|\to+\infty} |\Phi_1(x,y)| = 0.$$

Function $|\Phi_1(x,y)|$ is continuous for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus Z$, hence we check behavior around $x = y$. Using Taylor expansion of $\varphi'\left(|x|^{2-k}\right)$, we obtain

$$
\begin{aligned}
\lim_{x\to y} |\Phi_1(x,y)| \approx \varphi'\left(|y|^{2-k}\right) \lim_{x\to y}\left(\frac{x}{|x|^k} - \frac{y}{|y|^k}\right) \cdot \frac{x-y}{|x-y|^{2-k}} \\
+ (2-k)\varphi''\left(|y|^{2-k}\right) \lim_{x\to y} \frac{\left|\sum_{i,j=1}^d x_i y_j (x_i - y_i)(x_j - y_j)\right|}{|x|^k |y|^k |x-y|^{2-k}},
\end{aligned}
$$

where the first limit is zero, which can be proved by methods analogous to those in Lemma 2.19. Second limit is also zero due to the following estimate,

$$\left|\sum_{i,j=1}^d x_i y_j (x_i - y_i)(x_j - y_j)\right| \leq d \sup_{i,j\leq d} \{|x_i|, |y_j|\} |x-y|^2,$$

where this limit behaves like $|x-y|^k$ when $x \to y$. Estimates when $x = 0$ or $y = 0$ are analogous. We conclude that non-negative, continuous and well-defined function $|\Phi_1(x,y)|$, with finite limits at $+\infty$, attains its maximal value on $\mathbb{R}^d \times \mathbb{R}^d \setminus Z$, thus it is bounded. □

We combine this lemma with relations (2.37)-(2.38) to obtain

$$
\begin{aligned}
|J_\lambda(t) - J_{\lambda,1}(t)| &= \frac{k(2-k)}{2\lambda^{2-k}} \int_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus(B_{\lambda/2}\times B_{\lambda/2})} u(t,x)u(t,y)\,|\Phi_\lambda(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{k(2-k)\psi_k}{2\lambda^{2-k}} \int_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus(B_{\lambda/2}\times B_{\lambda/2})} u(t,x)u(t,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{k(2-k)\psi_k}{\lambda^{2-k}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_{\lambda/2}} u(t,x)u(t,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{k(2-k)\psi_k}{2^{k-2}\lambda^{2-k}} M I_\lambda(t).
\end{aligned}
\tag{2.40}
$$

Gathering identity (2.34) along with estimates (2.32), (2.39) and (2.40), we obtain the differential inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t} I_\lambda(t) \leq \frac{1}{\lambda^{2-k}} \left(\varepsilon\mathcal{D}_\lambda(u)(t) - \mu_k\omega_k M^2 + \omega_k M I_\lambda(t)\right) \tag{2.41}
$$

for all $t \geq 0$, where

$$
\omega_k = k(2-k)2^{2-k}\left(\psi_k + 2^{k-1}\right) \tag{2.42}
$$

and

$$
\mu_k = \frac{2^{2k-4}}{\psi_k + 2^{k-1}}. \tag{2.43}
$$

This inequality can be rewritten as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(I_\lambda(t)e^{-\omega_k Mt/\lambda^{2-k}}\right) \leq \left(\varepsilon\mathcal{D}_\lambda(u)(t) - \mu_k\omega_k M^2\right)\frac{e^{-\omega_k Mt/\lambda^{2-k}}}{\lambda^{2-k}}
$$

and after integration with respect to time, we obtain

$$
\begin{aligned}
I_\lambda(T)e^{-\omega_k MT/\lambda^{2-k}} - I_\lambda(0) \leq & \frac{\varepsilon}{\lambda^{2-k}} \int_0^T \mathcal{D}_\lambda(u)(t)e^{-\omega_k Mt/\lambda^{2-k}}\,\mathrm{d}t \\
& - \mu_k M\left(1 - e^{-\omega_k MT/\lambda^{2-k}}\right)
\end{aligned}
$$

for each $T > 0$. Omitting term with $I_\lambda(T)$, which is non-negative, we end up with inequality

$$
\frac{\varepsilon}{\lambda^{2-k}} \int_0^T \mathcal{D}_\lambda(u)(t)e^{-\omega_k Mt/\lambda^{2-k}}\,\mathrm{d}t \geq \mu_k M\left(1 - e^{-\omega_k MT/\lambda^{2-k}}\right) - I_\lambda(0). \tag{2.44}
$$

By the properties of function φ and assumption (2.25), we get

$$
I_\lambda(0) \leq \frac{1}{\lambda^{2-k}} \int_{\mathbb{R}^d} \left(\min\{|x|,\lambda\}\right)^{2-k} u_0(x)\,\mathrm{d}x < \mu_k M,
$$

thus we can define a positive number $\bar{L}_\lambda = \mu_k M - I_\lambda(0)$. Now consider a function

$$f(T) = \mu_k M \left(1 - e^{-\omega_k MT/\lambda^{2-k}}\right) - I_\lambda(0),$$

which is a RHS of inequality (2.44). It is a strictly increasing function with limit $\bar{L}_\lambda$ at $+\infty$ and we note that

$$f(T_0) = 0 \iff T_0 = \frac{\lambda^{2-k}}{\omega_k M} \log\left(\frac{\mu_k M}{\mu_k M - I_\lambda(0)}\right) > 0.$$

We conclude that for every $L_\lambda \in (0, \bar{L}_\lambda)$ identity

$$f(T_\lambda) = L_\lambda \tag{2.45}$$

is satisfied for some $T_\lambda > T_0$, which completes the proof. □

Remark 2.21. To obtain inequality (2.33), we integrate by parts twice, which in the case $d \geq 1$ and $k \in (0, 1)$ does not cause any difficulties. In fact, calculations obtained in this work are in a sense easier than in the paper [13], where in the one-dimensional case $\mathcal{D}_\lambda(u)(t) = 2u(0, t)$, due to the lack of regularity of $|x|$ around the origin. Possible extension of these results to case $k \in (1, 2)$ should take special care in defining quantity $\mathcal{D}_\lambda(u)$ in the one-dimensional case.

Remark 2.22. We note that constant $\psi_k > 0$ obtained in Lemma 2.20 depends on the choice of the cutoff function φ. Indeed, consider a family of functions

$$f_t(s) = \begin{cases} s, & 0 \leq s \leq t, \\ -\frac{(s+t-2)^2}{4(t-1)} + 1, & t \leq s \leq 2 - t, \\ 1, & s \geq 2 - t, \end{cases}$$

for $t \in [1/2, 1)$. Function f_t satisfies all the assumptions for fixed t, and moreover, $\|f_t''\|_\infty \to +\infty$ as $t \to 1$. In the case $d = 1$, one can easily calculate maximum value of the function $|\Phi_1(x, y)|$ and see the dependence on the values of f_t''. In this work, we do not consider the problem of choosing the optimal (in this sense - the smallest) constant ψ_k.

Now we prove the main result of this section.

Proof of Thorem 2.12. We estimate the following integral

$$\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} \frac{u(t, x)}{|x|^k} \, dx \, dt,$$

where we introduce a parameter $\nu > 0$ which value will be specified later. For an arbitrary $T > 0$, using twice the Hölder inequality, we get

$$\begin{aligned}
&\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} \frac{u(t,x)}{|x|^k} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} |x|^{-k\alpha_1} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\alpha_1}} \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x)^{\beta_1} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\beta_1}} \\
&\qquad \le \left(\frac{\sigma_d T}{d-k\alpha_1}(\nu\varepsilon)^{d/k-\alpha_1}\right)^{\frac{1}{\alpha_1}} \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x)^{(\beta_1-1/\beta_2)\alpha_2} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\beta_1\alpha_2}} \\
&\qquad\quad \times \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x) \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\beta_1\beta_2}},
\end{aligned}$$

where exponents $\alpha_1, \beta_1, \alpha_2, \beta_2 \ge 1$ satisfy $k\alpha_1 < d$ and $1/\alpha_i + 1/\beta_i = 1$, $i \in \{1,2\}$, and will also be determined later.

We denote $p = (\beta_1 - 1/\beta_2)\alpha_2$ and depending on its value, we use L^p-estimates from Lemma 2.9 or Remark 2.10 to obtain

$$\begin{aligned}
\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} \frac{u(t,x)}{|x|^k} \,\mathrm{d}x\,\mathrm{d}t &\le \left(\frac{\sigma_d T}{d-k\alpha_1}(\nu\varepsilon)^{d/k-\alpha_1}\right)^{\frac{1}{\alpha_1}} T^{\frac{1}{\beta_1\alpha_2}} \left(\sup_{t\in[0,T]} \|u(t)\|_p\right)^{1-\frac{1}{\beta_1\beta_2}} \\
&\quad \times \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x) \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\beta_1\beta_2}} \\
&\le C T^{\frac{1}{\alpha_1}+\frac{1}{\beta_1\alpha_2}} \nu^{\frac{d}{k\alpha_1}-1} \varepsilon^{\frac{d}{k\alpha_1}-1-\frac{1}{\eta}+\frac{1}{\eta\beta_1\beta_2}} \\
&\quad \times \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x) \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\beta_1\beta_2}},
\end{aligned}$$

where constant $C > 0$ is independent of T, ν and ε. We recall that $\eta = kp/(d(p-1))$ and by straightforward calculation, we get coefficient ε^{-1} regardless of the choice of the exponents, as long as they fulfill the assumptions. Thus, we obtain

$$\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} \frac{u(t,x)}{|x|^k} \,\mathrm{d}x\,\mathrm{d}t \le C T^{\alpha} \nu^{\beta} \varepsilon^{-1} \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x) \,\mathrm{d}x\,\mathrm{d}t\right)^{\gamma} \tag{2.46}$$

for some numbers $\alpha, \beta, \gamma > 0$. Note that the above inequality is valid only for ε such that $0 < \varepsilon < \varepsilon_1$, where ε_1 is depending on d, k, p and u_0, and can be calculated from the L^p-estimates.

We recall that $\lambda > 0$ is a number such that condition (2.25) is satisfied. Thus for a fixed $\nu > 0$ and sufficiently small $\varepsilon > 0$, such that $3\lambda/2 > (\nu\varepsilon)^{1/k}$, using the conservation of mass (2.13), we obtain

$$\int_0^T \int_{B_{3\lambda/2}\setminus B_{(\nu\varepsilon)^{1/k}}} \frac{u(t,x)}{|x|^k}\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{\nu\varepsilon}\int_0^T \int_{B_{3\lambda/2}\setminus B_{(\nu\varepsilon)^{1/k}}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t \le \frac{MT}{\nu\varepsilon}. \tag{2.47}$$

From inequalities (2.46)-(2.47), using definition (2.28) of the quantity $\mathcal{D}_\lambda(u)$, we deduce that

$$\frac{\varepsilon}{(2-k)(d-k)}\int_0^T \mathcal{D}_\lambda(u)(t)\,\mathrm{d}t \le \frac{MT}{\nu} + CT^\alpha \nu^\beta \left(\int_0^T \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^\gamma .$$

Setting $T = T_\lambda$, we can use Proposition 2.17 without the exponential term (due to the positivity of ω_k), to obtain

$$\frac{L_\lambda \lambda^{2-k}}{(2-k)(d-k)} - \frac{MT_\lambda}{\nu} \le CT_\lambda^\alpha \nu^\beta \left(\int_0^{T_\lambda} \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^\gamma . \tag{2.48}$$

We choose number $\nu > 0$ such that

$$\nu > \frac{MT_\lambda(2-k)(d-k)}{L_\lambda \lambda^{2-k}} = \nu_*,$$

which assures that LHS of inequality (2.48) is positive. For such $\nu > \nu_*$, we calculate second condition for $\varepsilon > 0$,

$$\varepsilon < \frac{1}{\nu}\left(\frac{3\lambda}{2}\right)^k = \varepsilon_2,$$

thus setting $\varepsilon_* = \min\{\varepsilon_1, \varepsilon_2\}$, we conclude the proof. For the sake of clarity we note that $T_* = T_\lambda$ and number $C_* > 0$ can be calculated from inequality (2.48) as long as $\nu > \nu_*$. □

We finish this section with the proof of Corollary 2.13.

Proof of Corollary 2.13. Let $p \in [1, +\infty]$. By Theorem 2.12 and the Hölder inequality with $p < +\infty$,

$$\begin{aligned} C_* &\le \int_0^{T_*} \int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \left(\frac{\sigma_d}{d}\right)^{\frac{p-1}{p}} (\nu\varepsilon)^{\frac{d(p-1)}{kp}} \int_0^{T_*} \left(\int_{B_{(\nu\varepsilon)^{1/k}}} u(t,x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \mathrm{d}t, \end{aligned}$$

where constant $C_{**}(p) > 0$ can be calculated directly from this inequality. For $p = +\infty$ we proceed analogously, obtaining coefficient $\varepsilon^{d/k}$. □

2.5 Moment estimates

In the last section of this chapter, we fix $\varepsilon = 1$ in equation (2.1) and investigate long time behavior of the solutions. Even though they are global-in-time and p-norms are bounded from above, the aggregation effect is still relevant and radial solutions do not tend to 0 as $t \to +\infty$.

Theorem 2.23. *Let $d \geq 1$ and $k \in (0,1)$. Let u be a radial, non-negative, global-in-time solution to problem (2.2), with radial, non-negative initial condition $u_0 \in L^1\big(\mathbb{R}^d, (1+|x|^2)\,\mathrm{d}x\big) \cap L^\infty(\mathbb{R}^d)$. Then*

$$\lim_{t\to+\infty} \|u(t)\|_p \neq 0$$

for all $p \in (1, +\infty]$.

By the global-in-time solution u we understand $u \in C\big([0,T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\big)$ for every $T > 0$, in the sense of Theorem 2.7. Theorem 2.23 suggests, that there exists a steady state to problem (2.2), but this is consistent with the results described in Section 1.3. Before proving Theorem 2.23, we present some auxiliary results and introduce notation for the moment of order b of solution u,

$$m_b(t) = \int_{\mathbb{R}^d} u(t,x)|x|^b\,\mathrm{d}x,$$

and the following lemma assures us of the existence of $m_2(t)$ for problem (2.2).

Lemma 2.24. *Let $d \geq 1$ and $k \in (0,1)$. Let u be a non-negative, global-in-time solution to problem (2.2), with non-negative initial condition $u_0 \in L^1\big(\mathbb{R}^d, (1+|x|^2)\,\mathrm{d}x\big) \cap L^\infty(\mathbb{R}^d)$. Then $m_2(t) < +\infty$ for all $t \in [0,T]$.*

Proof. This lemma can be proved by the same reasoning as shown in [60, Lemma 3.2(i)]. It is sufficient to show that $\nabla W_k * u \in L^\infty\big([0,T], L^\infty(\mathbb{R}^d)\big)$, but this follows from Lemma 2.4 and the fact that u is a global-in-time solution. □

The following interpolation lemma is a direct consequence of the Hölder inequality.

Lemma 2.25. *Let p_1, p_2 and p_3 be numbers satisfying $0 \leq p_1 < p_2 < p_3$ and let $v \in L^1\big(\Omega, (|x|^{p_1} + |x|^{p_3})\,\mathrm{d}x\big)$ for some $\Omega \subset \mathbb{R}^d$. Then*

$$\int_\Omega |v(x)||x|^{p_2}\,\mathrm{d}x \leq \left(\int_\Omega |v(x)||x|^{p_1}\,\mathrm{d}x\right)^{\frac{p_3-p_2}{p_3-p_1}} \left(\int_\Omega |v(x)||x|^{p_3}\,\mathrm{d}x\right)^{\frac{p_2-p_1}{p_3-p_1}}.$$

Proof. We use the Hölder inequality $\|fg\|_r \leq \|f\|_p \|g\|_q$ with exponents

$$r = \sqrt{\frac{(p_3 - p_2)(p_2 - p_1)}{p_3 - p_1}}, \quad p = \sqrt{\frac{(p_3 - p_1)(p_2 - p_1)}{p_3 - p_2}}, \quad q = \sqrt{\frac{(p_3 - p_2)(p_3 - p_1)}{p_2 - p_1}}.$$

Notice that $p_2/r = p_1/p + p_3/q$, thus by substitution

$$f(x) = |v(x)|^{1/p} |x|^{p_1/p} \quad \text{and} \quad g(x) = |v(x)|^{1/q} |x|^{p_3/q},$$

we obtain the result. □

In the following results, we consider function $|x - y|^k$ for $x, y \in \mathbb{R}^d$, thus we intruduce few lemmas on the estimates of this function.

Lemma 2.26. *Let $x, y \in \mathbb{R}^d$. For $k > 0$ the following inequality holds*

$$|x - y|^k \leq C_k \left(|x|^k + |y|^k \right),$$

where $C_k = 1$ if $k \in (0, 1]$ and $C_k = 2^{k-1}$ if $k > 1$.

Lemma 2.27. *Let $x, y \in \mathbb{R}$. For $k > 0$ the following inequality holds*

$$|x - y|^k \geq \begin{cases} c_k \left(|x|^k + |y|^k \right) & \text{if } xy \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $c_k = 2^{k-1}$ if $k \in (0, 1)$ and $c_k = 1$ if $k \geq 1$.

Proof of these results is rather standard thus we omit it, however we prove the multidimensional version of Lemma 2.27, which is slightly different from the one-dimensional case.

Lemma 2.28. *Let $x, y \in \mathbb{R}^d$, $d \geq 2$. For $k > 0$ the following inequality holds*

$$|x - y|^k \geq \begin{cases} c_k \left(|x|^k + |y|^k \right) & \text{if } x \cdot y \leq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $c_k = 2^{k/2-1}$ if $k \in (0, 2)$ and $c_k = 1$ if $k \geq 2$.

Proof. Consider inequality

$$|x - y|^2 = |x|^2 - 2x \cdot y + |y|^2 \geq |x|^2 + |y|^2 \tag{2.49}$$

which is valid for x, y such that $x \cdot y \leq 0$. Taking both sides of inequality (2.49) to the power $k_1 > 0$ and using Lemma 2.27 we obtain

$$|x - y|^{2k_1} \geq \left(|x|^2 + |y|^2 \right)^{k_1} \geq c_{k_1} \left(|x|^{2k_1} + |y|^{2k_1} \right),$$

where c_{k_1} is defined as in Lemma 2.27. Taking $k_1 = k/2$ we obtain the result. □

Now, we are going to derive equation for the derivative of the second moment.

Lemma 2.29. *Let $d \geq 1$ and $k \in (0,1)$. Let u be a non-negative, global-in-time solution to problem (2.2), with non-negative initial condition $u_0 \in L^1\big(\mathbb{R}^d, (1+|x|^2)\,\mathrm{d}x\big) \cap L^\infty(\mathbb{R}^d)$. Then*

$$m_2'(t) = dM - I(t) \tag{2.50}$$

for all $t \in [0,T]$, where M is mass of the solution and

$$I(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(t,x)u(t,y)|x-y|^k \,\mathrm{d}y\,\mathrm{d}x.$$

Proof. We follow the same reasoning as given in paper [19, Lemma 3]. We multiply the equation in problem (2.2) by a smooth function $\psi_\varepsilon(|x|)$ with compact support, that grows nicely to $|x|^2$ as $\varepsilon \to 0$ and integrate on $\mathbb{R}^d$. Integral

$$\int_{\mathbb{R}^d} \nabla\cdot(u\nabla W_k * u)\psi_\varepsilon(|x|)\,\mathrm{d}x,$$

after integrating by parts and using symmetrization, is equivalent to

$$-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(t,x)u(t,y)\left(\nabla\psi_\varepsilon(|x|) - \nabla\psi_\varepsilon(|y|)\right)\cdot(x-y)|x-y|^{k-2}\,\mathrm{d}y\,\mathrm{d}x. \tag{2.51}$$

Taking function $\nabla\psi_\varepsilon(x)$ Lipschitz continuous with constant $L \leq 2$, the integrand is bounded for all $\varepsilon > 0$ by function $u(t,x)u(t,y)|x-y|^k$, which is integrable on $\mathbb{R}^d$ by Lemmas 2.26, mass conservation property (2.13), Lemma 2.24 and 2.25 for all $t \in [0,T]$. Thus, by the dominated convergence theorem, integral (2.51) converges to $I(t)$ as $\varepsilon \to 0$. □

Before proving the main result of this section, we introduce another interpolation lemma.

Lemma 2.30. *Let $p \in [1,+\infty]$ and $b \geq 0$. For all $v \in L^1\big(\mathbb{R}^d, (1+|x|^b)\,\mathrm{d}x\big) \cap L^p(\mathbb{R}^d)$ the following inequality holds,*

$$\|v\|_1 \leq C\left(\int_{\mathbb{R}^d}|v(x)||x|^b\,\mathrm{d}x\right)^\beta \|v\|_p^{1-\beta},$$

where $C = C(p) \geq 2$ and $\beta = (p-1)/(bp+p-1) \in [0,1]$.

Proof. For

$$R = \left(\int_{\mathbb{R}^d}|v(x)||x|^b\,\mathrm{d}x\right)^{\frac{p}{bp+p-1}} \|v\|_p^{-\frac{p}{bp+p-1}},$$

using Hölder inequality, we obtain

$$
\begin{aligned}
\|v\|_1 &= \int_{|x|<R} |v(x)|\,\mathrm{d}x + \int_{|x|\geq R} |v(x)|\,\mathrm{d}x \\
&\leq (2R)^{1-1/p}\|v\|_p + R^{-b}\int_{|x|/R\geq 1} |v(x)||x|^b\,\mathrm{d}x \\
&\leq (2R)^{1-1/p}\|v\|_p + R^{-b}\int_{\mathbb{R}^d} |v(x)||x|^b\,\mathrm{d}x \\
&\leq \left(2^{1-\frac{1}{p}}+1\right)\left(\int_{\mathbb{R}^d} |v(x)||x|^b\,\mathrm{d}x\right)^{\frac{p-1}{bp+p-1}} \|v\|_p^{\frac{bp}{bp+p-1}},
\end{aligned}
$$

where for $p = +\infty$ exponents are valid by taking limit $p \to +\infty$. □

Proof of Theorem 2.23. We begin with showing the following inequality,

$$
m_2'(t) \leq dM - c_k M m_k(t), \tag{2.52}
$$

where constant $c_k > 0$ follows from Lemma 2.27 or 2.28, depending on the dimension $d \geq 1$.

We estimate integral $I(t)$ in equation (2.50) from below,

$$
I(t) \geq c_k \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(t,x)u(t,y)\left(|x|^k + |y|^k\right)\mathbf{1}_{\{x\cdot y\leq 0\}}(x,y)\,\mathrm{d}y\,\mathrm{d}x,
$$

where $\mathbf{1}_{\{x\cdot y\leq 0\}}$ denotes indicator function. We notice that for non-negative, radial functions $v, w \in L^1(\mathbb{R}^d)$,

$$
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(x)w(y)\mathbf{1}_{\{x\cdot y\leq 0\}}(x,y)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\left(\int_{\mathbb{R}^d} v(x)\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^d} w(y)\,\mathrm{d}y\right),
$$

thus by the assumptions on u, $I(t) \geq c_k M m_k(t)$. Notice that this estimate is not optimal due to the integration only over the half of $\mathbb{R}^{2d}$ space.

We assume by contradiction that for all $\varepsilon > 0$ there exists $T \in \mathbb{R}$ such that for all $t > T$ we have $\|u(t)\|_p < \varepsilon$. By Lemma 2.30, for all $t > T$,

$$
m_k(t) > C\varepsilon^{1-\frac{1}{\beta}}, \tag{2.53}
$$

where $C > 0$ is independent of t and ε. Combining inequalities (2.52) and (2.53), and integrating from T to $T+s$ for some $s > 0$, we obtain

$$
0 \leq m_2(T+s) < m_2(T) + sM\left(d - C\varepsilon^{1-\frac{1}{\beta}}\right). \tag{2.54}
$$

For ε sufficiently small, the coefficient of s is negative. Thus, for s sufficiently large, the RHS of inequality (2.54) is negative, which contradicts the non-negativity of the second moment. □

Remark 2.31. By the same arguments as in the proof of Theorem 2.23, one can conclude that $\lim_{t\to+\infty} m_b(t) \neq +\infty$ for all $b \in (0,k]$.

Remark 2.32. We can estimate integral $I(t)$ in equality (2.50) from above, using Lemma 2.26, to obtain

$$m_2'(t) \geq dM - 2C_k M m_k(t), \tag{2.55}$$

where solution u is not necessarily radial. One can conclude from this inequality, that $m_2(t)$ is bounded from below for all $t \geq 0$. Indeed, using Lemma 2.25, we obtain

$$m_2'(t) \geq dM - Cm_2(t)^{k/2}$$

for some $C > 0$. We deduce, that evolution of the moment $m_2(t)$ satisfies an estimate $m_2(t) \geq y(t)$ for all $t \geq 0$, where $y(t)$ is a solution to the following ODE

$$y'(t) = 2M - Cy(t)^{k/2}, \quad y(0) = m_2(0),$$

whose solution meets $\lim_{t\to+\infty} y(t) < +\infty$.

Remark 2.33. Both inequalities (2.52) and (2.55) can be considered as an equivalent of the well-known identity

$$m_2'(t) = 4M - \frac{M^2}{2\pi},$$

derived from the two-dimensional Keller–Segel chemotaxis model. This model in fact corresponds to the critical case $k = 0$ of the convolution kernel W_k.

Remark 2.34. Applying the same technique in the case $k \geq 2$, assuming sufficient regularity on u, one can obtain the following differential inequality

$$m_k'(t) \leq M^{\frac{2}{k}} \left(k(d+k-2)m_k(t)^{1-\frac{2}{k}} - \frac{kc_{k-2}}{2} m_k(t)^{2-\frac{2}{k}} \right),$$

from which follows $m_k(t) \leq C$ for some number $C > 0$ independent of $t \geq 0$.

Chapter 3

Special solutions to ADE

3.1 Statement of the problem

This chapter is devoted to the following ADE,

$$u_t - \Delta u^m = \nabla \cdot (u \nabla W_k * u), \quad t > 0,\ x \in \mathbb{R}^d, \tag{3.1}$$

where $d \geq 1$, $m \geq 1$ and W_k is the power-law interaction kernel, defined by

$$W_k(x) = \frac{|x|^k}{k}, \quad k > 0. \tag{3.2}$$

We investigate existence and properties of special solutions to problem (3.1), i.e., solutions with some specific qualitative properties. In the following lemma, we show scaling of this equation.

Lemma 3.1 (Scaling property)**.** *Let $d \geq 1$, $m \geq 1$, $k > 0$, $\lambda > 0$ and let parameters $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy the following equations,*

$$\alpha(2 - m) = \gamma(d + k) \quad \textit{and} \quad \beta = 2\gamma + \alpha(m - 1). \tag{3.3}$$

Then equation (3.1) has scaling $u_\lambda(t, x) = \lambda^\alpha u(\lambda^\beta t, \lambda^\gamma x)$, i.e., if u is a solution to equation (3.1), then u_λ is so.

Proof. We calculate each term in equation (3.1) for u_λ, where for derivatives we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\lambda(t, x) = \lambda^{\alpha+\beta} u_t(\lambda^\beta t, \lambda^\gamma x) \quad \text{and} \quad \Delta u_\lambda(t, x)^m = \lambda^{\alpha m + 2\gamma} (\Delta u^m)(\lambda^\beta t, \lambda^\gamma x).$$

For the convolution, using change of variables, we obtain

$$
\begin{aligned}
\frac{\partial W_k}{\partial x_i} * u_\lambda(t,x) &= \lambda^\alpha \int_{\mathbb{R}^d} (x_i - y_i)|x-y|^{k-2} u(\lambda^\beta t, \lambda^\gamma y)\,\mathrm{d}y \\
&= \lambda^{\alpha-\gamma d} \int_{\mathbb{R}^d} (x_i - \lambda^{-\gamma} z_i)|x-\lambda^{-\gamma} z|^{k-2} u(\lambda^\beta t, z)\,\mathrm{d}z \\
&= \lambda^{\alpha-\gamma d-\gamma(k-1)} \int_{\mathbb{R}^d} (\lambda^\gamma x_i - z_i)|\lambda^\gamma x - z|^{k-2} u(\lambda^\beta t, z)\,\mathrm{d}z \\
&= \lambda^{\alpha-\gamma(d+k-1)} \frac{\partial W_k}{\partial x_i} * u(\lambda^\beta t, \lambda^\gamma x)
\end{aligned}
\tag{3.4}
$$

and furthermore

$$
\nabla \cdot (u_\lambda(t,x)\nabla W_k * u_\lambda(t,x)) = \lambda^{2\alpha-\gamma(d+k-2)} \nabla \cdot \left(u(\lambda^\beta t, \lambda^\gamma x)\nabla W_k * u(\lambda^\beta t, \lambda^\gamma x)\right).
$$

Gathering all the exponents we obtain the result. □

Remark 3.2 (Self-similar solution). Considering problem (3.1), a natural question arises on the existence of the self-similar solution i.e., satisfying $u_\lambda(t,x) = u(t,x)$ along with condition

$$
\int_{\mathbb{R}^d} u_\lambda(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x.
$$

For a self-similar solution to exist, parameters d, m, k have to satisfy a relation $m = 1 - k/d$, which in literature is known as a *fair competition regime* [29]. A particular problem considered in this setting is the classical parabolic-elliptic Keller-Segel model in $\mathbb{R}^2$ (see, e.g., [70]).

3.2 Stationary solutions

In this section, we consider equation for the stationary solutions to problem (3.1),

$$
\Delta u^m + \nabla \cdot (u \nabla W_k * u) = 0, \quad x \in \mathbb{R}^d, \tag{3.5}
$$

where we assume that u is a positive and sufficiently fast-decreasing radial function. Moreover, its regularity depends on the parameter $m \geq 1$.

Let $m = 1$ and consider the one-dimensional case of equation (3.5),

$$
u_{xx} + (uW_k' * u)_x = 0, \quad x \in \mathbb{R}, \tag{3.6}
$$

which for a function u satisfying the assumptions, can be rewritten as equation

$$
u = \exp(-W_k * u + D), \tag{3.7}
$$

where $D \in \mathbb{R}$ is an arbitrary constant. Indeed, integrating equation (3.6) and dividing by u, we obtain

$$\frac{u_x}{u} = -(W_k * u)_x,$$

and integrating once again we rewrite the result as equation (3.7). Note that in the first integration step we assumed that

$$\lim_{x\to-\infty} u_x(x) + u(x)W_k' * u(x) = 0. \tag{3.8}$$

Analogous procedure can by applied in the case $m > 1$, where one-dimensional version of problem (3.5) can be reformulated into equation

$$u = (D - \bar{m}W_k * u)_+^{\frac{1}{m-1}}, \tag{3.9}$$

where $\bar{m} = (m-1)/m$ and function u satisfies

$$\lim_{x\to-\infty} (u^m)_x(x) + u(x)W_k' * u(x) = 0. \tag{3.10}$$

Constant $D \in \mathbb{R}$ is arbitrary and $v_+(x) = \max\{v(x), 0\}$ denotes non-negative part of a function v.

In examples (3.17) and (3.19), we describe explicit multidimensional solutions to problem (3.5) with $k = 2$, which are also solutions to the following system of partial differential equations,

$$\nabla u^m + u\nabla(W_k * u) = 0, \quad x \in \mathbb{R}^d. \tag{3.11}$$

We use this observation as a justification for searching d-dimensional solutions, $d \geq 2$, to problem (3.5) by solving system (3.11). By the same arguments as in the one-dimensional case, one can rewrite this system as equation (3.7) or (3.9), depending on the parameter m. We assume that constant of integration $D \in \mathbb{R}$ is the same in every dimension and independent of variable x_i.

We conclude this introduction with two results following from Lemma 3.1, which highlight certain properties of stationary solutions and are particularly useful in this context.

Lemma 3.3. *Let $d \geq 1$, $m = 1$ and $k > 0$. Assume that U is a solution to problem (3.5) such that*

$$\int_{\mathbb{R}^d} |x|^k U(x)\,\mathrm{d}x < +\infty.$$

Then for all $\lambda > 0$,

$$\int_{\mathbb{R}^d} |x|^k U_\lambda(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |x|^k U(x)\,\mathrm{d}x = \mu_{d,k},$$

where U_λ is defined in Lemma 3.1 and constant $\mu_{d,k} > 0$ is independent of λ.

Proof. By a straightforward calculation, using substitution $y = \lambda^\gamma x$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^k U_\lambda(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} |x|^k \lambda^\alpha U(\lambda^\gamma x)\,\mathrm{d}x \\ &= \lambda^\alpha \int_{\mathbb{R}^d} |\lambda^{-\gamma} y|^k U(y)\,\lambda^{-\gamma d}\mathrm{d}y \\ &= \lambda^{\alpha-\gamma(k+d)} \int_{\mathbb{R}^d} |y|^k U(y)\,\mathrm{d}y\end{aligned}$$

and from equations (3.3) with $m = 1$, we get $\alpha - \gamma(k + d) = 0$. □

Remark 3.4. Let U_1 and U_λ be solutions to problem (3.5), where

$$\int_{\mathbb{R}^d} U_1(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} U_\lambda(x)\,\mathrm{d}x = M.$$

Then $U_M(x) = M^{\alpha_1} U_1(M^{\gamma_1} x)$ is also a solution to problem (3.5), where

$$\alpha_1 = \frac{d+k}{d(m-1)+k} \quad \text{and} \quad \gamma_1 = \frac{2-m}{d(m-1)+k}.$$

3.2.1 Properties of the operator $\mathcal{F}$

Equation (3.7) is an equation for a fixed point, thus we introduce the operator

$$\mathcal{F}_1(\varphi) = \exp(-W_k * \varphi + D), \quad D \in \mathbb{R}, \tag{3.12}$$

where function φ is defined for all $x \in \mathbb{R}^d$. We study properties of operator $\mathcal{F}_1$, which is well-defined for appropriate functions, see proposition below.

Proposition 3.5. *Let $d \geq 1$, $k > 0$ and*

$$\Phi = \Big\{\varphi \in L^1\big(\mathbb{R}^d, (1 + |x|^k)\,\mathrm{d}x\big) \cap C(\mathbb{R}^d) : \\ \varphi \text{ is a positive, radially decreasing function}\Big\}, \tag{3.13}$$

then operator $\mathcal{F}_1$, defined by formula (3.12), satisfies $\mathcal{F}_1 : \Phi \to \Phi$ and $\mathcal{F}_1(\varphi) \in C^1(\mathbb{R}^d)$. Moreover, for all $\varphi \in \Phi$, there exist numbers $a_1, a_2, b_1, b_2 > 0$, depending on φ, such that

$$a_1 \exp\left(-b_1|x|^k\right) \leq \mathcal{F}_1(\varphi)(x) \leq a_2 \exp\left(-b_2|x|^k\right) \tag{3.14}$$

for all $x \in \mathbb{R}^d$.

Estimate (3.14) seems natural due to the form of operator (3.12). In particular, behavior for large $|x|$ is like $\exp(-|x|^k)$. For the record, we notice the same for explicit stationary solutions to problem (3.5) with $m = 1$, described in Subsection 3.2.3. Also, function $\mathcal{F}_1(\varphi)$ has improved regularity compared to φ, see Remark 3.7, below. Before proving this proposition, we present lemma on the radial decay of the convolution.

Lemma 3.6. *Let $d \geq 1$, $k > 0$, W_k be defined by formula (3.2) and φ be such that convolution $W_k * \varphi$ exists for all $x \in \mathbb{R}^d$. If φ is a continuous, positive and radially decreasing function, then $W_k * \varphi$ is continuous, positive and radially increasing.*

Proof. Continuity and positivity of convolution is obvious. Radial symmetry

$$W_k * \varphi(Sx) = W_k * \varphi(x)$$

follows from a suitable substitution in the integral and properties of $S \in SO(d)$.

For the radial monotonicity, fix $x \in \mathbb{R}^d$, $x \neq 0$, and consider the difference

$$W_k * \varphi(rx) - W_k * \varphi(x) = \frac{1}{k} \int_{\mathbb{R}^d} \varphi(y) g(y) \, \mathrm{d}y, \tag{3.15}$$

where $r > 1$ is fixed and $g(y) = |rx - y|^k - |x - y|^k$. Function g is anti-symmetric with respect to the hyperplane $H = \{y \in \mathbb{R}^d : g(y) = 0\}$, i.e., for all $y \in \mathbb{R}^d$, $g(y) = -g(y + 2\overrightarrow{yy'})$, where y' is an orthogonal projection of y on H. Moreover, $g(0) > 0$, and by monotonicity of function φ we have

$$\int_{\{g(y)>0\}} \varphi(y) \, \mathrm{d}y > \int_{\{g(y)<0\}} \varphi(y) \, \mathrm{d}y,$$

thus $W_k * \varphi(rx) - W_k * \varphi(x) > 0$. Proof for $x = 0$ is analogous, where one should consider $g(y) = |z - y|^k - |y|^k$ for any $z \in \mathbb{R}^d$. □

Proof of Proposition 3.5. We begin with φ satisfying only $\varphi \in L^1\big(\mathbb{R}^d, (1 + |x|^k) \, \mathrm{d}x\big)$. We estimate the convolution term in formula (3.12) from below, using Lemma 2.27 and 2.28, depending on the dimension d,

$$W_k * \varphi(x) \geq |x|^k \frac{c_k}{k} \int_{D(x)} \varphi(y) \, \mathrm{d}y + \frac{c_k}{k} \int_{D(x)} |y|^k \varphi(y) \, \mathrm{d}y,$$

where for fixed $x \in \mathbb{R}^d$, we define the set $D(x) = \{y \in \mathbb{R}^d : \langle x, y \rangle \leq 0\}$. It is easy to see that for a non-negative, radially symmetric function $v \in L^1(\mathbb{R}^d)$,

$$\int_{D(x)} v(y) \, \mathrm{d}y = \frac{1}{2} \int_{\mathbb{R}^d} v(y) \, \mathrm{d}y,$$

thus we obtain

$$\mathcal{F}_1(\varphi)(x) \leq \exp\left(-|x|^k \frac{c_k}{2k} \int_{\mathbb{R}^d} \varphi(y) \, \mathrm{d}y - \frac{c_k}{2k} \int_{\mathbb{R}^d} |y|^k \varphi(y) \, \mathrm{d}y + D \right). \tag{3.16}$$

We note that, by the definition, $\mathcal{F}_1(\varphi)(x) \geq 0$ for all $x \in \mathbb{R}^d$. From inequality (3.16) we conclude, using finiteness of the integrals, that $\|\mathcal{F}_1(\varphi)\|_\infty \leq e^D$, thus we can assume $\varphi \in L^\infty(\mathbb{R}^d)$. Furthermore, if $\varphi_0 \equiv 0$, then $\mathcal{F}_1(\varphi_0) \equiv e^D$, thus $\mathcal{F}_1(\varphi) \in L^1\left(\mathbb{R}^d, (1+|x|^k)\right)$ as long as we exclude φ_0 from this set.

Assuming continuity of φ, $\mathcal{F}_1(\varphi)$ is obviously continuous from the properties of convolution and composition of functions. Radial decay of $\mathcal{F}_1(\varphi)$, assuming φ is radially decreasing, follows from Lemma 3.6.

To get lower estimate in inequality (3.14), we proceed analogously as in the upper estimate, using Lemma 2.26 to obtain

$$\mathcal{F}_1(\varphi)(x) \geq \exp\left(-|x|^k \frac{C_k}{k} \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y - \frac{C_k}{k} \int_{\mathbb{R}^d} |y|^k \varphi(y)\,\mathrm{d}y + D\right).$$

To show that $\mathcal{F}_1(\varphi) \in C^1(\mathbb{R}^d)$, we consider convolution $\partial W_k/\partial x_i * \varphi$, which for $k > 1$ is continuous because $\partial W_k/\partial x_i = x_i|x|^{k-2}$ is continuous. This convolution has a polynomial bound

$$\left|\frac{\partial W_k}{\partial x_i} * \varphi(x)\right| \leq C_1|x|^{k-1} + C_2 \tag{3.17}$$

for some $C_1, C_2 > 0$, from Lemma 2.26. For $k \in (0,1]$, we show continuity of convolution by the distributive property,

$$\frac{\partial W_k}{\partial x_i} * \varphi = \left.\frac{\partial W_k}{\partial x_i}\right|_{B_R(0)} * \varphi + \left.\frac{\partial W_k}{\partial x_i}\right|_{\mathbb{R}^d \setminus B_R(0)} * \varphi, \tag{3.18}$$

where each term in the above formula is a convolution with either L^1- or L^∞-function, thus continuous. In addition, this convolution is bounded by Lemma 2.3.

Due to the continuity and estimates for the convolution $\partial W_k/\partial x_i * \varphi$, along with the exponential bound (3.14) for the continuous function $\mathcal{F}_1(\varphi)$, partial derivative

$$\frac{\partial \mathcal{F}_1(\varphi)}{\partial x_i} = -\left(\frac{\partial W_k}{\partial x_i} * \varphi\right) \mathcal{F}_1(\varphi) \tag{3.19}$$

is a continuous and bounded function. □

Remark 3.7. We explain the idea of improved regularity on the following example. Consider second derivative of the function $\mathcal{F}_1(\varphi)$,

$$\frac{\partial^2 \mathcal{F}_1(\varphi)}{\partial x_j \partial x_i} = -\left(\frac{\partial W_k}{\partial x_i} * \frac{\partial \varphi}{\partial x_j}\right) \mathcal{F}_1(\varphi) - \left(\frac{\partial W_k}{\partial x_i} * \varphi\right) \frac{\partial \mathcal{F}_1(\varphi)}{\partial x_j}.$$

We assume that $\partial\varphi/\partial x_i \in L^1\big(\mathbb{R}^d, (1+|x|^{\max\{0,k-1\}})\,\mathrm{d}x\big) \cap L^\infty(\mathbb{R}^d)$ and we note, that derivative $\partial\mathcal{F}_1(\varphi)/\partial x_i$ has an exponential bound, analogous to estimate (3.14), which follows from the definition (3.19). Now, one can follow arguments from the proof of Proposition 3.5 to show that $\mathcal{F}_1(\varphi) \in C^2(\mathbb{R}^d)$.

Higher regularity of $\mathcal{F}(\varphi)$ can be shown in an analogous way, assuming more regularity on the function φ. In particular, if $\varphi \in \mathcal{S}(\mathbb{R}^d)$ then $\mathcal{F}(\varphi) \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the space of rapidly decreasing functions on $\mathbb{R}^d$.

Remark 3.8. One can formulate analogous results for the operator arising from the equation (3.9),

$$\mathcal{F}_m(\varphi) = (D - \bar{m} W_k * \varphi)_+^{\frac{1}{m-1}} ,$$

with $m > 1$ and $D > 0$. For Φ defined in (3.13), $\mathcal{F}_m : \Phi \to \Phi$ and there exist numbers $a_1, a_2, b_1, b_2 > 0$, depending on φ, such that

$$\left(a_1 - b_1|x|^k\right)_+^{\frac{1}{m-1}} \leq \mathcal{F}_m(\varphi)(x) \leq \left(a_2 - b_2|x|^k\right)_+^{\frac{1}{m-1}} \tag{3.20}$$

for all $x \in \mathbb{R}^d$. In particular, $\mathcal{F}_m(\varphi)$ is a compactly supported function. On the other hand, assuming $D \leq 0$, $\mathcal{F}_m(\varphi) \equiv 0$ for every $\varphi \in \Phi$.

Improved regularity of the function $\mathcal{F}_m(\varphi)$ with respect to φ depends essentially on $m > 1$. For $m \in (1, 2)$, we take an arbitrary function $\varphi \in \Phi$ and consider the derivative

$$\frac{\partial \mathcal{F}_m(\varphi)}{\partial x_i} = -\frac{1}{m} \left(\frac{\partial W_k}{\partial x_i} * \varphi \right) \mathcal{F}_m(\varphi)^{2-m}. \tag{3.21}$$

From the proof of Proposition 3.5, convolution $\partial W_k / \partial x_i * \varphi$ is a continuous function on $\mathbb{R}^d$ and $\mathcal{F}_m(\varphi)^{2-m}$ is a continuous and compactly supported function, thus $\partial \mathcal{F}_m(\varphi)/\partial x_i \in C(\mathbb{R}^d)$ for all $k > 0$. Formula (3.21) is also valid for $m = 2$, however, this derivative is non-zero only on a compact set, thus $\partial \mathcal{F}_m(\varphi)/\partial x_i$ is bounded, but not necessarily continuous function on $\mathbb{R}^d$.

Let $m > 2$. Function $\mathcal{F}_m(\varphi)^{2-m}$ is unbounded by estimate (3.20), thus we show only Hölder continuity of $\mathcal{F}_m(\varphi)$. We recall a well-known inequality with $t, s \geq 0$ and $m \geq 2$,

$$|t^{\frac{1}{m-1}} - s^{\frac{1}{m-1}}| \leq |t - s|^{\frac{1}{m-1}},$$

which we use, to obtain

$$|\mathcal{F}_m(\varphi)(x) - \mathcal{F}_m(\varphi)(y)| \leq \bar{m}^{\frac{1}{m-1}} \, |W_k * \varphi(x) - W_k * \varphi(y)|^{\frac{1}{m-1}}$$

for $x, y \in \operatorname{supp} \mathcal{F}_m(\varphi)$. To estimate the difference on the RHS of the above inequality, we use multi-variable version of the mean value theorem,

$$|W_k * \varphi(x) - W_k * \varphi(y)| \leq |x - y| \sup_{t \in [0,1]} |\nabla W_k * \varphi((1-t)x + ty)| \, .$$

Recalling estimates (3.17) and (3.18), quantity $|\nabla W_k * \varphi((1-t)x + ty)|$ can be estimated from above by some constant $C > 0$ independent of $x, y \in \operatorname{supp} \mathcal{F}_m(\varphi)$ and $t \in [0, 1]$. Thus, function $\mathcal{F}_m(\varphi)$ is Hölder continuous with exponent $1/(m-1) \in (0, 1)$.

Higher regularity of $\mathcal{F}_m(\varphi)$, with $m \in (1,2)$, can be shown in analogous way as in Remark 3.7, by analysing successive derivatives of $\mathcal{F}_m(\varphi)$ and imposing more regularity on φ.

A standard approach to solve equations (3.7) and (3.9) is to use one of the two following theorems: either Banach or Schauder fixed-point theorem. The essential feature of these two tools is the existence of an invariant set in the neighborhood of the fixed point. In the next section we will show, based on a concrete example, that these theorems may not apply in the case of operators $\mathcal{F}_1$ and $\mathcal{F}_m$.

3.2.2 Case $k = 2$

Now, we describe explicit behavior of the operator $\mathcal{F}_1$ when $k = 2$, and we begin with the following lemma, which is crucial for the analysis.

Lemma 3.9. *Let $d \geq 1$ and $k = 2$. We say that function φ is exponential if there exist some numbers $a, b > 0$, such that $\varphi(x) = a \exp\left(-b|x|^2\right)$. Then for φ exponential, $\mathcal{F}_1(\varphi)$ is also exponential.*

Proof. We calculate the convolution $W_k * \varphi$,

$$\begin{aligned} W_k * \varphi(x) &= \int_{\mathbb{R}^d} \frac{|x-y|^2}{2} \varphi(y)\,\mathrm{d}y \\ &= \frac{1}{2}|x|^2 \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y - \sum_{i=1}^{d} x_i \int_{\mathbb{R}^d} y_i \varphi(y)\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^d} |y|^2 \varphi(y)\,\mathrm{d}y \\ &= \frac{1}{2}|x|^2 \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^d} |y|^2 \varphi(y)\,\mathrm{d}y, \end{aligned}$$

where sum of the integrals is equal to 0 due to the radial symmetry of function φ. Substituting convolution into operator $\mathcal{F}_1$, we obtain the result. □

Now we introduce the following parametrization for an exponential function,

$$\varphi(x) = M^{1+\frac{d}{2}} \left(\frac{d}{2\pi m}\right)^{\frac{d}{2}} \exp\left(-\frac{Md}{2m}|x|^2\right),$$

which satisfies

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x = M \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \varphi(x)\,\mathrm{d}x = m.$$

Therefore, we can represent any exponential function by coordinates $(M, m) \in \mathbb{R}^2$ and moreover, by Lemma 3.9, function

$$\mathcal{F}_1(\varphi)(x) = \exp\left(-\frac{M}{2}|x|^2 - \frac{m}{2} + D\right)$$

can be described in the same manner by pair $(M_F, m_F) \in \mathbb{R}^2$, where

$$\int_{\mathbb{R}^d} \mathcal{F}_1(\varphi)(x)\,\mathrm{d}x = M_F \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \mathcal{F}_1(\varphi)(x)\,\mathrm{d}x = m_F.$$

We conclude this analysis with the definition of operator $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$,

$$F_2(M, m) = e^{-\frac{m}{2}+D} \left(\frac{2\pi}{M}\right)^{\frac{d}{2}} \left(1, \frac{d}{M}\right), \quad D \in \mathbb{R}. \tag{3.22}$$

Before showing existence and properties of the fixed point of operator F_2, we introduce the definition of the *hyperbolic* fixed point.

Definition 3.10. We say that fixed point p of operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ is *hyperbolic*, if eigenvalues of the Jacobian matrix $DF(p)$ satisfy $|\lambda_1| < 1 < |\lambda_2|$.

Hyperbolic fixed points are also known in the literature as *saddle points* and have the following property. There exists a neighborhood V of point p and a non-empty set $V_2 \subset V$ such that for every ball $B_R(p) \subset V$, $B_R(p) \cap V_2 \neq \emptyset$ and for all $x \in B_R(p) \cap V_2$, $F^n(x)$ leaves the ball $B_R(p)$ as n increases. For more details on the properties of the hyperbolic points we refer the reader to, e.g, [71].

Proposition 3.11. *Operator F_2, defined in formula (3.22), has a unique hyperbolic fixed point*

$$p = \left(\exp\left(\frac{-d+2D}{d+2}\right)(2\pi)^{d/(d+2)}, d\right)$$

for every $D \in \mathbb{R}$.

Proof. Existence of the fixed point follows from the equation $F_2(M, m) = (M, m)$. By the Grobman-Hartman theorem, we calculate eigenvalues of the Jacobian matrix $DF_2(p)$,

$$\lambda_1 = \frac{1}{2}\left(-d + \sqrt{d(d+2)}\right) \quad \text{and} \quad \lambda_2 = -\frac{1}{2}\left(d + \sqrt{d(d+2)}\right),$$

where they satisfy condition in Definition 3.10. □

From these considerations, we conclude that regarding operator F_2, there is no invariant closed neighborhood of the fixed point p. This behavior suggests, that either Banach or Schauder fixed-point theorem may also not apply for the operator $\mathcal{F}_1$ with $k > 0$. Below, we present a graphical representation of the set of points $x \in \mathbb{R}^2$ satisfying condition $|p - x| < |p - F_2(x)|$.

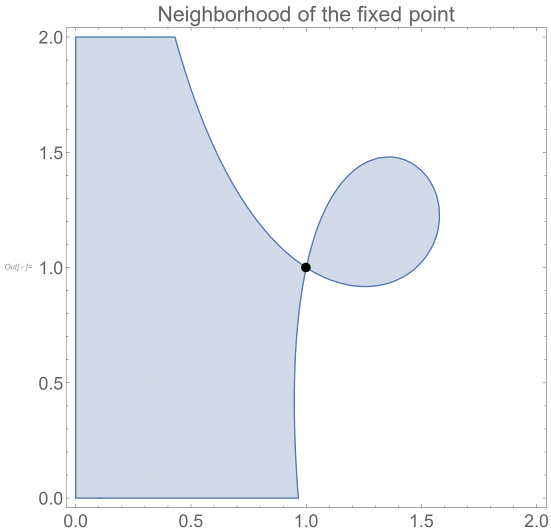

Figure 3.1: Points satisfying inequality $|p - x| < |p - F_2(x)|$ for fixed point $p = (1, 1)$.

Remark 3.12. Based on the numerical examination of the operator F_2, we notice existence of the invariant curves i.e., we suppose that there exist at least two solutions h_1, h_2 to the following functional equation,

$$\begin{cases} F_2(M, h(M)) = (M_h, m_h) \\ m_h = h(M_h). \end{cases}$$

Moreover, h_1 is a contraction around the fixed point, where h_2 manifests the opposite.

3.2.3 Explicit solutions

In this subsection, we show detailed calculations for certain steady states to problem (3.5), which can be expressed by an explicit formula. In the one-dimensional case this problem can be simplified to the following one

$$(u^m)_x + uW_k' * u = 0, \tag{3.23}$$

assuming condition (3.8) or (3.10), depending on the parameter $m \geq 1$. In the following two examples, we solve equation (3.23) directly for $k = 1$ using function $W_1'(x) = \mathrm{sgn}(x)$.

Example 3.13 ($d = 1$, $m = 1$, $k = 1$). We look for a solution $u \in L^1(\mathbb{R})$ to problem (3.23) such that

$$\int_{\mathbb{R}} u(x)\,\mathrm{d}x = M.$$

For such u we can define function v by

$$v(x) = \int_{-\infty}^{x} u(y)\,\mathrm{d}y,$$

then by the definition,

$$\lim_{x\to-\infty} v(x) = 0, \ \lim_{x\to+\infty} v(x) = M \text{ and } v'(x) = u(x).$$

We calculate convolution in equation (3.23) in the following way,

$$\begin{aligned} W_1' * u(x) &= \int_{\mathbb{R}} \operatorname{sgn}(x-y)v'(y)\,\mathrm{d}y \\ &= \int_{-\infty}^{x} v'(y)\,\mathrm{d}y - \int_{x}^{+\infty} v'(y)\,\mathrm{d}y \\ &= 2v(x) - M. \end{aligned} \tag{3.24}$$

Substituting $u = v'$ and $z = 2v - M$ in equation (3.23), we obtain a second-order nonlinear ordinary differential equation

$$z'' + zz' = 0$$

for functions z satisfying

$$\lim_{x\to+\infty} z(x) = -\lim_{x\to-\infty} z(x) = M. \tag{3.25}$$

This problem is well-known as a stationary version of viscous Burgers' equation [22], and can be solved explicitly, where solution is given by the formula

$$Z(x) = \sqrt{2a}\tanh\left(\frac{1}{2}\left(\sqrt{2a}x + b\sqrt{2a}\right)\right),$$

where $a > 0$ and $b \in \mathbb{R}$. For the sake of clarity, we omit the translation term by setting $b = 0$. Notice that $\lim_{x\to+\infty} Z(x) \geq 0$, thus this solution is valid only when $M \geq 0$.

Plugging the substitution back, we obtain a one-parameter family of solutions

$$U_M(x) = \frac{M^2}{4}\operatorname{sech}^2\left(\frac{M}{2}x\right).$$

Function U_M is smooth and symmetric, has scaling $U_M(x) = M^2U_1(Mx)$ consistent with Remark 3.4 and satisfies an exponential decay,

$$\lim_{|x|\to+\infty} U_M(x)\exp(M|x|) = M^2.$$

Example 3.14 ($d = 1$, $m = 2$, $k = 1$). Proceeding analogously as in Example 3.13, we obtain a first-order nonlinear ordinary differential equation

$$(z')^m + 2^{m-2}z^2 = C, \quad C \in \mathbb{R}, \tag{3.26}$$

for functions z satisfying condition (3.25).

This equation can be solved explicitly for $m = 2$ and $C > 0$, where we obtain

$$\arcsin\left(\frac{z(x)}{\sqrt{C}}\right) = x + t,$$

for $-\sqrt{C} \leq z(x) \leq \sqrt{C}$ and $-\pi/2 \leq x+t \leq \pi/2$. Parameter $t \in \mathbb{R}$ can be interpreted as translation, thus we fix $t = 0$. Function z can be extended to a continuous function on the whole domain in the following way,

$$Z(x) = \begin{cases} -\sqrt{C} & x < -\frac{\pi}{2} \\ \sqrt{C}\sin(x) & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \sqrt{C} & x > \frac{\pi}{2}, \end{cases}$$

therefore it satisfies condition at infinity for $\sqrt{C} = M \geq 0$.

Plugging back the substitution, we obtain a family of solutions

$$U_M(x) = \begin{cases} \frac{M}{2}\cos(x) & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 & \text{otherwise}, \end{cases} \tag{3.27}$$

where $U_M \in C(\mathbb{R})$, it is a symmetric function on the whole domain and smooth inside of the support. Moreover, all derivatives are bounded on $\mathbb{R}$, but not necessarily continuous. Notice, that support of a rescaled function $U_M(x) = MU_1(x)$ does not depend on the parameter.

Remark 3.15. When proceeding in Example 3.14, we omitted the case when derivative is negative, i.e., $z' = -\sqrt{C - z^2}$, from which one can obtain the following family of functions,

$$V_M(x) = \begin{cases} -\frac{M}{2}\cos(x) & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 & \text{otherwise}, \end{cases}$$

where $M \geq 0$, which also happens to solve equation (3.23).

According to the literature [77], nonlinear term Δu^m in problem (3.5) is considered only for the non-negative scalar function u, due to the physical motivation. Most used choice for properly defining Δu^m for negative functions is the so-called *Signed PME*, where nonnlinearity is of the form $\Delta(|u|^{m-1}u)$. We notice that function V_M does not satisfy equation (3.23) with such nonlinearity.

Remark 3.16 ($d = 1$, $m > 1$, $k = 1$). Assuming $C > 0$ and $m > 1$, we compute the integral corresponding to equation (3.26) numerically, using *Wolfram Mathematica*, and derive the following implicit equation,

$$C_m w(x)\,{}_2F_1\left(\frac{1}{2}, \frac{1}{m}; \frac{3}{2}; w(x)^2\right) = x + t,$$

where $z = 2^{1-m/2}C^{1/2}w$, ${}_2F_1$ is the Gaussian hypergeometric function [2], $C_m = 2^{1-m/2}C^{(m-2)/2m}$, $|x+t| \leq {}_2F_1(1/2, 1/m; 3/2; 1)C_m$ and $t \in \mathbb{R}$ can be interpreted as a translation.

Let $t = 0$ and

$$g_m(w) = w\,{}_2F_1\left(\frac{1}{2}, \frac{1}{m}; \frac{3}{2}; w^2\right).$$

By the same procedure as in Example 3.14, we calculate $M = C^{1/2}2^{1-m/2}$ and obtain a family of solutions to problem (3.23) in an implicit form,

$$U_M(x) = 2^{-\frac{2}{m}} M^{\frac{2}{m}} \left(g_m^{-1}\right)' \left(2^{1-\frac{m}{2}} M^{\frac{2}{m}-1} x\right),$$

where g_m^{-1} denotes inverse function. This representation is consistent with solution (3.27) and scaling from Remark 3.4.

Figures presented below illustrate numerical approximations of these solutions, and for a detailed explanation of the underlying computation, we refer to the Appendix A.

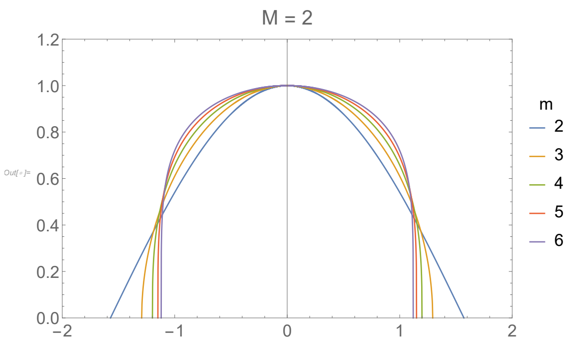

Figure 3.2: Simulations for a fixed $M = 2$ with m as a parameter.

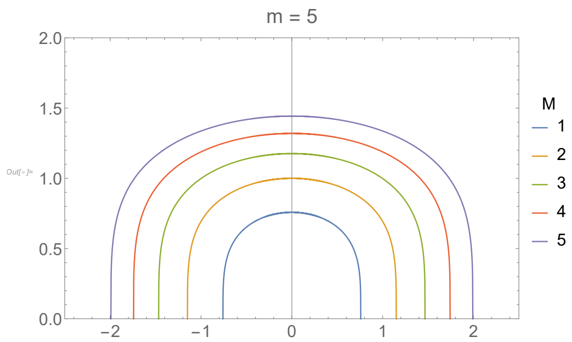

Figure 3.3: Simulations for a fixed $m = 5$ with mass M as a parameter.

In next two examples we solve equation (3.5) in higher dimensions as well, by looking for a solution in a specific form. We use the fact that for $k = 2$ gradient of the convolution kernel W_2 satisfies $\nabla W_2 = (x_1, ..., x_d)$. This approach, in contrast to Proposition 3.11, gives solutions directly, which are independent of constant $D \in \mathbb{R}$.

Example 3.17 ($d \geq 1$, $m = 1$, $k = 2$). We look for a solution in the following form,

$$U(x) = a \exp\left(-b|x|^2\right), \quad a, b > 0, \tag{3.28}$$

and we calculate the convolution $\nabla W_2 * U$. For the i-th variable we have

$$\begin{aligned}\frac{\partial W_2}{\partial x_i} * U(x) =& \int_{\mathbb{R}^d} (x_i - y_i) a \exp\left(-b|y|^2\right) \mathrm{d}y \\ =& x_i \int_{\mathbb{R}^d} a \exp\left(-b|y|^2\right) \mathrm{d}y \\ &- a \int_{\mathbb{R}} y_i \exp\left(-b y_i^2\right) \mathrm{d}y_i \left(\int_{\mathbb{R}} \exp\left(-bt^2\right) \mathrm{d}t \right)^{d-1} \\ =& M x_i,\end{aligned} \tag{3.29}$$

where second integral is equal to zero by the symmetry of function $\exp\left(-b y_i^2\right)$. Calculating the divergence, we obtain

$$\nabla \cdot (U \nabla W_2 * U) = M \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (x_i U(x)) = dMU(x) - 2bMU(x) \sum_{i=1}^{d} x_i^2$$

and for the i-th second derivative, we have

$$\frac{\partial^2 U}{\partial x_i^2} = 4b^2 x_i^2 U(x) - 2bU(x).$$

Substituting these results into equation (3.5), we obtain a system of two equations

$$\begin{cases} 4b^2 - 2bM = 0 \\ dM - 2db = 0, \end{cases}$$

which are equivalent. We conclude that $M = 2b$ and by calculating the mass of the function U,

$$M = \int_{\mathbb{R}^d} a \exp\left(-b|x|^2\right) \mathrm{d}x = a b^{-\frac{d}{2}} \pi^{\frac{d}{2}},$$

we obtain a relation between a, b and M. Thus, for fixed $d \geq 1$, we get a one-parameter family of solutions

$$U_M(x) = (2\pi)^{-\frac{d}{2}} M^{1+\frac{d}{2}} \exp\left(-\frac{M}{2}|x|^2\right), \quad M \geq 0,$$

where U_M is a symmetric smooth function satisfying $U_M(x) = \sqrt{M}^{d+2} U_1(\sqrt{M}x)$.

Remark 3.18. Based on the calculations in (3.29), we note that, assuming U takes the form (3.28), equation (3.5) with $m = 1$ and $k = 2$ resembles the Fokker-Planck equation, for which the existence of stationary solutions in exponential form is well-established (see, e.g., [73]).

Example 3.19 ($d \geq 1$, $m > 1$, $k = 2$). In this case, we look for a solution

$$U(x) = \left(a - b|x|^2\right)_+^{\frac{1}{m-1}}, \quad a, b > 0,$$

where for $m > 2$, function U belongs to the Hölder space $C^{0,1/(m-1)}(\mathbb{R}^d)$. Therefore, we show that U is a weak solution to problem (3.5), i.e., it satisfies

$$\int_{\mathbb{R}^d} \left(\nabla U^m + U\nabla(W_k * U)\right) \cdot \nabla\psi \, \mathrm{d}x = 0$$

for all $\psi \in C_c^\infty(\mathbb{R}^d)$.

Calculations for the convolution are analogous as in Example 3.17, providing $U(\partial W_2/\partial x_i) * U = Mx_iU(x)$, and for the nonlinear diffusion, we have

$$\frac{\partial U^m}{\partial x_i} = -\frac{2b}{\bar{m}} x_i U(x),$$

where $\bar{m} = (m-1)/m$. Comparing both terms, we get $M = 2b/\bar{m}$, and using d-spherical coordinates with adequate substitutions, we obtain

$$M = \int_{\mathbb{R}^d} \left(a - b|x|^2\right)_+^{\frac{1}{m-1}} \, \mathrm{d}x = \frac{1}{2} a^{\frac{1}{m-1}+\frac{d}{2}} b^{-\frac{d}{2}} \sigma_{d,m},$$

where

$$\sigma_{d,m} = \sigma_d B\left(\frac{d}{2}, \frac{1}{\bar{m}}\right),$$

thus constants $a, b > 0$ can be described by the parameter $M > 0$.

After substitution and rewriting the formula, we obtain scaling consistent with Remark 3.4,

$$U_M(x) = M^{\frac{d+2}{\kappa}} U_1\left(M^{\frac{2-m}{\kappa}} x\right),$$

where $\kappa = d(m-1) + 2$ and

$$U_1(x) = \left(2^{2-d}\sigma_{d,m}^{-2}\bar{m}^d\right)^{\frac{1}{\kappa}} \left(1 - \left(2^{-m}\sigma_{d,m}^{m-1}\bar{m}\right)^{\frac{2}{\kappa}} |x|^2\right)_+^{\frac{1}{m-1}}. \tag{3.30}$$

Notice that for $m \in (1, 2)$, $U_1 \in C^1(\mathbb{R}^d)$ and $U_1^m \in C^2(\mathbb{R}^d)$, thus it is the classical solution to problem (3.5).

Remark 3.20. Solutions obtained in Remark 3.16 and Example 3.19 resemble in shape the Barenblatt solution [3]. This is consistent with estimates (3.20) obtained for the operator $\mathcal{F}_m$.

Remark 3.21. It is well-known, that Barenblatt solution converges pointwise on $\mathbb{R}^d$ to the Gaussian kernel as $m \to 1$, for any given mass $M > 0$. Thus, one would expect the same relationship for solutions obtained in Examples 3.17 and 3.19. Indeed, let for simplicity $M = 1$ and we rewrite solution (3.30) as

$$U_m(x) = \left(2^{2-d}\sigma_{d,m}^{-2}\bar{m}^d\right)^{\frac{1}{\kappa}} \left(1 - \frac{\frac{1}{2}|x|^2}{\frac{1}{2}\left(2^{-m}\sigma_{d,m}^{m-1}\bar{m}\right)^{-\frac{2}{\kappa}}}\right)_+^{\frac{1}{m-1}}.$$

To show pointwise convergence, it is sufficient to notice that

$$\lim_{m\to 1} \frac{\Gamma\left(\frac{d}{2}+\frac{1}{\bar{m}}\right)}{\Gamma\left(\frac{1}{\bar{m}}\right)}\bar{m}^{\frac{d}{2}} = 1 \quad \text{and} \quad \lim_{m\to 1} \frac{2\left(2^{-m}\sigma_{d,m}^{m-1}\bar{m}\right)^{\frac{2}{\kappa}}}{m-1} = 1,$$

which follows from the properties of Γ function.

3.2.4 Integral equation

Recall equation (3.9) with $d = 1$ and $D > 0$. We assume that solution u to this equation is a symmetric, compactly supported function, thus by substitution $u^{m-1} = v$, we obtain

$$v(x) = D - \frac{\bar{m}}{k}\int_{-R}^{R} |x-y|^k v(y)^{\frac{1}{m-1}}\,\mathrm{d}y, \tag{3.31}$$

where $k > 0$, $\bar{m} = (m-1)/m$ and $R > 0$. Such equation is well-known in the literature as the nonlinear Fredholm integral equation of the second kind [78]. Existence of solutions to problem (3.31) with $m \in (1, 2]$, can be shown by the Banach fixed point theorem.

We are particularly interested in the description and numerical simulation of solutions to equation (3.31) in some cases. Assuming $k = 2l$, $l \in \mathbb{N}^+$ and $m > 1$ such that $1/(m-1) = n \in \mathbb{N}^+$, we can rewrite equation (3.31) as

$$v(x) = D - \frac{1}{2l(n+1)}\sum_{i=0}^{l} c_{2i}x^{2i},$$

where

$$c_{2i} = \int_{-R}^{R} v(y)^n y^{2l-2i}\,\mathrm{d}y.$$

Moreover, we assume $v(-R) = v(R) = 0$ and note that

$$c_{2l} = \int_{-R}^{R} v(y)^n\,\mathrm{d}y = \int_{-R}^{R} u(y)\,\mathrm{d}y,$$

thus we set $c_{2l} = M$.

We conclude these considerations by formulating problem (3.31) in terms of a system of l-nonlinear equations on the coefficients c_{2i} and number $R > 0$,

$$\begin{cases} v(x) = \dfrac{1}{2l(n+1)} \displaystyle\sum_{i=1}^{l} c_{2i} \left(R^{2i} - x^{2i}\right) \\ c_{2i} = \displaystyle\int_{-R}^{R} v(y)^n y^{2l-2i} \,\mathrm{d}y, \quad i \in \{1, ..., l-1\} \\ c_{2l} = M, \end{cases} \tag{3.32}$$

which we solve numerically for $M = 1$ in some cases, and present the results below. For the code solving this system, we refer to the Appendix A.

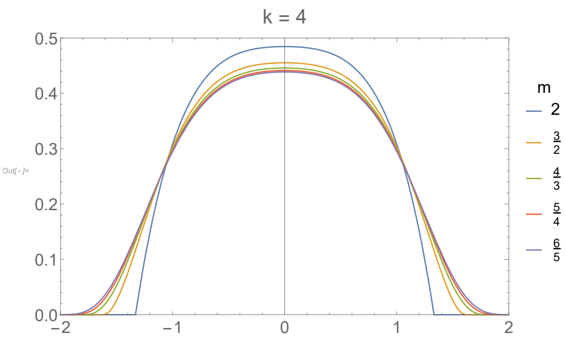

Figure 3.4: Simulations for a fixed $k = 4$ with m as a parameter.

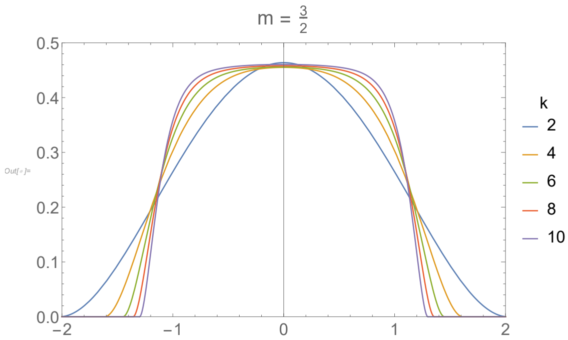

Figure 3.5: Simulations for a fixed $m = \frac{3}{2}$ with k as a parameter.

Remark 3.22. This approach can be extended to a two-dimensional case in the same manner. Moreover, one can consider more general class of kernels in the form $K(x, y) = \sum_{l=1}^{n} c_l g_l(x) h_l(y)$, which are known as *separable* kernels.

3.3 Sign-changing solution

In this section, we show derivation of a sign-changing solution to problem (3.1) for $d = m = k = 1$. For a sufficiently regular $u \in L^1(\mathbb{R})$, such that

$$\int_{\mathbb{R}} u(t, x) \,\mathrm{d}x = 0,$$

we recall procedure from Example 3.13 and obtain the viscous Burger's equation

$$z_t + z z_x = z_{xx},$$

where $u(t, x) = -z_x(t, -x)/2$ and $\lim_{x \to \pm\infty} z(t, x) = 0$ for $t > 0$. Existence of a sign changing solution for this equation is well-known (see, e.g., [79]), where it is obtained by the Hopf-Cole transformation. In the literature, it is referred to as N-wave solution, due to the characteristic shape of the letter N, and defined by

$$N(t,x) = \frac{x}{t}\left(1 + \sqrt{t}\exp\left(\frac{x^2}{4t}\right)\right)^{-1}.$$

Substituting this formula into the relation between u and z, we get

$$n(t,x) = -\frac{1}{2t}\left(1 + \sqrt{t}\exp\left(\frac{x^2}{4t}\right)\right)^{-1} + \frac{x^2}{4t^{\frac{3}{2}}}\exp\left(\frac{x^2}{4t}\right)\left(1 + \sqrt{t}\exp\left(\frac{x^2}{4t}\right)\right)^{-2},$$

and we note that scaling resulting from Lemma 3.1 do not apply in this case. Moreover, $n(t,x) \to 0$ pointwise on $\mathbb{R}$, as $t \to +\infty$. For a description of a work plan considering this problem, we refer the reader to Section 1.3. Here, we include plot of the evolution in time of solution n.

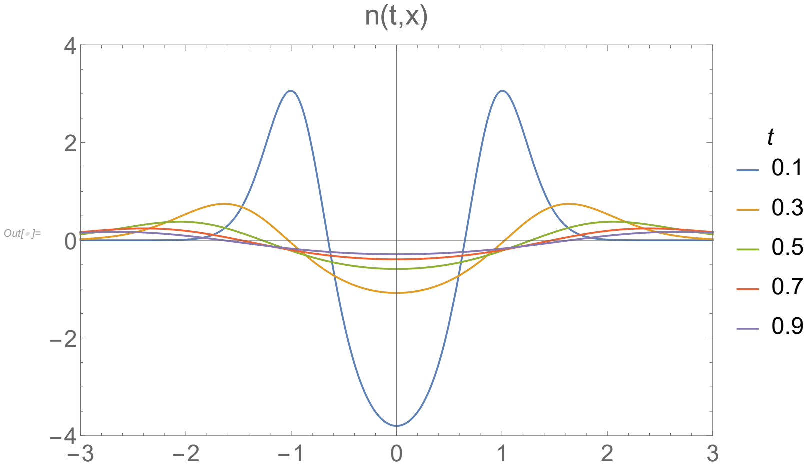

Figure 3.6: Evolution in time of the sing-changing solution n to problem (3.1).

Chapter 4

Chemotaxis model in the uniformly local L^p-spaces

4.1 Main results

This chapter is devoted to the results from the already published paper [42] of which summary we present below. For a detailed description of the author's main contribution to this publication, we refer the reader to Section 4.2.

Our goal is to study properties of solutions to the Cauchy problem for the simplified parabolic–elliptic Keller–Segel model of chemotaxis

$$\begin{cases} u_t - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, \ x \in \mathbb{R}^d, \\ -\Delta \psi + \psi = u, & t > 0, \ x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

with $d \geq 1$. We solve the second equation with respect to ψ to obtain $\psi = K * u$, where K is the Bessel function, described in the following lemma. In this result and throughout this chapter, we use standard definition of the Fourier transform,

$$\widehat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, \mathrm{d}x.$$

Lemma 4.1. *Denote by $\psi \in \mathcal{S}'(\mathbb{R}^d)$ a solution to the equation $-\Delta \psi + \psi = u$ for some $u \in \mathcal{S}'(\mathbb{R}^d)$ (the space of tempered distributions), then the following statements hold true.*

i) $\psi = K * u$, *where* $\widehat{K}(\xi) = \frac{1}{1+|\xi|^2}$.

ii) *For* $d = 1$, $K(x) = \frac{1}{2} e^{-|x|}$.

iii) *For* $d \geq 2$, $K \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ *for each* $p \in \left[1, \frac{d}{d-2}\right)$ *and* $\nabla K \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ *for each* $q \in \left[1, \frac{d}{d-1}\right)$.

iv) $|\nabla K(x)| = |K'(|x|)| = -K'(|x|)$ *is radially symmetric and decreasing in* $|x|$.

Thus, we can reduce problem (4.1) to the following one

$$\begin{cases} u_t - \Delta u + \nabla \cdot (u \nabla K * u) = 0, & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{4.2}$$

We begin by a result on the existence of local-in-time solutions to problem (4.2) in the uniformly local Lebesgue spaces $L^p_{\text{uloc}}(\mathbb{R}^d)$, which are defined as

$$L^p_{\text{uloc}}(\mathbb{R}^d) \equiv \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^d) : \|f\|_{p,\text{uloc}} \equiv \sup_{x \in \mathbb{R}^d} \left(\int_{B_1(x)} |f(y)|^p \,\mathrm{d}y \right)^{1/p} < +\infty \right\}$$

for $p \in [1, \infty)$ and $L^\infty_{\text{uloc}}(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$. In fact, there is an alternative definition of the norm in the uniformly local L^p-spaces,

$$\|f\|_{p,\text{uloc},\rho} \equiv \sup_{x \in \mathbb{R}^d} \left(\int_{B_\rho(x)} |f(y)|^p \,\mathrm{d}y \right)^{1/p}$$

for each $\rho > 0$, but by a simple scaling property one can show that these norms are in fact equivalent.

Theorem 4.2. *For each* p *satisfying*

$$\begin{aligned} p &\in [1, \infty] \quad \text{if} \quad d = 1, \\ p &\in \left[\frac{3}{2}, \infty\right] \quad \text{if} \quad d = 2, \\ p &\in \left(\frac{d}{2}, \infty\right] \quad \text{if} \quad d \geq 3, \end{aligned} \tag{4.3}$$

and every $u_0 \in L^p_{\text{uloc}}(\mathbb{R}^d)$*, there exists* $T > 0$ *and a unique mild solution*

$$u \in L^\infty\big([0,T), L^p_{\text{uloc}}(\mathbb{R}^d)\big) \cap C\big((0,T), L^p_{\text{uloc}}(\mathbb{R}^d)\big)$$

of problem (4.2)*. Moreover, if* $u_0 \geq 0$*, then* $u(t,x) \geq 0$ *almost everywhere in* $[0,T) \times \mathbb{R}^d$.

The standard proof of Theorem 4.2 is based on the Banach contraction principle (see Proposition 2.5) applied to an integral representation of solutions to problem (4.2), namely

$$u(t) = e^{t\Delta} u_0 - \int_0^t \nabla e^{(t-s)\Delta} \cdot (u(s) \nabla K * u(s)) \,\mathrm{d}s.$$

Constraints imposed on the exponent p in Theorem 4.2 result from the estimation of the nonlinear term in the space $L^p_{\text{uloc}}(\mathbb{R}^d)$.

This restriction on p is not needed in the paper [76], where the solution is constructed in a smaller space

$$\mathcal{L}^p_{\text{uloc}}(\mathbb{R}^d) = \overline{BUC(\mathbb{R}^d)}^{\|\cdot\|_{p,\text{uloc}}},$$

where $BUC(\mathbb{R}^d)$ is a space of all bounded uniformly continuous functions on $\mathbb{R}^d$. Solutions constructed in this space have improved spatial regularity and are also continuous at time $t = 0$ (see, e.g., [69, Proposition 2.2]), in contrast to the ones obtained in Theorem 4.2. Notice that mapping

$$t \mapsto e^{t\Delta} f \in L^\infty\big((0, \infty), L^p_{\text{uloc}}(\mathbb{R}^d)\big) \text{ for each } f \in L^p_{\text{uloc}}(\mathbb{R}^d)$$

is in general not continuous. However, one can prove continuity in the weak sense i.e., showing that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ the following mapping

$$t \mapsto \int_{\mathbb{R}^d} e^{t\Delta} f(x)\varphi(x)\,\mathrm{d}x$$

is continuous.

Next, we study a solution $u = u(t, x)$ of problem (4.2) which is a perturbation of the constant stationary solution $A \in \mathbb{R}$. Thus, the function

$$v(t, x) = u(t, x) - A$$

satisfies the following nonlinear problem.

$$\begin{cases} v_t - \Delta v + A\Delta K * v + \nabla \cdot (v\nabla K * v) = 0, & t > 0,\ x \in \mathbb{R}^d, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{4.4}$$

In fact, we consider a mild solution to this problem satisfying the integral equation

$$v(t) = S_A(t)v_0 - \int_0^t \nabla S_A(t-\tau) \cdot (v(\tau)\nabla K * v(\tau))\,\mathrm{d}\tau,$$

where the semigroup $\{S_A(t)\}_{t\geq 0}$ is studied in Section 4.2. In particular, this semigroup commutes with $\nabla\cdot$ by the same reasoning as for heat semigroup, i.e., we have a relation

$$\nabla e^{t\Delta} v_0 = e^{t\Delta}\nabla v_0$$

which can be justified by applying Fourier transform. Notice that this relation is usually not valid in a bounded domain with a suitable boundary condition.

In this setting one can show existence and uniqueness of the local-in-time mild solution to problem (4.4) for $v_0 \in L^p(\mathbb{R}^d)$, in the space $C\big([0,T), L^p(\mathbb{R}^d)\big)$, for p satisfying condition (4.5). The following corollary is an immediate consequence of the uniqueness of solutions established in Theorem 4.2 combined with the uniqueness result of solutions to the perturbed problem (4.4).

Corollary 4.3. *Let p satisfy conditions (4.3). For every $A \in \mathbb{R}$ and every $v_0 \in L^p(\mathbb{R}^d)$ there exists a unique local-in-time mild solution $u = u(x,t)$ of problem (4.2) (as stated in Theorem 4.2) corresponding to the initial datum $u_0 = A + v_0 \in L^p_{\text{uloc}}(\mathbb{R}^d)$. This solution satisfies $u - A \in C\big([0,T), L^p(\mathbb{R}^d)\big)$.*

Next, we show that one can construct global-in-time solutions around each constant solution $A \in [0,1)$.

Theorem 4.4. *Let $A \in [0,1)$ and let p satisfy condition*

$$p = 1 \text{ if } d = 1 \quad \text{or} \quad p \in \left(\frac{d}{2}, d\right] \text{ if } d \geq 2. \tag{4.5}$$

Fix $q \in (d, 2p]$. There exists $\varepsilon > 0$ such that for every $v_0 \in L^p(\mathbb{R}^d)$ with $\|v_0\|_p < \varepsilon$, problem (4.2) with the initial condition $u_0 = A + v_0$ has a unique global-in-time mild solution $u(x,t)$ satisfying $u - A \in C\big([0,\infty), L^p(\mathbb{R}^d)\big)$ and

$$\|u(t) - A\|_p + t^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|u(t) - A\|_q \leq C \|u_0 - A\|_p$$

for a constant $C > 0$ independent of t and all $t > 0$.

The estimate from Theorem 4.4 can be interpreted as a stability of each constant solution $A \in [0,1)$ in $L^p(\mathbb{R}^d)$ and asymptotic stability in $L^q(\mathbb{R}^d)$.

The smallness assumption imposed on initial conditions in Theorem 4.4 seems to be necessary. This is clear in the case $A = 0$, where sufficiently large initial data lead to solutions which blow-up in finite time, see, e.g., [14, 40, 55, 59] for blow-up results for solutions for system (1.7) considered in the whole space.

Next, we deal with case $A > 1$ which appears to be the unstable constant stationary solution.

Theorem 4.5. *The constant stationary solution $A > 1$ of problem (4.2) is not stable in the Lyapunov sense under small perturbations from $L^p(\mathbb{R}^d)$ for each p satisfying condition (4.3) except $p = 1$ and $p = \infty$.*

In this theorem, we do not claim that solutions corresponding to L^p-perturbations of $A > 1$ are global-in-time. We show only that if they are global-in-time then they cannot be stable.

To conclude this overview, we notice that a constant $A < 0$ is linearly stable which we comment in Remark 4.14, below. The proof of nonlinear stability of this constant can be obtained by the method used in the proof of Theorem 4.4. We add some comments on the linear stability of the constant solution $A = 1$ in Remark 4.15, below.

4.2 Linearized problem

We consider problem (4.4) without nonlinear part, namely

$$\begin{cases} v_t - \Delta v + A\Delta K * v = 0, & t > 0,\ x \in \mathbb{R}^d, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $A \in \mathbb{R}$ is an arbitrary constant and operator $\Delta - A\Delta K*$ can be expressed by the Fourier transform as follows

$$(\Delta\varphi - A\Delta K * \varphi)\widehat{\ }(\xi) = \left(-|\xi|^2 + A\frac{|\xi|^2}{1+|\xi|^2}\right)\widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^d. \tag{4.6}$$

We begin by presenting preliminary properties of this operator.

Lemma 4.6. *There exists a constant $L > 0$ such that for each $p \in [1,\infty]$,*

$$\|-\Delta K * v\|_p \le L\|v\|_p \quad \textit{for all} \quad v \in L^p(\mathbb{R}^d). \tag{4.7}$$

Sketch of the proof. From identity (4.6), we conclude that the action of the operator $-\Delta K*$ can be represented by convolution with a finite measure on $\mathbb{R}^d$, described by a function $1 - 1/(1+|\xi|^2)$ in the Fourier variable. For more details, we refer the reader to e.g., [74, Lemma 2.(i), p.133].

Lemma 4.7. *For each $A \in \mathbb{R}$, a closure in $L^p(\mathbb{R}^d)$ of the operator $\Delta - A\Delta K*$ generates an analytic semigroup $\{S_A(t)\}_{t\ge 0}$ on $L^p(\mathbb{R}^d)$ for every $p \in [1,\infty)$. This semigroup is defined by the Fourier transform by the formula*

$$\widehat{S_A(t)v_0}(\xi) = \widehat{\mu}_A(t,\xi)\widehat{v_0}(\xi),$$

where

$$\widehat{\mu}_A(t,\xi) = e^{-t\left(|\xi|^2 - A\frac{|\xi|^2}{1+|\xi|^2}\right)}. \tag{4.8}$$

Proof. It is well-known that Laplacian generates an analytic semigroup of linear operators on $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty)$ and bounded perturbation of such operator maintains the same property (see, e.g., [43, Example 4.10, p.107 and Theorem 2.10, p.176]). The Fourier representation of this semigroup can be obtained by routine calculations. □

Lemma 4.8. *Assume that $A \in \mathbb{R}$ and choose the constant L from inequality (4.7). Then for each $1 \le q \le p \le \infty$, there exists a constant $C = C(d, p, q) > 0$ such that*

$$\|S_A(t)v_0\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}e^{|A|Lt}\|v_0\|_q$$

and

$$\|\nabla S_A(t)v_0\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}e^{|A|Lt}\|v_0\|_q$$

for all $t > 0$ and $v_0 \in L^q(\mathbb{R}^d)$.

Proof. Here, we use the notation $S_A(t)v_0 = e^{t\Delta}(e^{-A\Delta K*}v_0)$. Using the L^p-L^q estimates of the heat semigroup (see Lemma 2.2) and Lemma 4.6, we obtain

$$\|e^{t\Delta}(e^{-tA\Delta K*}v_0)\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|e^{-tA\Delta K*}v_0\|_q \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}e^{|A|Lt}\|v_0\|_q.$$

The proof for the second inequality is analogous. □

The following theorem improves estimates from Lemma 4.8 in the case of $A \in [0, 1)$ and it plays a crucial role in the proof of stability of constant solutions to problem (4.2).

Theorem 4.9. *Assume that $A \in [0, 1)$. For all exponents satisfying $1 \le q \le p \le \infty$ there exist constants $C = C(d, A, p, q) > 0$ such that*

$$\|S_A(t)v_0\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|v_0\|_q \tag{4.9}$$

and

$$\|\nabla S_A(t)v_0\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}\|v_0\|_q \tag{4.10}$$

for all $t > 0$ and $v_0 \in L^q(\mathbb{R}^d)$.

The proof of this theorem is based on the following lemmas.

Lemma 4.10. *Let $\widehat{D^N v}(\xi) \equiv |\xi|^N \widehat{v}(\xi)$ for all $N \in \mathbb{R}$. For all $v \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class of smooth rapidly decreasing functions) and for every $N > \frac{d}{2}$, the following inequality holds*

$$\|v\|_1 \le C\|\widehat{v}\|_2^{1-\frac{d}{2N}}\|D^N\widehat{v}\|_2^{\frac{d}{2N}},$$

with a constant $C = C(d, N) > 0$.

Proof. We calculate the L^1-norm of the function v in the following way for some $R > 0$,

$$\begin{aligned}\|v\|_1 =& \int_{|x|\le R} |v(x)|\,\mathrm{d}x + \int_{|x|>R} |v(x)|\,\mathrm{d}x \\ \le& \left(\int_{|x|\le R} \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{|x|\le R} |v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &+ \left(\int_{|x|>R} |x|^{-2N}\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{|x|>R} |x|^{2N}|v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ \le& |\mathbb{S}^{d-1}|^{\frac{1}{2}}(R^{\frac{d}{2}}\|v\|_2 + R^{\frac{d}{2}-N}\|D^N\widehat{v}\|_2).\end{aligned}$$

Taking $R = \left(\frac{2N-d}{d}\frac{\|D^N\widehat{v}\|_2}{\|\widehat{v}\|_2}\right)^{1/N}$ we obtain the result. □

Lemma 4.11. *Let $D^N v$ be defined as before. For all $v \in \mathcal{S}(\mathbb{R}^d)$ and for every $N \in \mathbb{N}$, the following inequality holds*

$$\|D^N v\|_2^2 \le C \sum_{|\alpha|=N} \|\partial^\alpha v\|_2^2,$$

with a constant $C = C(N) > 0$, where α denotes a multi-index.

Proof. By the Plancherel formula and the definition of $D^N v$,

$$\|D^N v\|_2^2 = \|\widehat{D^N v}\|_2^2 = \int_{\mathbb{R}^d} |\xi|^{2N}|\widehat{v}(\xi)|^2\,d\xi = \int_{\mathbb{R}^d} (\xi_1^2 + \dots + \xi_n^2)^N|\widehat{v}(\xi)|^2\,d\xi,$$

where for $N \in \mathbb{N}$ last integral is equivalent to the sum of integrals

$$\sum_{k_1+\dots+k_n=N} \binom{N}{k_1, \dots, k_n} \int_{\mathbb{R}^d} \prod_{j=1}^{d} (\xi_j^2)^{k_j}|\widehat{v}(\xi)|^2\,d\xi.$$

Using multi-index notation with multi-index α and once again Plancherel formula, we obtain

$$\|D^N v\|_2^2 \le C \sum_{|\alpha|=N} \int_{\mathbb{R}^d} (\xi^\alpha)^2|\widehat{v}(\xi)|^2\,d\xi = C \sum_{|\alpha|=N} \|\xi^\alpha\widehat{v}\|_2^2 = C \sum_{|\alpha|=N} \|\partial^\alpha v\|_2^2,$$

with some constant $C > 0$. □

Lemma 4.12. *Assume that $A \in [0, 1)$. For the function $\widehat{\mu}_A$ defined by formula (4.8) and for every multi-index α with $|\alpha| = N$, $N \in \mathbb{N}$, there exists a constant $C = C(d, A, N) > 0$ such that*

$$\|\partial_\xi^\alpha \widehat{\mu}_A(t)\|_2^2 \le Ct^{N-\frac{d}{2}} \quad \textit{for all} \quad t \ge 1.$$

Proof. For $N = 0$, by the inequality $|\xi|^2/(1+|\xi|^2) \le |\xi|^2$ ($\xi \in \mathbb{R}^d$), we obtain

$$\|\widehat{\mu}_A(t)\|_2^2 = \int_{\mathbb{R}^d} e^{-2t|\xi|^2 + 2At\frac{|\xi|^2}{1+|\xi|^2}}\, \mathrm{d}\xi \le \int_{\mathbb{R}^d} e^{-2t(1-A)|\xi|^2}\, \mathrm{d}\xi = Ct^{-\frac{d}{2}}.$$

For $N \ge 1$, we introduce the C^∞-function $h(\xi) \equiv |\xi|^2 - A|\xi|^2/(1+|\xi|^2)$ which satisfies estimates $|\partial_{\xi_j} h(\xi)| \le C|\xi|$ and $|\partial_\xi^\beta h(\xi)| \le C$ for every j, $1 \le j \le d$, and multi-index β with $|\beta| \ge 2$. We use the multivariate Faà di Bruno's formula (see, e.g., [47, formula (4)]),

$$\partial_\xi^\alpha e^{-th(\xi)} = e^{-th(\xi)} \sum_{k=1}^{N} (-t)^k H_k(\xi),$$

where $H_k(\xi)$ is a sum of products of partial derivatives of the function $h(\xi)$ satisfying $|H_k(\xi)| \le C(1+|\xi|^k)$. We prove the following inequality by induction in $N \in \mathbb{N}^+$,

$$\left|\partial_\xi^\alpha e^{-th(\xi)}\right| \le Ce^{-th(\xi)} \sum_{k-\frac{\ell}{2} \le \frac{N}{2}} t^k \left(1+|\xi|^\ell\right).$$

For $N = 1$, by straightforward calculation and properties of function $h(\xi)$,

$$\left|\partial_{\xi_j} e^{-th(\xi)}\right| \le e^{-th(\xi)} t |\partial_{\xi_j} h(\xi)| \le Ce^{-th(\xi)} t\left(1+|\xi|\right).$$

We show the induction step for $N+1$,

$$\begin{aligned}
\left|\partial_{\xi_j}\partial_\xi^\alpha e^{-th(\xi)}\right| &\le t\left|\partial_{\xi_j} h(\xi)\right| \left|\partial_\xi^\alpha e^{-th(\xi)}\right| + \left| e^{-th(\xi)} \sum_{k=1}^{N} (-t)^k \partial_{\xi_j} H_k(\xi)\right| \\
&\le Ct|\xi|e^{-th(\xi)} \sum_{k-\frac{\ell}{2}\le\frac{N}{2}} t^k\left(1+|\xi|^\ell\right) + Ce^{-th(\xi)} \sum_{k-\frac{\ell}{2}\le\frac{N}{2}} t^k \left(1+|\xi|^\ell\right) \\
&\le Ce^{-th(\xi)} \sum_{k-\frac{\ell}{2}\le\frac{N+1}{2}} t^k\left(1+|\xi|^\ell\right),
\end{aligned}$$

which holds true because $|\partial_{\xi_j} H_k(\xi)|$ satisfies the same estimate as $|H_k(\xi)|$ by the properties of function $h(\xi)$, and $k+1-(\ell+1)/2 \le (N+1)/2$.

We group coefficients t^k and $|\xi|^\ell$ in the following way: $t^k|\xi|^\ell t^{k-\ell/2}|\sqrt{t}\xi|^\ell$. Thus, by the assumption $t \ge 1$ and induction, we have $t^{k-\ell/2} \le t^{N/2}$. We obtain an estimate

$$|\partial_\xi^\alpha \widehat{\mu}_A(t)| \le Ct^{\frac{N}{2}} P(|\sqrt{t}\xi|) e^{-t|\xi|^2 + At\frac{|\xi|^2}{1+|\xi|^2}},$$

where $P(s)$ is a polynomial of degree N. By the same inequality as in the case $N = 0$ and properties of the exponential function,

$$|\partial_\xi^\alpha \widehat{\mu}_A(t)| \le t^{\frac{N}{2}} P(|\sqrt{t}\xi|) e^{-t(1-A)|\xi|^2} \le Ct^{\frac{N}{2}} e^{-\delta t|\xi|^2}, \tag{4.11}$$

for some $\delta \in (0, 1-A)$. Calculating the L^2-norm of both sides of inequality (4.11) we obtain the result. □

Lemma 4.13. *Assume that $A \in [0,1)$. For every $p \in [1,\infty]$ there exists a constant $C > 0$ such that*

$$\|S_A(t)v_0\|_p \le C\|v_0\|_p,$$

for all $t \ge 0$ and all $v_0 \in L^p(\mathbb{R}^d)$.

Proof. For $t \in [0,1]$ this is an immediate consequence of Lemma 4.8. For $t \ge 1$, function $\mu_A(t)$ is a Schwartz function in the variable ξ, and thus in the variable x. Therefore, by the properties of the Fourier transform, action of the operator $S_A(t)$ on the function v_0 can be expressed as a convolution with the function $\mu_A(t)$, and by the Young inequality,

$$\|S_A(t)v_0\|_p = \|\mu_A(t) * v_0\|_p \le \|\mu_A(t)\|_1\|v_0\|_p.$$

In order to estimate $\|\mu_A(t)\|_1$ we use Lemma 4.10, Lemma 4.11 and Lemma 4.12 with $N > d/2$ to obtain

$$\begin{aligned}\|\mu_A(t)\|_1 &\le C\|\widehat{\mu}_A(t)\|_2^{1-\frac{d}{2N}}\|D^N\widehat{\mu}_A(t)\|_2^{\frac{d}{2N}} \\ &\le C\|\widehat{\mu}_A(t)\|_2^{1-\frac{d}{2N}}\left(\sum_{|\alpha|=N}\|\partial^\alpha\widehat{\mu}_A(t)\|_2^2\right)^{\frac{d}{4N}} \\ &\le C\left(t^{-\frac{d}{4}}\right)^{1-\frac{d}{2N}}\left(t^{N-\frac{d}{2}}\right)^{\frac{d}{4N}} \\ &= C\end{aligned}$$

for all $t \ge 1$. □

Proof of Theorem 4.9. We begin with inequality (4.9) taking $\varepsilon \in (A,1)$. The following formula can be justified on the level of Fourier transform, thus by the standard heat semigroup estimates (see Lemma 2.2),

$$\|S_A(t)v_0\|_p = \|e^{(1-\varepsilon)t\Delta}(e^{\varepsilon t\Delta - tA\Delta K*}v_0)\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|e^{\varepsilon t\Delta - tA\Delta K*}v_0\|_q,$$

where constant $C > 0$ depends on ε but not on $t > 0$. Now we substitute $\tilde{t} = \varepsilon t$ to obtain $tA = \tilde{t}(A/\varepsilon) = \tilde{t}\tilde{A}$, where $0 < \tilde{A} < 1$. Thus, by Lemma 4.13,

$$\|S_A(t)v_0\|_p \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|e^{\tilde{t}\Delta - \tilde{t}\tilde{A}\Delta K*}v_0\|_q \le Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|v_0\|_q.$$

Using the formula

$$\nabla S_A(t)v_0 = \nabla e^{(1-\varepsilon)t\Delta}(e^{\varepsilon t\Delta - tA\Delta K*}v_0),$$

we prove analogously inequality (4.10). □

Remark 4.14. The L^p-L^q estimates (4.9)-(4.10) of the semigroup $\{S_A(t)\}_{t\geq 0}$ hold true for $A < 0$ as well. They can be proved by the same reasoning as above using the obvious inequality

$$e^{-t\left(|\xi|^2 - A\frac{|\xi|^2}{1+|\xi|^2}\right)} \leq e^{-t|\xi|^2} \quad \text{for each } A < 0.$$

Remark 4.15. For the completeness of this work, we notice that a constant solution $A = 1$ is linearly stable in $L^2(\mathbb{R}^d)$. Indeed, since $e^{-t(|\xi|^2 - |\xi|^2/(1+|\xi|^2))} \leq 1$ for all $\xi \in \mathbb{R}^d$ and $t \geq 0$, by the Plancherel formula and Hölder inequality we obtain

$$\|S_1(t)v_0\|_2 = \|\widehat{\mu}_1(t)\widehat{v_0}\|_2 \leq \|\widehat{\mu}_1(t)\|_\infty \|\widehat{v_0}\|_2 \leq \|v_0\|_2$$

for all $v_0 \in L^2(\mathbb{R}^d)$. We skip a discussion of the stability of this constant solution for $p \neq 2$.

Appendix A

Numerical results

A.1 Stationary solutions from Remark 3.16

Due to the difficulty of obtaining the inverse function g_m^{-1} in Remark 3.16 directly, we proceed as follows. Let $C_1 = 2^{-2/m}M^{2/m}$ and $C_2 = 2^{1-m/2}M^{2/m-1}$. We can represent solution U_M as a parametric function $(x, U_M(x)) \in \mathbb{R}^2$, with $x \in \mathbb{R}$ satisfying $|C_2 x| \leq {}_2F_1(1/2, 1/m; 3/2; 1)$. For fixed x, using the inverse function theorem, we obtain

$$(x, U_M(x)) = \left(x, \frac{C_1}{g_m'(g_m^{-1}(C_2 x))}\right) = \left(\frac{g_m(w)}{C_2}, \frac{C_1}{g_m'(w)}\right),$$

where $w \in [-1, 1]$ satisfy $g_m(w) = C_2 x$.

We approximate the solution by calculating values in these points for certain w, and visualize it by connecting them. Below, we provide an implementation of this approach, where the parameter n is responsible for the accuracy of the approximation (the higher – the better).

```
SteadyStateList[m_, M_, n_] := Module[
{$gm, $dgm, $C1, $C2, $S, $R, $T, $x},

(*definition of hypergeometric function*)
$gm[w_, m] = w*Hypergeometric2F1[0.5, 1/m, 1.5, w^2];
$dgm[w_] = D[$gm[w, m], w];

(*definition of scaling constants*)
$C1 = 4^(-1/m) M^(2/m);
$C2 = 2^((-2 + m)/m) M^(-1 + 2/m);
$S = $C1/$dgm[0];
$R = $gm[1, m]/$C2;

(*definition of point representation*)
$x = Join[{-1}, Table[-(1 - w/n)^(m - 1), {w, n}],
```

```
Table[(w/n)^(m - 1), {w, n}]];
$T = Table@Evaluate[{$gm[w, m]/$C2, $C1/$dgm[w]}, {w, $x}];
$T
]
```

A.2 Stationary solutions from Subsection 3.2.4

The following code is a direct application of the procedure described in Subsection 3.2.4.

```
SteadyState[n_, k_,M_] := Module[
    {l, c, X, v, System, Restrictions, Solutions, cS, RS, XS, u},

    (*definition of function v*)
    l = k/2;
    c = ReplacePart[Table[Symbol["c" <> ToString@(2*i)], {i, l}], l -> M];
    X = Table[R^(2*i) - x^(2*i), {i, l}];
    v[x] = 1/(2*l*(n + 1))*c.X;

    (*computation*)
    System = Table[Equal[c[[i]],
        Integrate[v[x]^n*x^(2*l - 2*i), {x, -R, R}]], {i, l}];
    Restrictions = Table[Greater[c[[i]], 0], {i, l - 1}];
    Solutions = NSolve[Join[System, Restrictions], Reals];
    cS = Join[(c[[1 ;; (l - 1)]] /. Solutions)[[1]], {1}];
    RS = (R /. Solutions)[[1]];
    XS = Table[RS^(2*i) - x^(2*i), {i, l}];

    (*result and validation*)
    {u[x] = (1/(2*l*(n + 1))*cS.XS)^n*Boole[Abs[x] <= RS],
        Integrate[u[x], {x, -RS, RS}]}
]
```

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