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O ciągach miar i ideałach w algebrach Boole'a

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On sequences of measures and ideals in Boolean algebras

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To my parents
Basia & Sławek

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Chapter 1

Introduction

Studying properties of ideals and filters in Boolean algebras, in particular in the algebra $\mathcal{P}(\omega)$ of subsets of the set of natural numbers, forms a rich branch of combinatorial set theory (see e.g. a survey by Hrušák [53] or a monograph by Sikorski [95]). The origin of the notion of an ideal is a fundamental set-theoretic concept of a family of subsets of a given set which are small in some particular way, and the notion of filter corresponds to the dual family of large sets.

The main topics of this dissertation are sequences of measures on Boolean algebras and their connections with ideals. This lies at the intersection of set theory and measure theory. Let us go into summarizing the results of the thesis.

Let $\mathcal{A}$ be a Boolean algebra. We say that a sequence $\langle \mu_n \colon n \in \omega \rangle$ of finitely additive signed measures on a Boolean algebra $\mathcal{A}$ is *pointwise bounded* if $\sup_{n\in\omega} |\mu_n(A)| < \infty$ for every $A \in \mathcal{A}$, and it is *norm bounded* if $\sup_{n\in\omega} \|\mu_n\| < \infty$. The classical theorem due to Nikodym and Andô asserts that if $\mathcal{A}$ is σ-complete, then every pointwise bounded sequence of measures on $\mathcal{A}$ is norm bounded. This theorem, due to its numerous applications, is one of the most important results in the theory of vector measures, see Diestel and Uhl [21, Section I.3], and hence motivates the following definition, introduced by Schachermayer [91].

Definition. A Boolean algebra $\mathcal{A}$ has the *Nikodym property* if every pointwise bounded sequence of finitely additive signed measures on $\mathcal{A}$ is norm bounded.

Equivalently, $\mathcal{A}$ has the Nikodym property if, for every sequence $\langle \mu_n \colon n \in \omega \rangle$ of finitely additive signed measures on $\mathcal{A}$ such that $\mu_n(A) \to 0$ for all $A \in \mathcal{A}$, we have $\sup_{n\in\omega} \|\mu_n\| < \infty$ (see Proposition 2.5.3).

The Nikodym–Andô theorem has been generalized by many authors who introduced various weaker properties of Boolean algebras that still imply the Nikodym property, see e.g. Freniche [37], Haydon [50], Schachermayer [91]. On the other hand, countable infinite Boolean algebras do not have the Nikodym property, since the Stone space $St(\mathcal{A})$ of a countable infinite Boolean algebra $\mathcal{A}$ always contains a non-trivial convergent sequence, which implies the existence of a particular sequence of measures (see Examples 2.5.4 and 2.5.5). This example is a motivation for considering the following type of topological spaces.

Let $\mathcal{F}$ be a free filter on ω. We consider the set $N_\mathcal{F} = \omega \cup \{p_\mathcal{F}\}$, where $p_\mathcal{F}$ is a fixed point not belonging to ω, endowed with the topology in which ω is discrete in $N_\mathcal{F}$ and the open neighborhoods of $p_\mathcal{F}$ are of the form $A \cup \{p_\mathcal{F}\}$ for some $A \in \mathcal{F}$.

The space N_{Fr}, where by Fr is the Fréchet filter on ω, is homeomorphic to a convergent sequence in $\mathbb{R}$ together with its limit point. Thus, for a Boolean algebra $\mathcal{A}$, the Stone space $St(\mathcal{A})$ contains a non-trivial convergent sequence if and only if it contains a homeomorphic copy of N_{Fr}. Consequently, if N_{Fr} embeds into $St(\mathcal{A})$, then $\mathcal{A}$ does not have the Nikodym property. In Chapter 3, we study for which other free filters on ω we also have such implication.

We will say that a free filter $\mathcal{F}$ on ω has the *fs-Nikodym property* if there is no sequence $\langle \mu_n\colon\ n \in \omega \rangle$ of finitely supported measures on the space $N_{\mathcal{F}}$ such that $\|\mu_n\| \to \infty$ and $\mu_n(A) \to 0$ for every $A \in \mathrm{Clopen}(N_{\mathcal{F}})$ (we call such a sequence of measures *fs-anti-Nikodym*; see also Definition 3.1.4). We denote by $\mathcal{AN}$ the class of all ideals on ω whose dual filters do not have the fs-Nikodym property. We prove that if $\mathcal{A}$ is a Boolean algebra, $\mathcal{F}$ is a free filter on ω which does not have the fs-Nikodym property, and $N_{\mathcal{F}}$ homeomorphically embeds into $St(\mathcal{A})$, then $\mathcal{A}$ does not have the Nikodym property (see Corollary 4.2.2 for a stronger statement). This result generalizes the aforementioned fact about embedding N_{Fr} into the Stone spaces of Boolean algebras.

For any non-trivial convergent sequence $\langle x_n\colon\ n \in \omega \rangle$ in a given space $N_{\mathcal{F}}$, the sequence $\langle \mu_n\colon\ n \in \omega \rangle$ of finitely supported non-negative measures defined for each $n \in \omega$ by the equality $\mu_n = n \cdot \delta_{x_n}$ satisfies the conditions $\lim_{n\to\infty} \mu_n(\omega) = \infty$ and $\lim_{n\to\infty} \mu_n(\omega \setminus A) = 0$ for every $A \in \mathcal{F}$. We show that the fs-Nikodym property for filters is in fact characterized by similar conditions (Theorem 3.2.1). This characterization can be translated in a way which shows a connection between the class $\mathcal{AN}$ and exhaustive ideals associated to lower semicontinuous (*lsc*) submeasures on ω (Theorem 3.2.2). Finally, we obtain a useful characterization of the class $\mathcal{AN}$ as the family of subideals of summable ideals on ω (Corollary 3.2.5), which constitutes an important class of ideals in $\mathcal{P}(\omega)$.

There are several other important classes of ideals in $\mathcal{P}(\omega)$. One of them is the class of density ideals (generalizing the density zero ideal $\mathcal{Z}$), and another is its subclass: the Erdős–Ulam ideals (see Section 2.6 for their definitions). Using the above-mentioned characterizations of $\mathcal{AN}$, we present several conditions for a density ideal equivalent with being in the class $\mathcal{AN}$, for example the condition of being isomorphic to an Erdős–Ulam ideal (Theorem 3.3.3).

We also investigate the strucure of the class $\mathcal{AN}$ ordered by $\leq_K$, the *Katětov order*, which is an important tool for comparing the structural complexity of ideals and filters on ω (see e.g. Hrušák [54]). In particular, we study the cofinal structure of $(\mathcal{AN}, \leq_K)$ (Section 3.3.1), and we conclude by computing the Tukey type of the order $(\mathcal{AN}, \leq_K)$ (Corollary 3.4.4). We compare the fs-Nikodym property with the bounded Josefson–Nissenzweig property of filters (Proposition 3.5.1), which was introduced by Marciszewski and Sobota in [76]. Finally, in Section 3.6, we analyze some families of ideals that are contained in the class $\mathcal{AN}$ which give us rich substructures in the order $(\mathcal{AN}, \leq_K)$.

Our results also apply to the Grothendieck property of Boolean algebras, which is closely related to the Nikodym property. The space of measures on a Boolean algebra normed by the variation norm is a Banach space, thus we have a notion of weak convergence of measures.

Definition. A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if every norm bounded sequence $\langle \mu_n\colon\ n \in \omega \rangle$ of finitely additive signed measures on $\mathcal{A}$ such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{A}$ is weakly convergent to 0.

This property has its origin in Banach space theory, from the notion of a *Grothendieck space*, see González and Kania [42] and Section 2.5.

The connection between the Grothendieck property and the Nikodym property of Boolean algebras has been a subject of an extensive research (see e.g. [37], [50] or [91]), but it is still not fully understood, e.g. it is not easy to find an algebra having exactly one of these properties. The Boolean algebra $\mathscr{J}$ of Jordan measurable subsets of the unit interval $[0,1]$ was the first known example of a Boolean algebra with the Nikodym property but without the Grothendieck property (Schachermayer [91]; see also Graves and Wheeler [43] and Valdivia [109] for generalizations). Boolean algebras with the Grothendieck property but without the Nikodym property have been so far constructed only consistently—by Talagrand [106] under Continuum Hypothesis, by Sobota and Zdomskyy [100] under Martin's Axiom, and by Głodkowski and Widz [40] in a specific forcing extension; no ZFC example is known.

In Chapter 4 we provide further ZFC examples of Boolean algebras with the Nikodym property and without the Grothendieck property. The constructed examples are subalgebras of the Boolean algebra $\mathcal{P}(\omega)$, the power set of ω, and have the form $\mathcal{A}_\mathcal{F} = \mathcal{F} \cup \mathcal{F}^*$, where $\mathcal{F}$ is a filter on ω and $\mathcal{F}^*$ is its dual ideal. First, we prove that there is a family $\mathcal{H}_1$ of $\mathfrak{c}$ many pairwise non-isomorphic Boolean algebras, each of them having the Nikodym property but not the Grothendieck property, which are of the form $\mathcal{A}_\mathcal{F}$ where $\mathcal{F}^*$ is an Erdős–Ulam ideal (Theorem 4.4.26). This yields that the constructed Boolean algebras are of low Borel complexity—they are $\mathbb{F}_{\sigma\delta}$ sets when thought of as subsets of the Cantor space 2^ω.

The cardinality of $\mathcal{H}_1$, i.e. the continuum $\mathfrak{c}$, is optimal, as there are only $\mathfrak{c}$ many Borel subsets of 2^ω. By using appropriate non-Borel filters we obtain, however, a larger family. We prove that there is a family $\mathcal{H}_2$ of $2^\mathfrak{c}$ many pairwise non-isomorphic non-Borel Boolean algebras of $\mathcal{P}(\omega)$, each of them having the Nikodym property but not the Grothendieck property (Theorem 4.4.30).

Let us now briefly describe the strategy for proving the above results, consisting of several steps. Our first step relies on the study of so-called Nikodym concentration points of unbounded sequences of measures on Boolean algebras without the Nikodym property, initiated by Aizpuru [1] and Sobota [97]. Briefly, given a Boolean algebra $\mathcal{A}$, we say that a sequence $\langle \mu_n \colon n \in \omega \rangle$ of measures on $\mathcal{A}$ is *anti-Nikodym* if $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{A}$ and $\sup_{n\in\omega} \|\mu_n\| = \infty$, and that in this case a point t in the Stone space $St(\mathcal{A})$ of $\mathcal{A}$ (i.e. an ultrafilter t in $\mathcal{A}$) is its *Nikodym concentration point* if for each $A \in t$ we have $\sup_{n\in\omega} \|\mu_n \restriction A\| = \infty$. If a Boolean algebra does not have the Nikodym property, then it carries an anti-Nikodym sequence, and every anti-Nikodym sequence has a Nikodym concentration point (Lemma 4.1.3). In Section 4.1 we will investigate anti-Nikodym sequences of measures having *strong* Nikodym concentration points, that is, Nikodym concentration points satisfying a slightly technical yet natural condition described in Definition 4.1.7. In particular, we prove in Theorem 4.1.18 that if an anti-Nikodym sequence of measures on a Boolean algebra $\mathcal{A}$ has a strong Nikodym concentration point, then $\mathcal{A}$ carries an anti-Nikodym sequence of measures with pairwise disjoint supports (or even contained in pairwise disjoint clopen subsets of $St(\mathcal{A})$), which, as one will see below, will appear to be an extremely useful feature. In particular, the first important consequence is that if $\mathcal{A}$ carries an anti-Nikodym sequence of measures with a strong Nikodym concentration point then $\mathcal{A}$ does not have the Grothendieck property (Theorem 4.1.20).

In the second step towards the proofs of Theorems 4.4.26 and 4.4.30, we prove in Proposition 4.3.12 that if a Boolean algebra $\mathcal{A}$ has the Nikodym property but its subalgebra $\mathcal{A}\langle\mathcal{I}\rangle$ generated by $\mathcal{I}$ (i.e. $\mathcal{A}\langle\mathcal{I}\rangle = \mathcal{I} \cup \mathcal{I}^*$) does not, then every anti-Nikodym sequence of measures on $\mathcal{A}\langle\mathcal{I}\rangle$ has a unique strong Nikodym concentration point, namely, the point $p_{\mathcal{I}^*} \in St(\mathcal{A}\langle\mathcal{I}\rangle)$ corresponding to the filter $\mathcal{I}^*$ (which is an ultrafilter in $\mathcal{A}\langle\mathcal{I}\rangle$). This, together with aforementioned Theorem 4.1.18, enables us to characterize the Nikodym property for Boolean algebras of the form $\mathcal{A}\langle\mathcal{I}\rangle$ in terms of non-negative measures with pairwise disjoint supports—see Theorem 4.3.15 for a general statement.

The third step in proving Theorems 4.4.26 and 4.4.30 relies on decomposing the Nikodym property of Boolean algebras of the form $\mathcal{A}_{\mathcal{F}}$ into two parts. We prove, using a variant of Theorem 4.3.15 for a free filter $\mathcal{F}$ on ω and the algebra $\mathcal{A}_{\mathcal{F}}$, that the algebra $\mathcal{A}_{\mathcal{F}}$ has the Nikodym property if and only if the quotient algebra $\mathcal{A}_{\mathcal{F}}/\operatorname{Fin}$ has the Nikodym property and the filter $\mathcal{F}$ has the fs-Nikodym property (Theorem 4.4.7).

Consequently, using this decomposition result and Theorem 4.4.15, asserting that for a P-filter $\mathcal{F}$ the Nikodym property of the algebra $\mathcal{A}_{\mathcal{F}}$ is equivalent to the fs-Nikodym property of the filter $\mathcal{F}$, we get that for every density ideal $\mathcal{I}$ the Boolean algebra $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property if and only if $\mathcal{I}$ is isomorphic to an Erdős–Ulam ideal. We then use this result, together with the fact that there are $\mathfrak{c}$ many pairwise non-isomorphic Erdős–Ulam ideals on ω, to finally obtain the family $\mathcal{H}_1$ of Boolean algebras in Theorem 4.4.26.

The argument for Theorem 4.4.30 is a bit different—instead of using Erdős–Ulam ideals, we consider a family of $2^{\mathfrak{c}}$ many pairwise non-isomorphic non-analytic ideals of the form $\mathcal{U}^* \oplus \mathcal{Z}$, where each $\mathcal{U}$ is a free ultrafilter on ω and $\mathcal{Z}$ is the standard asymptotic density zero ideal.

The remaining part of Chapter 4 is Section 4.5, where we study relations between various notions of boundedness of ideals on ω and of the Nikodym property. In connection with this, we also investigate the so-called *hypergraph ideals*, concluding with a construction of a family of $\mathfrak{c}$ many non-isomorphic hypergraph ideals which are not *totally bounded* but have the Nikodym property (Theorem 4.5.26).

In the last part of the thesis, we will study combinatorial problems related to Rosenthal's lemma ([88], [89]) which is an important result concerning sequences of measures on Boolean algebras (see Koszmider and Martínez-Celis [69, Section 6] and Kąkol *et al.*[63, Section 19.4.1] for the discussion about its various classical forms and uses).

Rosenthal's lemma. *Suppose that $\langle A_n\colon\ n \in \omega\rangle$ is an antichain in a Boolean algebra $\mathcal{A}$, and $\langle \mu_k\colon\ k \in \omega\rangle$ is a sequence of non-negative measures on $\mathcal{A}$ such that the set of sums $\{\sum_{n\in\omega} \mu_k(A_n)\colon k \in \omega\}$ is bounded. For every $\varepsilon > 0$, there exists infinite set $B \subseteq \omega$ such that for every $k \in B$ we have:*

$$\sum_{n\in B\setminus\{k\}} \mu_k(A_n) < \varepsilon.$$

This lemma, in particular, is a key step in proving Grothendieck's result that all σ-complete Boolean algebras have the Grothendieck property ([44]), and similarly in the proofs of its subsequent generalizations, e.g. by Haydon [50] (see [63, Section 19.4] for more information on the role of Rosenthal's lemma in proving the Grothendieck property for various Boolean algebras). As it is showed in [69, Section 6], the core of Rosenthal's lemma is a purely combinatorial statement: that $[\omega]^\omega$ is a *Rosenthal family*, which we define below.

An infinite matrix $M = (m_{k,n})_{k,n\in\omega}$ consisting of non-negative reals is called a *Rosenthal matrix* if the set of sums $\{\sum_{n\in\mathbb{N}} m_{k,n} : k \in \omega\}$ is bounded. Following Koszmider and Martínez-Celis [69], given $\varepsilon > 0$ and a Rosenthal matrix M, we say that an infinite subset

B of ω *ε-fragments* M if for every $k \in B$ we have

$$\sum_{n \in B \setminus \{k\}} m_{k,n} < \varepsilon.$$

A family $\mathcal{F}$ of infinite subsets of ω is a *Rosenthal family* if for any $\varepsilon > 0$ and every Rosenthal matrix M there is an $B \in \mathcal{F}$ such that it ε-fragments M (Sobota [96, Definition 1.3]).

Rosenthal families have been recently studied in [69], [86] and [96]. In [69], the authors proved that every ultrafilter on ω is a Rosenthal family, and that $\mathfrak{r}$ is the smallest size of a Rosenthal family. Moreover, they found a connection between Rosenthal families and free sets (see Section 5.1). Functions from ω to ω without fixed points are also related to this topic, as they correspond to Rosenthal matrices of zeros and ones (see [69, Section 2.2]).

We define in Chapter 5 two dual schemes of cardinal invariants: Δ_φ and $\mathfrak{ros}_\varphi$, parametrized by a property φ of functions from ω to ω (see Definition 5.1.2). Their definition is related to the notion of freeness and to functions without fixed points, and so to the combinatorics of Rosenthal families. We will study the following invariants from these schemes: $\Delta_{\mathbf{1}}$, $\Delta_{\text{Fin-1}}$, $\Delta_{\text{1-1}}$ and $\Delta_{\text{involution}}$ (and their dual versions for $\mathfrak{ros}$), where e.g. $\varphi = \mathbf{1}$ corresponds to all functions from ω to ω and $\varphi = \text{Fin-1}$ corresponds to finite-to-one functions.

In Section 5.2 we compute lower and upper bounds for $\Delta_{\mathbf{1}}$ and $\Delta_{\text{Fin-1}}$ in terms of the classical cardinal invariants $\mathfrak{b}$, $\mathfrak{s}$ and their variants (Corollaries 5.2.6 and 5.2.11). Then, in Section 5.3, we show that $\Delta_{\text{1-1}} = \Delta_{\text{involution}}$ and $\mathfrak{ros}_{\text{1-1}} = \mathfrak{ros}_{\text{involution}}$ (Theorem 5.3.3). This also connects our results with the research of Banakh [7] in the topic of large-scale topology, as this easily implies that $\Delta_{\text{1-1}}$ and $\mathfrak{ros}_{\text{1-1}}$ are equal respectively to the invariants Δ and $\widehat{\Delta}$ defined in [7] (Corollary 5.3.4).

Then, we further concentrate on $\Delta_{\text{1-1}}$ and $\mathfrak{ros}_{\text{1-1}}$. We define two dual families of cardinal invariants: $\mathfrak{e}_g$ and $\widehat{\mathfrak{e}_g}$ for a given stricly increasing function $g \in \omega^\omega$, and we prove that $\Delta_{\text{1-1}} \geq \mathfrak{e}_g$ and $\mathfrak{ros}_{\text{1-1}} \leq \widehat{\mathfrak{e}_g}$ for every such function g, which improves the previously known bounds $\max\{\mathfrak{b}, \mathfrak{s}, \text{cov}(\mathcal{N})\} \leq \Delta$ and $\min\{\mathfrak{d}, \mathfrak{r}, \text{non}(\mathcal{N})\} \geq \widehat{\Delta}$ (Theorem 5.4.3). Finally, using the method of forcing, we show that the strict inequalities $\max\{\mathfrak{b}, \mathfrak{s}, \text{cov}(\mathcal{N})\} < \Delta_{\text{1-1}}$ and $\min\{\mathfrak{d}, \mathfrak{r}, \text{non}(\mathcal{N})\} < \mathfrak{ros}_{\text{1-1}}$, are consistent with ZFC (Theorems 5.5.4 and 5.5.5).

Sources and contribution. Chapter 3, with the exception of the characterization from Corollary 3.2.5, is based on the author's paper:

- T. Żuchowski, *The Nikodym property and filters on ω*, Arch. Math. Logic 64 (2025), no. 5–6, 705–735.

Chapter 4, together with this characterization, is based on the author's joint preprint with Damian Sobota:

- D. Sobota, T. Żuchowski, *The Nikodym and Grothendieck properties of Boolean algebras and rings related to ideals*, preprint (2025), `arXiv:2510.19744`.

Chapter 5 is based on the author's joint paper with Arturo Martínez-Celis:

- A. Martínez-Celis, T. Żuchowski, *On cardinal invariants related to Rosenthal families and large scale topology*, Ann. Pure Appl. Logic 176 (2025), no. 8, paper no. 103607.

Chapter 2

Preliminaries

2.1 Set-theoretic basics

Let X be a set. By $\mathcal{P}(X)$ we denote the family of all subsets of X. $|X|$ stands for the cardinality of X. If κ is a cardinal number, then by $[X]^{<\kappa}$ and $[X]^{\kappa}$ we denote the families of all subsets of X of cardinality strictly less than κ and of all subsets of X of cardinality equal to κ, respectively. For two sets A, B we write $A \subseteq^* B$ if $A \setminus B$ is finite.

By $\aleph_n$ we denote the n-th infinite cardinal number (where $\aleph_0$ is the cardinality of $\mathbb{N}$). When considered as an ordinal number, we will write it as ω_n, where $\omega = \omega_0$. For a cardinal number κ by κ^+ we denote the cardinal successor of κ. By $\mathfrak{c}$ we denote the *continuum*, i.e. the cardinality of the real line $\mathbb{R}$ and of $\mathcal{P}(\omega)$. By the *Continuum Hypothesis* (in short, CH), we mean the statement $2^{\aleph_0} = \aleph_1$.

We will identify ω with the set $\mathbb{N}$ of all non-negative integers. We also denote $\omega_+ = \omega \setminus \{0\} = \{1, 2, 3, \ldots\}$ and $\mathrm{Fin} = [\omega]^{<\omega}$. We say that $\langle I_n\colon\ n \in \omega\rangle$ is an *interval partition* of ω, if there exists a strictly increasing sequence $\langle x_n\colon\ n \in \omega\rangle$ of natural numbers such that $x_0 = 0$ and for every $n \in \omega$ we have $I_n = \{x_n, \ldots, x_{n+1} - 1\}$.

Let A be a non-empty set. A family $\mathcal{F} \subseteq \mathcal{P}(A)$ is a *filter on A* if $\emptyset \notin \mathcal{F}$, $A \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections and taking supersets. We say that a filter $\mathcal{F}$ on A is an *ultrafilter* if there is no filter $\mathcal{G}$ on A such that $\mathcal{F} \subsetneq \mathcal{G}$. By $Fr = Fr(\omega)$ we denote the *Fréchet filter on ω*, where $Fr(A) = \{B \subseteq A\colon A \setminus B \text{ is finite}\}$. A filter $\mathcal{F}$ on A is *free* if $Fr(A) \subseteq \mathcal{F}$. Throughout this thesis, all considered filters are assumed to be free. A filter $\mathcal{F}$ on ω is a *P-filter* if for every sequence $\langle A_n\colon\ n \in \omega\rangle$ of sets from $\mathcal{F}$ there is $A \in \mathcal{F}$ such that $A \subseteq^* A_n$ for each $n \in \omega$.

For a family $S \subseteq \mathcal{P}(A)$ we put $S^* = \{X\colon A \setminus X \in S\}$—$S^*$ is said to be *dual to S*. A family $\mathcal{I} \subseteq \mathcal{P}(A)$ is an *ideal on A* if $\mathcal{I}^*$ is a free filter on A. Note that this implies that $[A]^{<\omega} \subseteq \mathcal{I}$. An ideal $\mathcal{I}$ on ω is a *P-ideal* if the dual filter $\mathcal{I}^*$ is a P-filter, i.e. if for every sequence $\langle A_n \in \mathcal{I}\colon\ n \in \omega\rangle$ there is $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for every $n \in \omega$. We say that an ideal $\mathcal{I}$ on ω is *tall* if for every infinite $A \subseteq \omega$ there exists infinite $B \subseteq A$ such that $B \in \mathcal{I}$.

For sets A, B we denote by A^B the set of all functions from B to A. In particular, ω^ω consists of functions from ω to ω. Also, we denote $A^{<\omega} = \bigcup_{k\in\omega} A^k$. We define the preorder $\leq^*$ on ω^ω by setting $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. There are two cardinal invariants related to this preorder:

- $\mathfrak{d}$ is the smallest size of a family $\mathcal{F} \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ with $g \leq^* f$ (such a family is called *dominating* in ω^ω);

- $\mathfrak{b}$ is the smallest size of a family $\mathcal{F} \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ with $f \nleq^* g$ (such a family is called *unbounded* in ω^ω).

The cardinal numbers $\mathfrak{d}$ and $\mathfrak{b}$ are examples of the *cardinal characteristics of the continuum*, i.e. cardinal invariants which are between $\aleph_1$ and $\mathfrak{c}$. Such cardinal numbers are studied intensively in the field of infinitary combinatorics, see Blass [12]. We will also use in the sequel other classical cardinal invariants: $\mathfrak{s}$ and $\mathfrak{r}$, which are defined as:

- $\mathfrak{s}$ is the smallest size of a family $\mathcal{S} \subseteq [\omega]^\omega$ such that for every $A \in [\omega]^\omega$ there is $B \in \mathcal{S}$ such that both $A \cap B$ and $A \setminus B$ are infinite (such a family is called *splitting* in $[\omega]^\omega$);

- $\mathfrak{r}$ is the smallest size of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that for every $A \in [\omega]^\omega$ there is $B \in \mathcal{R}$ such that either $B \subseteq^* A$ or $B \cap A$ is finite (such a family is called *reaping* in $[\omega]^\omega$).

Moreover, we will use some cardinal invariants related to topology and measure on 2^ω, which we will define later in this chapter.

2.2 Boolean algebras, topology and Stone spaces

Let $\mathcal{A}$ be a Boolean algebra (with the operations $\wedge$, $\vee$, $\setminus$, ^{c}, zero element $0_\mathcal{A}$, and unit element $1_\mathcal{A}$). A Boolean algebra is naturally a partially ordered set. We denote $\mathcal{A}^+ = \mathcal{A} \setminus \{0_\mathcal{A}\}$. For a subset $\mathcal{X} \subseteq \mathcal{A}$ by $\mathcal{A}\langle\mathcal{X}\rangle$ we denote the Boolean subalgebra of $\mathcal{A}$ generated by $\mathcal{X}$; we also set $\mathcal{X}^* = \{1_\mathcal{A} \setminus A\colon A \in \mathcal{X}\}$ and call $\mathcal{X}^*$ *dual* to $\mathcal{X}$. For an element $A \in \mathcal{A}$ we set $\mathcal{A}_A = \{A \wedge B\colon B \in \mathcal{A}\}$; of course, $\mathcal{A}_A$ is also a Boolean algebra (with the obvious operations and constants). A collection $\langle A_i\colon\ i \in I\rangle$ of elements of $\mathcal{A}$ is an *antichain* if $A_i \wedge A_j = 0_\mathcal{A}$ for every $i \neq j \in I$. A family $\mathcal{S} \subseteq \mathcal{A}^+$ is called a *splitting family* if for every $A \in \mathcal{A}^+$ there is $S \in \mathcal{S}$ such that $A \wedge S \neq 0_\mathcal{A}$ and $A \setminus S \neq 0_\mathcal{A}$.

We define ideals on $\mathcal{A}$ analogously as the ideals of sets, i.e. $\mathcal{I}$ is an *ideal in the Boolean algebra* $\mathcal{A}$ if $1_\mathcal{A} \notin \mathcal{I}$, $A \vee B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and $\mathcal{I}$ is downward closed with respect to the natural order $\leq$ on $\mathcal{A}$. Note that if $\mathcal{I}$ is an ideal in $\mathcal{A}$, then for every $A \in \mathcal{I}$ we have $\mathcal{A}_A = (\mathcal{A}\langle\mathcal{I}\rangle)_A$. For $A \in \mathcal{A}$ and a fixed ideal $\mathcal{I}$ in $\mathcal{A}$, by $A^\bullet$ we denote the equivalence class of A in the quotient Boolean algebra $\mathcal{A}/\mathcal{I}$ (see [68, Section 5.4] for the definition of a quotient algebra). Also, we analogously define filters, i.e. $\mathcal{F}$ is a *filter in* $\mathcal{A}$ if $\mathcal{F}^*$ is an ideal in $\mathcal{A}$. Again, a filter $\mathcal{F}$ in $\mathcal{A}$ is an *ultrafilter* if there is no filter $\mathcal{G}$ on A such that $\mathcal{F} \subsetneq \mathcal{G}$.

A Boolean algebra $\mathcal{A}$ is said to be *σ-complete* if every countable subset of $\mathcal{A}$ has supremum, that is, the least upper bound with respect to the natural order $\leq$ on $\mathcal{A}$. For example, for every set A the Boolean algebra of sets $\mathcal{P}(A)$ is obviously σ-complete, with the set union as supremum.

In this thesis, all considered topological spaces are assumed to be Hausdorff. Let X be a topological space. By $\mathrm{Clopen}(X)$ we denote the family of all clopen subsets of X, i.e. all subsets which are both open and closed. Recall that X is *zero-dimensional* if it has a base consisting of clopen sets. For a Boolean algebra $\mathcal{A}$, by $St(\mathcal{A})$ we denote the *Stone space* of $\mathcal{A}$, i.e. the space of all ultrafilters in $\mathcal{A}$ with the topology defined by the following basic clopen sets:

$$[A]_\mathcal{A} = \{U\colon \text{ an ultrafilter in } \mathcal{A} \text{ such that } A \in U\}$$

for $A \in \mathcal{A}$. Recall that $St(\mathcal{A})$ is a compact zero-dimensional space, and the mapping $\mathcal{A} \ni A \mapsto [A]_{\mathcal{A}} \in \mathrm{Clopen}(St(\mathcal{A}))$ is an isomorphism between the Boolean algebras $\mathcal{A}$ and $\mathrm{Clopen}(St(\mathcal{A}))$. It follows that two Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if $St(\mathcal{A})$ and $St(\mathcal{B})$ are homeomorphic (see Koppelberg [68, Chapter 3] for more information about Stone spaces and the Stone duality).

Recall also that every compact Hausdorff space K is *normal*, i.e. for every disjoint closed $C, C' \subseteq K$ there exist disjoint open $U, U' \subseteq K$ such that $C \subseteq U$ and $C' \subseteq U'$.

By $C(X)$ we denote the set of all continuous real-valued functions on a topological space X and by $C_b(X)$ we denote its subset consisting of all bounded functions. If X is a compact space, then we endow $C(X)$ with the supremum norm, which makes it a Banach space. Recall that a subspace Y of X is *C^*-embedded* if every $f \in C_b(Y)$ extends to some $f' \in C_b(X)$.

For a Tychonoff (for example, zero-dimensional) space X, by βX we denote the *Čech–Stone compactification* of X (see Gillman and Jerison [39, Chapter 6] for more information about the Čech–Stone compactification). Recall that $\beta\omega$, i.e. the Čech–Stone compactification of the space ω (equipped as usual with the discrete topology), can be associated with the Stone space of the algebra $\mathcal{P}(\omega)$ (see e.g. [68, Theorem 8.9]). By ω^* we denote the *remainder* $\beta\omega \setminus \omega$, which can be associated with the Stone space of the quotient algebra $\mathcal{P}(\omega)/\,\mathrm{Fin}$.

A point x in X is a *P-point* if for every sequence $\langle U_n\colon\ n \in \omega\rangle$ of open neighborhoods of x in X the interior of the intersection $\bigcap_{n\in\omega} U_n$ contains x.

By $Bor(X)$ we denote the σ-algebra of all Borel subsets of X (the smallest σ-algebra containing all open sets). Two particular subfamilies of Borel sets will be of importance to us: the family of $\mathbb{F}_\sigma$ sets, i.e. of countable unions of closed sets, and the family of $\mathbb{F}_{\sigma\delta}$ sets, i.e. of countable intersections of $\mathbb{F}_\sigma$ sets.

Every subset of ω can be naturally identified with a point in the Cantor space 2^ω equipped with the standard product topology (which is compact metrizable, hence Polish). Therefore, subsets of $\mathcal{P}(\omega)$ correspond in a natural way to subsets of 2^ω. Consequently, when considering subsets of $\mathcal{P}(\omega)$ we may investigate their Borel complexity, analyticity, etc.

A subset $M \subseteq X$ of a topological space X is *meager* if it can be covered by a countable union of nowhere-dense sets. There are several cardinal invariants associated with the ideal $\mathcal{M}$ of meager subsets of 2^ω: $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, $\mathrm{add}(\mathcal{M})$, $\mathrm{cof}(\mathcal{M})$. We will mainly use the two following ones:

$$\mathrm{non}(\mathcal{M}) = \min\left\{|A|\colon A \in \mathcal{P}(2^\omega) \setminus \mathcal{M}\right\};$$

$$\mathrm{cov}(\mathcal{M}) = \min\left\{|\mathcal{F}|\colon \mathcal{F} \subseteq \mathcal{M} \text{ and } \bigcup \mathcal{F} = 2^\omega\right\}.$$

It is folklore that the values of these invariants will not change if we use the ideal of meager sets of arbitrary Polish space X without isolated points, instead of $\mathcal{M}$ (e.g. with $X = \omega^\omega$ equipped with the standard product topology). This follows from the fact that, for any such a space X, there is a bijection $f\colon X \to 2^\omega$ such that both f and f^{-1} send meager sets to meager sets (see e.g. [65, Ex. 8.32]).

2.3 Banach spaces

A normed space $(X, \|\cdot\|)$ is a *Banach space* if X is complete in the metric induced by the norm $\|\cdot\|$. Recall that a subset A of a normed space X is *norm bounded* if $\sup_{x\in A}\|x\| < \infty$.

Given two normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, we will use the following standard norms on the direct sum $E \oplus F$: $\|(x,y)\|_1 = \|x\|_E + \|y\|_F$ and $\|(x,y)\|_\infty = \max(\|x\|_E, \|y\|_F)$ (for $x \in E$, $y \in F$). We assume by default that $E \oplus F$ is endowed with the norm $\|\cdot\|_1$.

For two normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, the norm $\|T\|$ of a linear operator $T\colon E \to F$ is defined as $\|T\| = \sup\{\|T(x)\|_F\colon \|x\|_E \leq 1\}$, and T is *bounded* if $\|T\| < \infty$. We say that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are *isomorphic* if there is a bounded linear operator $T\colon E \to F$ which is a bijection and the inverse operator $T^{-1}\colon F \to E$ is also bounded, and we say that they are *isometrically isomorphic* if moreover T satisfies $\|x\|_E = \|T(x)\|_F$ for every $x \in E$.

By X^* we denote the space of all bounded linear functionals on a normed space X (i.e. operators from X to $\mathbb{R}$), with the operator norm. For every normed space X, X^* is a Banach space, called the *space dual to X*. We say that a sequence $\langle x_n\colon\ n \in \omega\rangle$ in X is *weakly convergent* to $x \in X$ if $x^*(x_n) \to x^*(x)$ for every $x^* \in X^*$. A sequence $\langle x_n^*\colon\ n \in \omega\rangle$ in X^* is *weak* convergent* to $x^* \in X^*$ if $x_n^*(x) \to x^*(x)$ for every $x \in X$. By the fact that X embeds into X^{**} via evaluations, it follows that the weak convergence implies the weak* convergence in X^*.

2.4 Measures

Let $\mathcal{A}$ be a Boolean algebra $\mathcal{A}$ and let $\mathcal{I}$ be an ideal in the algebra $\mathcal{A}$. Let $\mathcal{X} \in \{\mathcal{A}, \mathcal{I}\}$. A function $\mu\colon \mathcal{X} \to \mathbb{R}$ is a *measure* on $\mathcal{X}$ if μ is finitely additive and bounded, i.e. for the *variation norm*

$$\|\mu\| = \sup\left\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \mathcal{X},\ A \wedge B = 0_{\mathcal{A}}\right\}$$

we have $\|\mu\| < \infty$. By $\mathrm{ba}(\mathcal{X})$ we denote the vector space of all measures on $\mathcal{X}$ endowed with the variation norm, which makes $\mathrm{ba}(\mathcal{X})$ a Banach space. More generally, for $\mu \in \mathrm{ba}(\mathcal{X})$ and $X \in \mathcal{X}$ we define the *variation of μ* by setting

$$|\mu|(X) = \sup\left\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \mathcal{X},\ A \wedge B = 0_{\mathcal{A}},\ A \vee B \leq X\right\},$$

and, in the case $\mathcal{X} = \mathcal{A}$, we have $\|\mu\| = |\mu|\big(1_{\mathcal{A}}\big)$. In the case of the Boolean algebra $\mathcal{P}(\omega)$, an element μ of $\mathrm{ba}(\mathcal{P}(\omega))$ will be also called a *measure on ω*.

For a measure $\mu \in \mathrm{ba}(\mathcal{X})$, an element $A \in \mathcal{X}$, and a subset $\mathcal{Y} \subseteq \mathcal{X}$, we define the *restricted* measures $\mu \restriction A \in \mathrm{ba}(\mathcal{X})$ and $\mu \restriction \mathcal{Y} \in \mathrm{ba}(\mathcal{Y})$ as follows: $(\mu \restriction A)(B) = \mu(A \wedge B)$ for every $B \in \mathcal{X}$, and $(\mu \restriction \mathcal{Y})(B) = \mu(B)$ for every $B \in \mathcal{Y}$. Note that $\|\mu \restriction A\| = \|\mu \restriction \mathcal{A}_A\|$ for every $A \in \mathcal{A}$. For measures $\mu \in \mathrm{ba}(\mathcal{A})$ and $\nu \in \mathrm{ba}(\mathcal{I})$ we will say that μ *extends* ν if $\mu \restriction \mathcal{I} = \nu$. Of course, since $\mathcal{I}$ is downward closed, we have $|\nu|(A) = |\mu|(A)$ for every $A \in \mathcal{I}$ and so $\|\nu\| \leq \|\mu\|$.

We say that a measure μ on $\mathcal{X}$ is *non-negative* if $\mu(A) \geq 0$ for every $A \in \mathcal{X}$, and that a measure μ on $\mathcal{A}$ is a *probability measure* if it is non-negative and $\mu\big(1_{\mathcal{A}}\big) = 1$. The following simple measure-theoretic observation will be useful later.

Remark 2.4.1. Let $\mu\colon \mathcal{X} \to [0, \infty)$ be a non-negative finitely additive function on $\mathcal{X}$ (not necessarily bounded). If $A = \bigvee_{i\in\omega} A_i$, where $A \in \mathcal{X}$ and $\langle A_i\colon\ i \in \omega\rangle$ is an antichain in $\mathcal{X}$, then we have

$$\mu(A) \geq \mu\Big(\bigvee_{i<n} A_i\Big) = \sum_{i<n} \mu(A_i)$$

for every $n \in \omega$, and so $\mu(A) \geq \sum_{i\in\omega} \mu(A_i)$.

Let X be a topological space. For any finitely additive function $\mu\colon Bor(X) \to \mathbb{R}$, we define the *variation of* μ, denoted by $|\mu|$, by setting

$$|\mu|(Y) = \sup\big\{|\mu(A)| + |\mu(B)|\colon\ A, B \in Bor(X),\ A \cap B = \emptyset,\ A \cup B \subseteq Y\big\}$$

for each $Y \in Bor(X)$. A function $\mu\colon Bor(X) \to \mathbb{R}$ is a *measure* on X if μ is countably additive, *regular*, and bounded, i.e. for every $A \in Bor(X)$ we have

$$|\mu|(A) = \sup\big\{|\mu|(K)\colon K \subseteq A, K \text{ - compact}\big\},$$

and, for the *variation norm*

$$\|\mu\| = \sup\big\{|\mu(A)| + |\mu(B)|\colon\ A, B \in Bor(X),\ A \cap B = \emptyset\big\},$$

we have $\|\mu\| < \infty$. By definition, we have $\|\mu\| = |\mu|(X)$. By $M(X)$ we denote the vector space of all measures on X endowed with the variation norm, which makes $M(X)$ a Banach space.

For a compact space K, the regularity of measures implies the *outer regularity* (see [49, Theorem 52.E]), i.e. for every $\mu \in M(K)$ and $A \in Bor(K)$ we have

$$|\mu|(A) = \inf\big\{|\mu|(U)\colon A \subseteq U, U \text{ - open}\big\}.$$

It is well-known that for every $\mu \in \mathrm{ba}(\mathcal{A})$ there is a unique measure $\widehat{\mu} \in M(St(\mathcal{A}))$, which we will call the *Radon extension* of μ, such that $\mu(A) = \widehat{\mu}\big([A]_{\mathcal{A}}\big)$ for every $A \in \mathcal{A}$ and $\|\mu\| = \|\widehat{\mu}\|$ (see e.g. Semadeni [93, Chapter 17 and Section 18.7]). Similarly, every $\mu \in M(St(\mathcal{A}))$ induces a measure $\check{\mu} \in \mathrm{ba}(\mathcal{A})$ by setting $\check{\mu}(A) = \mu\big([A]_{\mathcal{A}}\big)$ for every $A \in \mathcal{A}$; we also have $\|\mu\| = \|\check{\mu}\|$. Consequently, the Banach spaces $\mathrm{ba}(\mathcal{A})$ and $M(St(\mathcal{A}))$ are isometrically isomorphic.

If $A \in Bor(X)$, then the *restricted* measure $\mu \restriction A$ is defined by the formula $(\mu \restriction A)(B) = \mu(A \cap B)$ for every $B \in Bor(X)$. We say that μ is a *non-negative measure* if $\mu(A) \geq 0$ for every $A \in Bor(X)$, and that it is a *probability measure* if it is non-negative and $\mu(X) = 1$. Note that $|\mu| = \mu$ for any non-negative measure μ.

We denote by λ the standard product probability measure on 2^ω, and by $\mathcal{N}$ the ideal of λ-*null* Borel subsets of 2^ω, i.e. the ideal of Borel sets of measure 0. There are cardinal invariants associated with the ideal $\mathcal{N}$, analogous to the ones for $\mathcal{M}$; we will use in the sequel $\mathrm{non}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{N})$:

$$\mathrm{non}(\mathcal{N}) = \min\big\{|A|\colon A \in \mathcal{P}(2^\omega) \setminus \mathcal{N}\big\},$$

$$\mathrm{cov}(\mathcal{N}) = \min\big\{|\mathcal{F}|\colon \mathcal{F} \subseteq \mathcal{N} \text{ and } \bigcup \mathcal{F} = 2^\omega\big\}.$$

It is folklore that the values of these invariants will not change if we use the ideal of μ-null Borel sets for arbitrary probability measure μ on a Polish space X satisfying $\mu(\{x\}) = 0$ for each $x \in X$, instead of $\mathcal{N}$. It follows from the fact that, for any such X and μ, there is a bijection $f\colon X \to 2^\omega$ such that $\lambda(f[A]) = \mu(A)$ for every $A \in Bor(X)$ (see Kechris [65, Theorem 17.41]).

If X is a topological space and $x \in X$, then δ_x denotes the one-point measure on X concentrated at x. A measure $\mu \in M(X)$ is said to be *finitely supported* if there are a finite subset $F \subseteq X$ and non-zero real numbers α_x, $x \in F$, such that $\mu = \sum_{x\in F} \alpha_x \delta_x$. Note that in this case we have $\|\mu\| = \sum_{x\in F} |\alpha_x|$, and that for every $A \in Bor(X)$ we have $\mu \restriction A = \sum_{x\in F\cap A} \alpha_x \delta_x$ and $|\mu|(A) = \|\mu \restriction A\| = \sum_{x\in F\cap A} |\alpha_x|$.

For each $\mu \in M(K)$ by $\operatorname{supp}(\mu)$ we denote the *support* of μ, i.e. the smallest closed subset F of K such that $|\mu|(F) = \|\mu\|$. Note that if μ is finitely supported, then $\operatorname{supp}(\mu) = F$, where F is as in the above definition of a finitely supported measure. A sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of measures on K is *disjointly supported* if $\operatorname{supp}(\mu_n) \cap \operatorname{supp}(\mu_{n'}) = \emptyset$ for every $n \neq n' \in \omega$. Let us also say that a sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of measures on a Boolean algebra $\mathcal{A}$ is *disjointly supported* if the sequence $\langle \widehat{\mu}_n \colon\ n \in \omega \rangle$ of the respective Radon extensions on $St(\mathcal{A})$ is disjointly supported.

For a non-negative measure μ supported on a finite set X we define the following numbers (cf. Farah [27, Section 1.13]):

$$\operatorname{at}^+(\mu) = \max \big\{ \mu(\{x\}) \colon x \in X \big\},$$

$$\operatorname{at}^-(\mu) = \min \big\{ \mu(\{x\}) \colon x \in X \big\}.$$

If μ is a measure on X and $f \in C(X)$, then we set $\mu(f) = \int_X f d\mu$ if the integral exists. Note that if μ is of the form $\sum_{i=1}^n \alpha_i \cdot \delta_{x_i}$, then $\mu(f) = \sum_{i=1}^n \alpha_i \cdot f(x_i)$. If K is a compact space, then by the classical Riesz–Markov–Kakutani Representation Theorem $M(K)$ is isometrically isomorphic to the dual Banach space $C(K)^*$ by the mapping $\mu \mapsto \mu(\cdot)$.

Let $\mathcal{R}$ be a ring of subsets of a set S. A function $\mu \colon \mathcal{R} \to \mathbb{R}$ is a *measure* on $\mathcal{R}$ if μ is additive and bounded, i.e. for the variation norm

$$\|\mu\| = \sup \big\{ |\mu(A)| + |\mu(B)| \colon\ A, B \in \mathcal{R},\ A \cap B = \emptyset \big\}$$

we have $\|\mu\| < \infty$. By $\operatorname{ba}(\mathcal{R})$ we denote the Banach space of all measures on $\mathcal{R}$ endowed with the variation norm.

For an ideal $\mathcal{I}$ in a Boolean algebra $\mathcal{A}$ put $\mathcal{R}(\mathcal{I}) = \big\{ [A]_{\mathcal{A}} : A \in \mathcal{I} \big\}$ and note that $\mathcal{R}(\mathcal{I})$ is a ring of clopen subsets of $St(\mathcal{A})$. Similarly as above, every $\mu \in \operatorname{ba}(\mathcal{I})$ induces a measure $\mathring{\mu} \in \operatorname{ba}(\mathcal{R}(\mathcal{I}))$ with $\|\mu\| = \|\mathring{\mu}\|$ by setting $\mathring{\mu}\big([A]_{\mathcal{A}} \big) = \mu(A)$ for every $A \in \mathcal{I}$, and every $\mu \in \operatorname{ba}(\mathcal{R}(\mathcal{I}))$ induces a measure $\tilde{\mu} \in \operatorname{ba}(\mathcal{I})$ with $\|\mu\| = \|\tilde{\mu}\|$ by setting $\tilde{\mu}(A) = \mu\big([A]_{\mathcal{A}} \big)$ for every $A \in \mathcal{I}$. Consequently, the Banach spaces $\operatorname{ba}(\mathcal{I})$ and $\operatorname{ba}(\mathcal{R}(\mathcal{I}))$ are isometrically isomorphic.

Let $\mathcal{R}$ be a ring of subsets of a set S, $\mathcal{A}$ a Boolean algebra, and $\mathcal{I}$ an ideal in $\mathcal{A}$. Let $\mathcal{X} \in \{\mathcal{R}, \mathcal{A}, \mathcal{I}\}$. We say that a sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of measures in $\operatorname{ba}(\mathcal{X})$ is *pointwise convergent* to some measure $\mu \in \operatorname{ba}(\mathcal{X})$ if $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every $A \in \mathcal{X}$. Moreover, a sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of measures in $\operatorname{ba}(\mathcal{A})$ is *weak* convergent* to $\mu \in \operatorname{ba}(\mathcal{A})$ if the sequence $\langle \widehat{\mu}_n \colon\ n \in \omega \rangle$ is weak* convergent to $\widehat{\mu}$ in $C(St(\mathcal{A}))^*$. We have the following well-known characterizations of weak and weak* convergence in $\operatorname{ba}(\mathcal{A})$.

Proposition 2.4.2. *A sequence $\langle \mu_n \colon\ n \in \omega \rangle$ in $\operatorname{ba}(\mathcal{A})$ is weak* convergent to $\mu \in \operatorname{ba}(\mathcal{A})$ if and only if $\langle \mu_n \colon\ n \in \omega \rangle$ is norm bounded and pointwise convergent to μ.*

Sketch of proof. On one hand, weak* convergence implies pointwise convergence since $\chi_{[A]_{\mathcal{A}}}$ is continuous for every $A \in \mathcal{A}$. Moreover, a weak* convergent sequence has to be norm bounded by the Banach–Steinhaus theorem. On the other hand, by the zero-dimensionality of $St(\mathcal{A})$ and the Stone–Weierstrass theorem, the linear span of the set $\big\{ \chi_{[A]_{\mathcal{A}}} \colon A \in \mathcal{A} \big\}$ is norm dense in $C(St(\mathcal{A}))$, and so we get the reverse implication (cf. Proposition 3.5.1). □

Proposition 2.4.3. *A sequence $\langle \mu_n \colon\ n \in \omega \rangle$ is weakly convergent to μ in $\operatorname{ba}(\mathcal{A})$ if and only if $\lim_{n\to\infty} \widehat{\mu}_n(B) = \widehat{\mu}(B)$ for every $B \in Bor(St(\mathcal{A}))$.*

Proof. See [63, Proposition 19.2.5]. □

2.5 The Nikodym and Grothendieck properties of Boolean algebras

Let $\mathcal{A}$ be a Boolean algebra. We say that a sequence $\langle\mu_n\colon n\in\omega\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is *pointwise bounded* if $\sup_{n\in\omega}|\mu_n(A)|<\infty$ for every $A\in\mathcal{A}$. The classical Nikodym Boundedness Theorem says that if $\mathcal{A}$ is σ-complete, then every pointwise bounded sequence of countably additive measures on $\mathcal{A}$ with finite norms is norm bounded. It was later generalized by Andô to finitely additive measures. Therefore, we have:

Theorem 2.5.1 (Nikodym–Andô [84], [3])**.** *If $\mathcal{A}$ is a σ-complete Boolean algebra, then every every pointwise bounded sequence of measures on $\mathcal{A}$ is norm bounded in $ba(\mathcal{A})$.*

This theorem, due to its numerous applications, is one of the most important results in the theory of vector measures, see Diestel and Uhl [21, Section I.3], and hence motivates the following definition.

Definition 2.5.2 (Schachermayer [91])**.** A Boolean algebra $\mathcal{A}$ has the *Nikodym property* if every pointwise bounded sequence of measures on $\mathcal{A}$ is norm bounded in $ba(\mathcal{A})$.

It is a folklore fact that $\mathcal{A}$ has the Nikodym property if and only if it satisfies a formally weaker property, stated in the following proposition (after [99, Proposition 2.4]).

Proposition 2.5.3. *Let $\mathcal{A}$ be a Boolean algebra. The following are equivalent:*

(1) $\mathcal{A}$ has the Nikodym property;

(2) for every sequence $\langle\mu_n\colon n\in\omega\rangle$ of measures on $\mathcal{A}$ such that $\mu_n(A)\to 0$ for all $A\in\mathcal{A}$, we have $\sup_{n\in\omega}\|\mu_n\|<\infty$.

Proof. (1) $\Rightarrow$ (2) Obvious, since $\mu_n(A)\to 0$ implies $\sup_{n\in\omega}|\mu_n(A)|<\infty$ for any $A\in\mathcal{A}$.

(2) $\Rightarrow$ (1) Assume that $\mathcal{A}$ satisfies (2), yet there is a pointwise bounded sequence $\langle\mu_n\colon n\in\omega\rangle$ of measures on $\mathcal{A}$ such that $\sup_{n\to\infty}\|\mu_n\|=\infty$. By taking a suitable subsequence of $\langle\mu_n\colon n\in\omega\rangle$, we may assume that $\lim_{n\to\infty}\|\mu_n\|=\infty$. For every $n\in\omega$ we define a measure:

$$\nu_n=\mu_n/\sqrt{\|\mu_n\|}.$$

Then, for every $A\in\mathcal{A}$ and $n\in\omega$ we have:

$$|\nu_n(A)|=\frac{|\mu_n(A)|}{\sqrt{\|\mu_n\|}}\le\frac{\sup_{m\in\omega}|\mu_m(A)|}{\sqrt{\|\mu_n\|}},$$

so $\lim_{n\to\infty}\nu_n(A)=0$ for every $A\in\mathcal{A}$. On the other hand, since $\|\nu_n\|=\sqrt{\|\mu_n\|}$ for every $n\in\omega$, we have $\lim_{n\to\infty}\|\nu_n\|=\infty$, a contradiction with (2). □

We will use both of the above equivalent statements for the definition of the Nikodym property in this work.

Countable infinite Boolean algebras do not have the Nikodym property, which is connected with the existence of a particular subspace in their Stone spaces.

Example 2.5.4. If $\mathcal{A}$ is a countable infinite Boolean algebra, then $\mathcal{A}$ is a homomorphic image of the free Boolean algebra with $\aleph_0$ generators (by [68, Corollary 9.8]). Then, as a consequence of the Stone duality ([68, Theorem 8.2]), the Stone space $St(\mathcal{A})$ is homeomorphic to an infinite closed subspace of the Stone space of the free algebra—i.e. of the Cantor space 2^ω (see [68, Corollary 9.7]). Thus, $St(\mathcal{A})$ contains a non-trivial convergent sequence.

Example 2.5.5. If $St(\mathcal{A})$ contains a non-trivial sequence $\langle x_n\colon\ n \in \omega\rangle$ convergent to some $x \in St(\mathcal{A})$, then the sequence $\langle n(\delta_{x_n} - \delta_x)\colon\ n \in \omega\rangle$ of Borel measures on $St(\mathcal{A})$ induces a pointwise bounded but not norm bounded sequence of measures on $\mathcal{A}$:

- $\|n(\delta_{x_n} - \delta_x)\| = 2n$ for each $n \in \omega$;
- for every $A \in \mathcal{A}$ and for almost all $n \in \omega$ we have $\delta_{x_n}\big([A]_{\mathcal{A}}\big) = \delta_x\big([A]_{\mathcal{A}}\big)$, and so the sequence $\langle n(\delta_{x_n} - \delta_x)\colon\ n \in \omega\rangle$ is pointwise convergent to 0.

This example is a motivation for studying the spaces $N_{\mathcal{F}}$ in Chapter 3.

Let us now introduce the Grothendieck property of Boolean algebras, which is closely related to the Nikodym property. This property has its origin in Banach space theory, see [42].

Definition 2.5.6. A Banach space X is a *Grothendieck space* if every every weak* convergent sequence in its dual X^* is weakly convergent.

By the characterization of the weak* convergence in the spaces of the form $C(St(\mathcal{A}))^*$ for a Boolean algebra $\mathcal{A}$ (see Proposition 2.4.2), the following property of $\mathcal{A}$ is equivalent to $C(St(\mathcal{A}))$ being a Grothendieck space.

Definition 2.5.7 (Schachermayer [91])**.** A Boolean algebra $\mathcal{A}$ has the *Grothendieck property* if every norm bounded sequence $\langle \mu_n\colon\ n \in \omega\rangle$ in $\mathrm{ba}(\mathcal{A})$ such that $\lim_{n\to\infty}\mu_n(A) = 0$ for every $A \in \mathcal{A}$ is weakly convergent to 0.

It is again a classical result, proved by Grothendieck, that σ-complete Boolean algebras have this property.

Theorem 2.5.8 (Grothendieck [44])**.** *Every σ-complete Boolean algebra has the Grothendieck property.*

On the other hand, we have the analogous condition that forbids the Grothendieck property as for the Nikodym property.

Example 2.5.9. If the Stone space $St(\mathcal{A})$ of a Boolean algebra $\mathcal{A}$ contains a non-trivial sequence $\langle x_n\colon\ n \in \omega\rangle$ convergent to some $x \in St(\mathcal{A})$, then the sequence $\langle \delta_{x_n} - \delta_x\colon\ n \in \omega\rangle$ of Borel measures on $St(\mathcal{A})$ induces a norm bounded sequence which is pointwise convergent to 0 but not weakly convergent to 0:

- $\|\delta_{x_n} - \delta_x\| = 2$ for each $n \in \omega$;
- for every $A \in \mathcal{A}$ and for almost all $n \in \omega$ we have $\delta_{x_n}\big([A]_{\mathcal{A}}\big) = \delta_x\big([A]_{\mathcal{A}}\big)$, and so the sequence $\langle \delta_{x_n} - \delta_x\colon\ n \in \omega\rangle$ is pointwise convergent to 0;
- for a Borel set $U = \{x\}$, the sequence $\langle \delta_{x_n}(U) - \delta_x(U)\colon\ n \in \omega\rangle$ is not convergent to 0.

2.6 Submeasures and standard classes of ideals on ω

We will now introduce several standard definitions concerning submeasures and ideals associated with them.

Definition 2.6.1. A function $\varphi\colon \mathcal{P}(A) \to [0,\infty]$ is a *submeasure on a set* A if $\varphi(\emptyset) = 0$, $\varphi(\{a\}) < \infty$ for every $a \in A$, $\varphi(X) \leq \varphi(Y)$ whenever $X \subseteq Y \subseteq A$, and $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for every $X, Y \subseteq A$.

A submeasure φ on A is *lower semicontinuous* (in short, *lsc*) if

$$\varphi(X) = \sup\left\{\varphi(Y)\colon Y \in [X]^{<\omega}\right\}$$

for every $X \subseteq A$.

The following folklore lemma (see e.g. [80, Lemma 1.3]) enables us to combine several submeasures into one. We prove it here for the sake of completeness.

Lemma 2.6.2. *Let be $\varphi_1, \ldots,\ \varphi_k$ be submeasures respectively on $A_1, \ldots, A_k$. Then, the function*

$$X \mapsto \max\left\{\varphi_i(X \cap A_i)\colon i = 1, \ldots, k\right\}$$

is a submeasure on $A_1 \cup \ldots \cup A_k$. Moreover, if $\varphi_1, \ldots,\ \varphi_k$ are all lsc, then it is an lsc submeasure.

Proof. By induction, it is sufficient to prove the case $k = 2$. It is easy to see that the function is monotone and satisfies both $\max\{\varphi_1(\emptyset), \varphi_2(\emptyset)\} = 0$ and $\max\{\varphi_1(\{a\}), \varphi_2(\{a\})\} < \infty$ for every $a \in A_1 \cup A_2$. Let $X, Y \subseteq A_1 \cup A_2$ and $i \in \{1, 2\}$. We have

$$\varphi_i\big((X \cup Y) \cap A_i\big) \leq \varphi_i(X \cap A_i) + \varphi_i(Y \cap A_i),$$

and so:

$$\begin{aligned}&\max\left\{\varphi_1\big((X \cup Y) \cap A_1\big), \varphi_2\big((X \cup Y) \cap A_2\big)\right\} \leq \\ &\qquad\leq \max\left\{\varphi_1(X \cap A_1), \varphi_2(X \cap A_2)\right\} + \max\left\{\varphi_1(Y \cap A_1), \varphi_2(Y \cap A_2)\right\}.\end{aligned}$$

Moreover, for every $X \subseteq A_1 \cup A_2$, if it is satisfied

$$\varphi_i(X \cap A_i) = \sup\left\{\varphi_i(Y \cap A_i)\colon Y \in [X]^{<\omega}\right\}$$

for $i = 1, 2$, then we have

$$\begin{aligned}&\max\left\{\varphi_1(X \cap A_1), \varphi_2(X \cap A_2)\right\} = \\ &= \sup\left\{\varphi_i(Y \cap A_i)\colon Y \in [X]^{<\omega}, i = 1, 2\right\} = \sup\left\{\max\{\varphi_1(Y \cap A_1), \varphi_2(Y \cap A_2)\}\colon Y \in [X]^{<\omega}\right\}.\end{aligned}$$

□

The following more general result about combining families of submeasures can be proved in essentially the same way.

Lemma 2.6.3. *Let be $\langle \varphi_i\colon\ i \in I\rangle$ be a family such that φ_i is submeasure on A_i for every $i \in I$. Then, the function*

$$X \mapsto \sup_{i \in I} \varphi_i(X \cap A_i)$$

is a submeasure on $\bigcup_{i \in I} A_i$. Moreover, if φ_i is lsc for each $i \in I$, then it is an lsc submeasure.

Let us concentrate now on submeasures on ω. Note that a submeasure φ on ω is lsc if and only if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap [0, n])$ for every $A \subseteq \omega$. We say that a submeasure φ on ω is *bounded* if $\varphi(\omega) < \infty$, otherwise φ is *unbounded*. Note that every non-negative measure μ on ω is a bounded finitely additive submeasure.

We consider the following two standard ideals associated with an lsc submeasure φ on ω:

$$\mathrm{Fin}(\varphi) = \big\{A \subseteq \omega\colon\ \varphi(A) < \infty\big\},$$
$$\mathrm{Exh}(\varphi) = \big\{A \subseteq \omega\colon\ \lim_{n\to\infty} \varphi\big(A \setminus [0, n]\big) = 0\big\},$$

the second one called the *exhaustive ideal* associated to φ. Trivially,

$$\mathrm{Fin} \subseteq \mathrm{Exh}(\varphi) \subseteq \mathrm{Fin}(\varphi).$$

It is also not difficult to show that $\mathrm{Fin}(\varphi)$ is an $\mathbb{F}_\sigma$ ideal and $\mathrm{Exh}(\varphi)$ is an $\mathbb{F}_{\sigma\delta}$ P-ideal (when, as usual, thought of as subsets of the Cantor space 2^ω). The following results of Mazur and Solecki characterize $\mathbb{F}_\sigma$ ideals and analytic P-ideals on ω in terms of submeasures.

Theorem 2.6.4 (Mazur [78])**.** *Let $\mathcal{I}$ be an ideal on ω. Then, $\mathcal{I}$ is an $\mathbb{F}_\sigma$ ideal if and only if there is an unbounded lsc submeasure φ such that $\mathcal{I} = \mathrm{Fin}(\varphi)$.*

Theorem 2.6.5 (Solecki [103])**.** *Let $\mathcal{I}$ be an ideal on ω. Then, $\mathcal{I}$ is an analytic P-ideal if and only if there is an lsc submeasure φ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$.*

For an lsc submeasure φ on ω, by the *core* of φ, denoted $\varphi^\bullet$ (see [24, Section 1]), we mean the submeasure defined by

$$\varphi^\bullet(A) = \lim_{n\to\infty} \varphi\big(A \setminus [0, n]\big)$$

for every $A \subseteq \omega$. We trivially have

$$\mathrm{Exh}(\varphi) = \big\{A \subseteq \omega\colon\ \varphi^\bullet(A) = 0\big\}.$$

Let us note that $\varphi^\bullet(\cdot)$ was denoted by $\|\cdot\|_\varphi$ in [30].

For submeasures φ and ψ on ω we write $\psi \leq \varphi$ if $\psi(A) \leq \varphi(A)$ for every $A \subseteq \omega$. Following Farah [27, page 21], we call a submeasure φ *non-pathological* if for every $A \subseteq \omega$ we have:

$$\varphi(A) = \sup\big\{\mu(A)\colon\ \mu \text{ is a non-negative measure on } \omega \text{ such that } \mu \leq \varphi\big\}.$$

Trivially, every non-negative measure on ω is non-pathological. An ideal $\mathcal{I}$ on ω is *non-pathological* if there is a non-pathological lsc submeasure φ on ω such that $\mathcal{I} = \mathrm{Exh}(\varphi)$.

The family of density submeasures is an important subclass of non-pathological lsc submeasures (see e.g. [27, Chapter 1.13]). We say that a submeasure φ on ω is a *density submeasure* if there exists a sequence $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ of non-negative measures on ω with pairwise disjoint supports contained in ω and such that

$$\varphi = \sup_{n\in\omega} \mu_n.$$

An ideal $\mathcal{I}$ on ω is a *density ideal* if there is a density submeasure $\varphi = \sup_{n\in\omega} \mu_n$ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$. We say in this situation that $\mathcal{I}$ is *generated by the sequence* $\big\langle \mu_n\colon\ n \in \omega\big\rangle$. Let us note an easy observation about density ideals.

Lemma 2.6.6. *For a density ideal $\mathcal{I}$ generated by the sequence $\langle \mu_n \colon\ n \in \omega \rangle$ we have:*

$$\mathcal{I} = \{ A \subseteq \omega \colon\ \lim_{n\to\infty} \mu_n(A) = 0 \}.$$

Proof. Denote by φ the density submeasure $\sup_{n\in\omega} \mu_n$, so that $\mathcal{I} = \mathrm{Exh}(\varphi)$.
($\subseteq$) Fix $A \in \mathrm{Exh}(\varphi)$ and $\varepsilon > 0$. There is $N \in \omega$ such that $\varphi\big(A \setminus [0, N]\big) < \varepsilon$. As the measures $\langle \mu_n \colon\ n \in \omega \rangle$ are disjointly supported, there is $M \in \omega$ such that for any $m \geq M$ the measure μ_m have support disjoint from $[0, N]$. Therefore, for each $m \geq M$ we have

$$\mu_m(A) = \mu_m\big(A \setminus [0, N]\big) \leq \varphi\big(A \setminus [0, N]\big) < \varepsilon.$$

($\supseteq$) Fix $\varepsilon > 0$ and $A \subseteq \omega$ such that $\lim_{n\to\infty} \mu_n(A) = 0$. There is $M \in \omega$ such that $\mu_m(A) < \varepsilon$ for each $m \geq M$. As the measures $\langle \mu_n \colon\ n \in \omega \rangle$ are finitely supported, there is $N \in \omega$ such that $[0, N]$ covers the supports of measures $\mu_0, \ldots, \mu_{M-1}$. Then, we have

$$\varphi\big(A \setminus [0, N]\big) = \sup_{m\geq M} \mu_m\big(A \setminus [0, N]\big) \leq \varepsilon.$$

$\square$

The prototypical ideal for the class of density ideals is the *(asymptotic) density zero ideal* $\mathcal{Z} = \mathrm{Exh}(\varphi_d)$, where we define the *asymptotic density submeasure* φ_d by

$$\varphi_d(A) = \sup_{n\in\omega} \frac{\big|A \cap [2^n, 2^{n+1})\big|}{2^n}$$

for every $A \subseteq \omega$. It is folklore that

$$\mathcal{Z} = \left\{ A \subseteq \omega \colon\ \lim_{n\to\infty} \frac{|A \cap [0, n)|}{n} = 0 \right\}. \tag{$*$}$$

Erdős–Ulam ideals, which were introduced by Just and Krawczyk [58], generalize the asymptotic density zero ideal. We say that $f \colon \omega \to \mathbb{R}^+$ is an *Erdős–Ulam function* if we have

$$\sum_{n\in\omega} f(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{f(n)}{\sum_{i\leq n} f(i)} = 0.$$

The *Erdős–Ulam ideal* associated with an Erdős–Ulam function f is defined as

$$\mathcal{EU}_f = \left\{ A \subseteq \omega \colon\ \lim_{n\to\infty} \frac{\sum_{i\in A\cap[0,n)} f(i)}{\sum_{i\in[0,n)} f(i)} = 0 \right\}.$$

By the above-mentioned equality ($*$), $\mathcal{Z}$ is an Erdős–Ulam ideal. Another example is the *logarithmic density zero ideal* $\mathcal{Z}_{\log} = \mathcal{EU}_f$ associated with the function $f \colon \omega \to \mathbb{R}^+$ given by $f(n) = 1/(n+1)$ for every $n \in \omega$. It turns out that every Erdős–Ulam ideal is a density ideal (see e.g. [27, Section 1.13]).

Summable ideals constitute yet another important class of non-pathological ideals (see e.g. [27, Section 1.12]). Given $f \colon \omega \to [0, \infty)$ such that $\sum_{n\in\omega} f(n) = \infty$, the *summable ideal* corresponding to f is the ideal

$$\mathcal{I}_f = \left\{ A \subseteq \omega \colon\ \sum_{n\in A} f(n) < \infty \right\}.$$

It is easy to see that $\mathcal{I}_f = \mathrm{Exh}(\mu_f) = \mathrm{Fin}(\mu_f)$, where μ_f is a non-negative countably additive (and so lsc) submeasure on ω defined for every $A \subseteq \omega$ by

$$\mu_f(A) = \sum_{n \in A} f(n).$$

It follows that each summable ideal is an $\mathbb{F}_\sigma$ P-ideal. A classical example is the ideal $\mathcal{I}_{1/n}$ corresponding to the function $f(n) = 1/(n+1)$.

Note that, for every lsc submeasure φ on ω there is a bounded lsc submeasure ψ such that $\mathrm{Exh}(\varphi) = \mathrm{Exh}(\psi)$, as it suffices to define ψ by setting:

$$\psi(A) = \min\{\varphi(A), 1\}$$

for every $A \subseteq \omega$. Hernández-Hernández and Hrušák studied such analytic P-ideals for which *every* representation as $\mathrm{Exh}(\psi)$ uses a bounded ψ.

Definition 2.6.7 (Hernández-Hernández and Hrušák [52])**.** An analytic P-ideal $\mathcal{I}$ on ω is said to be *totally bounded* if whenever φ is a lower semicontinuous submeasure on ω for which $\mathcal{I} = \mathrm{Exh}(\varphi)$, then $\varphi(\omega) < \infty$.

A typical example of a totally bounded ideal is $\mathcal{Z}$ ([52, Proposition 3.18]), whereas no summable ideal is totally bounded. Another example of a totally bounded ideal is $\mathrm{tr}(\mathcal{N})$, the *trace of the null ideal* (again [52, Proposition 3.18])—the ideal on $2^{<\omega}$ defined as

$$\mathrm{tr}(\mathcal{N}) = \Big\{A \subseteq 2^{<\omega} \colon \lambda\big(\{x \in 2^\omega \colon \exists^\infty n \in \omega \, (x \restriction \{0, \dots, n\} \in A)\}\big) = 0\Big\}.$$

It turns out that $\mathrm{tr}(\mathcal{N})$ is of the form $\mathrm{Exh}(\varphi)$ for some non-pathological lsc submeasure φ, see e.g. [14, page 5].

The last additional property of ideals which we will use in the sequel is connected with the classical Bolzano–Weierstrass theorem asserting that every bounded sequence of real numbers contains a convergent subsequence. It was first studied by Filipów *et al.* [30, Section 2].

Definition 2.6.8 ([30])**.** An ideal $\mathcal{I}$ on ω has the *Bolzano–Weierstrass property* (in short, *(BWP)*) if for any bounded sequence $\langle x_n \colon n \in \omega \rangle$ of real numbers there is $A \notin \mathcal{I}$ such that $\langle x_n \colon n \in A \rangle$ is $\mathcal{I}$-convergent, that is, there is $x \in \mathbb{R}$ such that $\{n \in A \colon |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$.

For more information on this property see [5], [30], or [33].

We will also use in the sequel the ideal conv, defined as the ideal on $\mathbb{Q} \cap [0, 1]$ generated by sequences in $\mathbb{Q} \cap [0, 1]$ convergent in $[0, 1]$. This ideal is of different type from the ideals associated with submeasures, as it has a descriptive complexity $\mathbb{F}_{\sigma\delta\sigma}$ (see [53, Section 3.4]).

The Katětov order $\leq_K$ is an important tool for comparing the structural complexity of ideals on ω, see e.g. Hrušák [54].

Definition 2.6.9. For two ideals $\mathcal{I}$ and $\mathcal{J}$ on ω, we say that $\mathcal{I}$ is *Katětov below* $\mathcal{J}$, denoted $\mathcal{I} \leq_K \mathcal{J}$, if there is a function $f \colon \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$; we call such a function f a *Katětov reduction.* If it is not true that $\mathcal{I} \leq_K \mathcal{J}$ or $\mathcal{J} \leq_K \mathcal{I}$, then we say that $\mathcal{I}$ and $\mathcal{J}$ are *Katětov incomparable.* Similarly, we say that $\mathcal{I}$ is *Katětov–Blass* *below* $\mathcal{J}$, which we denote by $\mathcal{I} \leq_{KB} \mathcal{J}$, if there is a finite-to-one function $f\colon \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$, and we call such a function f a *Katětov–Blass reduction*. We write $\mathcal{I} <_K \mathcal{J}$ if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \not\leq_K \mathcal{I}$. We say that $\mathcal{I}$ and $\mathcal{J}$ are *Katětov equivalent*, denoted by $\mathcal{I} \equiv_K \mathcal{J}$, if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$.

Of course, the above definitions translate in the natural way via complements to filters on ω.

Trivially, $\mathcal{I} \subseteq \mathcal{J}$ implies $\mathcal{I} \leq_{KB} \mathcal{J}$ (and so $\mathcal{I} \leq_{KB} \mathcal{J}$), with the identity function being the Katětov–Blass reduction. Therefore, in particular we have $\mathrm{Fin} \leq_{KB} \mathcal{J}$ for every ideal $\mathcal{J}$ on ω.

Remark 2.6.10. If $f\colon \omega \to \omega$ is a witness for $\mathcal{I} \leq_K \mathcal{J}$, then $A := f[\omega]$ cannot be finite, as this would imply that $A \in \mathcal{I}$ and so $\omega = f^{-1}[A] \in \mathcal{J}$.

Let us list here more basic observations about the Katětov order.

Lemma 2.6.11. *$\mathcal{I} \equiv_K \mathrm{Fin}$ if and only if $\mathcal{I}$ is not tall.*

Proof. ($\Rightarrow$) Let $f\colon \omega \to \omega$ be a witness for $\mathcal{I} \leq_K \mathrm{Fin}$. By Remark 2.6.10, $f[\omega]$ is an infinite subset of ω. By the definition of $\leq_K$, for every $B \in \mathcal{I}$ the set $f^{-1}[B]$ is finite, and so $B \cap f[\omega]$ is finite. Therefore, $\mathcal{I}$ is not tall.

($\Leftarrow$) As we wrote earlier, we have always $\mathrm{Fin} \leq_K \mathcal{I}$. Let $A \in [\omega]^\omega$ be such that $B \cap A$ is finite for every $B \in \mathcal{I}$. Let $f\colon \omega \to \omega$ be an arbitrary bijection between ω and A. Then, for every $B \in \mathcal{I}$, the set $f^{-1}[B] = f^{-1}[B \cap A]$ is finite, and so f is a witness for $\mathcal{I} \leq_K \mathrm{Fin}$. $\square$

For the next observation we need the notion of a *disjoint sum* of ideals. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on ω. Fix a partition (Ω_1, Ω_2) of ω into two infinite sets and bijections $b_i\colon \omega \to \Omega_i$, $i = 1, 2$. Then, we define the ideal $\mathcal{I}_1 \oplus \mathcal{I}_2$ on ω as follows: for every $A \in \mathcal{P}(\omega)$, $A \in \mathcal{I}_1 \oplus \mathcal{I}_2$ if and only if $b_i^{-1}\big[A \cap \Omega_i\big] \in \mathcal{I}_i$ for $i = 1, 2$.

Remark 2.6.12. For every ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on ω we have $\mathcal{I}_1 \oplus \mathcal{I}_2 \leq_K \mathcal{I}_1, \mathcal{I}_2$, as by the definition the function b_i is a witness for $\mathcal{I}_1 \oplus \mathcal{I}_2 \leq_K \mathcal{I}_i$ where $i = 1, 2$.

The following lemma is folklore.

Lemma 2.6.13. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. If $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J}$ is a P-ideal, then $\mathcal{I} \leq_{KB} \mathcal{J}$.*

Proof. [1] Let $f\colon \omega \to \omega$ be a Katětov reduction witnessing $\mathcal{I} \leq_K \mathcal{J}$. For every $n \in \omega$ the set $A_n = f^{-1}(\{n\})$ belongs to the ideal $\mathcal{J}$. By the assumption about $\mathcal{J}$, there exists $A \in \mathcal{J}$ such that $A_n \subseteq^* A$ for every $n \in \omega$. We define $g\colon \omega \to \omega$ by:

$$g(n) = \begin{cases} n, & \text{if } n \in A, \\ f(n), & \text{if } n \notin A. \end{cases}$$

The function f is finite-to-one on $\omega \setminus A$, as $(\omega \setminus A) \cap A_n$ is finite for every $n \in \omega$, thus g is finite-to-one on ω. The function g is a witness for $\mathcal{I} \leq_{KB} \mathcal{J}$, because for any $X \in \mathcal{I}$ we have $g^{-1}[X] \subseteq f^{-1}[X] \cup A \in \mathcal{J}$, hence also $g^{-1}[X] \in \mathcal{J}$. $\square$

Another basic notion for comparing two ideals on ω is being isomorphic. Again, obviously this definition translates via complements to filters on ω.

[1] This argument was presented to us by Jacek Tryba in personal communication.

Definition 2.6.14. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. We say that $\mathcal{I}$ is *isomorphic to* $\mathcal{J}$, which we denote by $\mathcal{I} \cong \mathcal{J}$, if there is a bijection $f: \omega \to \omega$ such that $A \in \mathcal{I}$ if and only if $f[A] \in \mathcal{J}$, for all $A \subseteq \omega$.

It is immediate that $\mathcal{I} \cong \mathcal{J}$ implies $\mathcal{I} \equiv_K \mathcal{J}$.

We will use the following two lesser-known cardinal invariants of tall ideals on ω, which are closely related to the Katětov order (see [52, Definition 1.1]).

Definition 2.6.15 (Hernández-Hernández and Hrušák [52]). The *covering number* $\mathrm{cov}^*(\mathcal{I})$ and the *uniformity number* $\mathrm{non}^*(\mathcal{I})$ of a tall ideal $\mathcal{I}$ on ω are given by the formulae:

$$\mathrm{cov}^*(\mathcal{I}) = \min\big\{|\mathcal{A}|\colon \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall X \in [\omega]^\omega\ \exists A \in \mathcal{A}\ (|A \cap X| = \aleph_0)\big\},$$

$$\mathrm{non}^*(\mathcal{I}) = \min\big\{|\mathcal{X}|\colon \mathcal{X} \subseteq [\omega]^\omega \text{ and } \forall A \in \mathcal{I}\ \exists X \in \mathcal{X}\ (|A \cap X| < \aleph_0)\big\}.$$

The following folklore result shows their relation to Katětov reducibility.

Proposition 2.6.16 (Proposition 3.1, [52]). *Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω. Then:*

- *if* $\mathcal{I} \leq_K \mathcal{J}$ *then* $\mathrm{cov}^*(\mathcal{J}) \leq \mathrm{cov}^*(\mathcal{I})$,
- *if* $\mathcal{I} \leq_{KB} \mathcal{J}$ *then* $\mathrm{non}^*(\mathcal{I}) \leq \mathrm{non}^*(\mathcal{J})$.

The ideals $\mathrm{tr}(\mathcal{N})$ and $\mathcal{Z}$ are related in the following way.

Fact 2.6.17. $\mathrm{tr}(\mathcal{N}) <_K \mathcal{Z}$.

Proof. By [52, Proposition 3.6] we have $\mathrm{tr}(\mathcal{N}) \leq_K \mathcal{Z}$. Moreover, by Proposition 2.6.16, $\mathrm{tr}(\mathcal{N}) \equiv_K \mathcal{Z}$ would imply that $\mathrm{cov}^*(\mathrm{tr}(\mathcal{N})) = \mathrm{cov}^*(\mathcal{Z})$. But, by [52, Theorem 3.15], this equality does not hold in the random model (obtained by forcing with the quotient algebra of Borel subsets of 2^{ω_2} modulo the ideal of measure zero sets of 2^{ω_2} over a model of CH), and the Katětov order between Borel ideals is absolute between transitive models of ZFC (see e.g. Meza-Alcántara [79, Prop. 1.3.5]). $\square$

2.7 Forcing

We refer the reader to Kunen's book [70] for the elementary theory of forcing. We will use standard notions and notations, following Jech [57].

Let us only briefly recall some basic terminology. A *forcing notion* $(\mathbb{P}, \leq)$ is a preordered set, and elements of $\mathbb{P}$ will be called *conditions*. If $p \leq q$ then we say that p is *stronger than* q. The conditions $p, q \in \mathbb{P}$ are *compatible* if $r \leq p, q$ for some $r \in \mathbb{P}$. Otherwise, the two conditions are *incompatible*.

A subset $A \subseteq \mathbb{P}$ is an *antichain* if any two distinct $p, q \in A$ are incompatible. Let κ be a cardinal. We will say that $\mathbb{P}$ satisfies κ-cc if $|A| < \kappa$ for every antichain $A \subseteq \mathbb{P}$, and we will use the name *ccc* instead of $\aleph_1$-cc. We will also say that $\mathbb{P}$ is *σ-centered* if $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$, where each $\mathbb{P}_n$ is *centered*, i.e. for every finite subset $p_0, \ldots, p_k \in \mathbb{P}_n$ there is $p \in \mathbb{P}_n$ such that $p \leq p_0, \ldots, p_k$. It is easy to see that every σ-centered forcing satisfies ccc.

A subset $D \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$ there is $d \in D$ such that $d \leq p$. A subset $G \subseteq \mathbb{P}$ is a *filter in* $\mathbb{P}$ if G is nonempty, upward closed (i.e. $p \in G$ and $p \leq q$ imply $q \in G$), and for every $p, q \in G$ there is $r \in G$ with $r \leq p, q$. A filter $G \subseteq \mathbb{P}$ is $\mathbb{P}$-*generic* if $G \cap D \neq \emptyset$ for every dense subset $D \subseteq \mathbb{P}$ being an element of the ground model.

Martin's Axiom (in short, MA) states that if $\mathbb{P}$ is ccc and $\mathcal{D}$ is a family of strictly less than $\mathfrak{c}$ dense subsets of $\mathcal{P}$, then there exists a $\mathcal{D}$-*generic* filter G on $\mathcal{P}$, i.e. such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Chapter 3

Spaces $N_{\mathcal{F}}$ and the Nikodym property

3.1 Fs-anti-Nikodym sequences of measures on spaces $N_{\mathcal{F}}$

In this section we study some properties of sequences of measures on spaces $N_{\mathcal{F}}$ connected with the Nikodym property. Let us first introduce the spaces $N_{\mathcal{F}}$ for filters $\mathcal{F}$ on ω.

Definition 3.1.1. Let $\mathcal{F}$ be a filter on ω. Endow the set $N_{\mathcal{F}} = \omega \cup \{p_{\mathcal{F}}\}$, where $p_{\mathcal{F}}$ is a fixed point not belonging to ω, with the topology $\tau_{\mathcal{F}}$ defined in the following way:

- every point of ω is isolated in $N_{\mathcal{F}}$, i.e. $\{n\}$ is open for every $n \in \omega$,
- a set $U \subseteq N_{\mathcal{F}}$ is an open neighborhood of $p_{\mathcal{F}}$ if and only if $U = A \cup \{p_{\mathcal{F}}\}$ for some $A \in \mathcal{F}$.

For every filter $\mathcal{F}$ the space $N_{\mathcal{F}}$ is a countable non-discrete zero-dimensional space. The simplest space of this form is:

Example 3.1.2. The space N_{Fr} is homeomorphic to a non-trivial convergent sequence in $\mathbb{R}$ together with its limit point.

The following observation shows that we can recover the isomorphism class of a filter $\mathcal{F}$ from the topology on $N_{\mathcal{F}}$.

Proposition 3.1.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be filters on ω. If $N_{\mathcal{F}}$ is homeomorphic to $N_{\mathcal{G}}$, then $\mathcal{F} \cong \mathcal{G}$.*

Proof. Let $f\colon N_{\mathcal{F}} \to N_{\mathcal{G}}$ be a homeomorphism. We will show that $f \restriction \omega$ is a witness for $\mathcal{F} \cong \mathcal{G}$. First, $f(p_{\mathcal{F}}) = f(p_{\mathcal{G}})$, as these are the only non-isolated points in $N_{\mathcal{F}}$ and $N_{\mathcal{G}}$, respectively. Thus, $f \restriction \omega$ is a bijection from ω onto ω. Now, let $A \in \mathcal{F}$. Then, $f\big[A \cup \{p_{\mathcal{F}}\}\big]$ is an open neighbourhood of $p_{\mathcal{G}}$, and so $f[A] \in \mathcal{G}$. Analogously, if $A \in \mathcal{G}$ then $f^{-1}\big[A \cup \{p_{\mathcal{G}}\}\big]$ is an open neighbourhood of $p_{\mathcal{F}}$, and so $f^{-1}[A] \in \mathcal{F}$. $\square$

We define now a property of finitely supported sequence of measures connected with the lack of the Nikodym property (cf. Corollary 4.2.2).

Definition 3.1.4. Let X be a zero-dimensional space. We call a sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of finitely supported measures on X *fs-anti-Nikodym*, or shortly an *fs-AN-sequence*, if

$\|\mu_n\| \to \infty$ and $\mu_n(A) \to 0$ for every $A \in \mathrm{Clopen}(X)$[1]. We say that X has the *finitely supported Nikodym property* if there is no fs-anti-Nikodym sequence of measures on X.

We also say that a filter $\mathcal{F}$ on ω has the *fs-Nikodym property* if the space $N_{\mathcal{F}}$ has the finitely supported Nikodym property. We denote by $\mathcal{AN}$ the class of all ideals on ω whose dual filters do not have the fs-Nikodym property.

The first example of a filter without the fs-Nikodym property is Fr, which connects Examples 3.1.2 and 2.5.5.

Example 3.1.5. Fr does not have the fs-Nikodym property.

Proof. We define a sequence of finitely supported measures on N_{Fr} by setting $\mu_n = n \cdot (\delta_n - \delta_{p_{Fr}})$. First, we have $\|\mu_n\| = 2n$ for each $n \in \omega$, and so $\|\mu_n\| \to \infty$. Next, let $A \in \mathrm{Clopen}(N_{Fr})$. If $A \subseteq \omega$, then A is finite, as $N_{Fr} \setminus A$ is an open neighbourhood of p_{Fr}. In the other case, A itself is an open neighbourhood of p_{Fr}, and so $A \cap \omega \in Fr$. Therefore, in both cases there is $N \in \omega$ such that $\mu_n(A) = 0$ for every $n \geq N$. Thus, we proved that $\langle \mu_n \colon n \in \omega \rangle$ is an fs-AN-sequence on N_{Fr}. □

Remark 3.1.6. If $\mathcal{F} \subseteq \mathcal{G}$ are filters on ω, then an fs-AN-sequence $\langle \mu_n \colon n \in \omega \rangle$ of measures on $N_{\mathcal{G}}$ is also fs-AN-sequence on $N_{\mathcal{F}}$, because the topology of $N_{\mathcal{G}}$ is *finer* than the topology of $N_{\mathcal{F}}$, so $\mathrm{Clopen}(N_{\mathcal{F}}) \subseteq \mathrm{Clopen}(N_{\mathcal{G}})$.

Marciszewski and Sobota studied in [76] another property of spaces of the form $N_{\mathcal{F}}$ that is connected with the Grothendieck property in a similar way as the finitely supported Nikodym property is connected with the Nikodym property of Boolean algebras.

Definition 3.1.7. Let X be a Tychonoff space. A sequence $\langle \mu_n \colon n \in \omega \rangle$ of finitely supported measures on X is a *bounded Josefson–Nissenzweig sequence*, or shortly a *BJN-sequence*, if $\|\mu_n\| = 1$ for every $n \in \omega$ and $\mu_n(f) \to 0$ for every $f \in C_b(X)$. We say that X has the *bounded Josefson–Nissenzweig property*, or shortly the *BJNP*, if there is a BJN-sequence of measures on X.

We also say that a filter $\mathcal{F}$ on ω has the *bounded Josefson–Nissenzweig property*, or shortly the *BJNP*, if the space $N_{\mathcal{F}}$ has the BJNP. We denote by $\mathcal{BJNP}$ the class of all ideals on ω whose dual filters have the BJNP.

We comment here about the possible generalization of the finitely supported Nikodym property. One may ask whether we could exchange the condition that $\mu_n(A) \to 0$ for every $A \in \mathrm{Clopen}(X)$ for the condition that $\mu_n(f) \to 0$ for every $f \in C_b(X)$, as in the definition of the bounded Josefson–Nissenzweig property, and in this way generalize the definition of the finitely supported Nikodym property to non-zero-dimensional spaces. However, this property is not satisfied for any topological space (X, τ) which has a weaker compact topology $\tau' \subseteq \tau$. As in Remark 3.1.6, this would imply that there is a sequence of measures $\langle \mu_n \colon n \in \omega \rangle$ on (X, τ') such that $\|\mu_n\| \to \infty$ and $\mu_n(f) \to 0$ for every $f \in C(X, \tau')$, i.e. that $\langle \mu_n \colon n \in \omega \rangle$ is weak* convergent to 0 and unbounded in norm in the dual space $C(X, \tau')^*$—that would contradict the Banach–Steinhaus theorem for $C(X, \tau')$. In particular, for any filter $\mathcal{F}$ the space $N_{\mathcal{F}}$ does not satisfy this stronger property, as $\mathcal{F}$ contains the Fréchet filter Fr and so the topology τ_{Fr} is weaker than $\tau_{\mathcal{F}}$, and N_{Fr} is compact being homeomorphic to a convergent sequence of real numbers together with its limit point.

[1] Let us note that from the technical reasons this definition uses stronger assumption on the norms of measures than the definition of an anti-Nikodym sequence on Boolean algebra.

The following result shows that we can always modify an fs-anti-Nikodym sequence of measures on a space $N_{\mathcal{F}}$ so that the measures have disjoint supports. It is known that the analogous result holds for BJN-sequences of measures on any Tychonoff space (see [76, Lemma 5.1.(3)]).

Proposition 3.1.8. *If a filter $\mathcal{F}$ on ω does not have the fs-Nikodym property, then there is an fs-AN-sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on $N_{\mathcal{F}}$ which is disjointly supported, that is,* $\operatorname{supp}(\mu_k)\cap\operatorname{supp}(\mu_l)=\emptyset$ *for every $k\neq l\in\omega$.*

Proof. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be an fs-AN-sequence of measures on $N_{\mathcal{F}}$.

Step 1. We will find an fs-AN-sequence $\langle\theta_k\colon\ k\in\omega\rangle$ on $N_{\mathcal{F}}$ such that $\operatorname{supp}(\theta_k)\cap\operatorname{supp}(\theta_l)\cap\omega=\emptyset$ for every $k\neq l\in\omega$, i.e. with supports disjoint on ω.

We first define inductively a strictly increasing sequence $\langle n_k\colon\ k\in\omega\rangle$ of natural numbers and an increasing sequence $\langle A_k\colon\ k\in\omega\rangle$ of subsets of $N_{\mathcal{F}}$ satisfying $A_k=\big[0,\max\big(\bigcup_{i=0}^{k-1}\operatorname{supp}(\mu_{n_i})\cap\omega\big)\big]$ and $|\mu_{n_k}|(A_k)<1/k$ for every $k\in\omega$. We start with $n_0=0$ and $A_0=\emptyset$. Assume now that for some $k\in\omega_+$ we have defined natural numbers $n_0<\ldots<n_{k-1}$ and subsets $A_0\subseteq\ldots\subseteq A_{k-1}$ of $N_{\mathcal{F}}$ as required. The set $A_k=\big[0,\max\big(\bigcup_{i=0}^{k-1}\operatorname{supp}(\mu_{n_i})\cap\omega\big)\big]$ is a finite subset of ω, and so a clopen subset of $N_{\mathcal{F}}$. As $\langle\mu_n\colon\ n\in\omega\rangle$ is an fs-AN-sequence, there exists $n_k>n_{k-1}$ such that $|\mu_{n_k}|(A_k)<1/k$.

We now define a measure

$$\theta_k=\mu_{n_k}\restriction(N_{\mathcal{F}}\setminus A_k)$$

for every $k\in\omega$. The sequence $\langle\theta_k\colon\ k\in\omega\rangle$ of measures on $N_{\mathcal{F}}$ will meet the requirements, as:

- for every pair of natural numbers $k<l$ we have $\operatorname{supp}(\theta_k)\cap\omega\subseteq A_{k+1}$ and $\operatorname{supp}(\theta_l)\cap A_{k+1}=\emptyset$, by the definition of the sequence $\langle A_k\colon\ k\in\omega\rangle$,
- for all $k\in\omega_+$ we have
$$\|\theta_k\|\geq\|\mu_{n_k}\|-\|\mu_{n_k}\restriction A_k\|>\|\mu_{n_k}\|-1/k,\text{ and so }\|\theta_k\|\to\infty,$$
- for all $A\in\operatorname{Clopen}(N_{\mathcal{F}})$ we have $\theta_k(A)\to 0$, as $\mu_{n_k}(A)\to 0$ and
$$\big|\theta_k(A)-\mu_{n_k}(A)\big|\leq\|\theta_k-\mu_{n_k}\|=\|\mu_{n_k}\restriction A_k\|<1/k\to 0.$$

Step 2. In Step 1 we constructed an fs-AN-sequence $\langle\theta_n\colon\ n\in\omega\rangle$ on $N_{\mathcal{F}}$ with supports disjoint on ω, and we will now "remove" the point $p_{\mathcal{F}}$ from the supports. We have two cases.

a) $\liminf_{n\to\infty}\big|\theta_n(\{p_{\mathcal{F}}\})\big|<\infty$:

   We can find a subsequence $\langle\theta_{n_k}\colon\ k\in\omega\rangle$ and $\alpha\in\mathbb{R}$ such that $\theta_{n_k}\big(\{p_{\mathcal{F}}\}\big)\to\alpha$. We define a sequence $\langle\nu_k\colon\ k\in\omega\rangle$ of measures by setting

$$\nu_k=(\theta_{n_{2k}}-\theta_{n_{2k+1}})\restriction\omega$$

   for every $k\in\omega$. Then, $\langle\nu_k\colon\ k\in\omega\rangle$ is a disjointly supported fs-AN-sequence on $N_{\mathcal{F}}$, as:

   - for every $k\in\omega$ we have
$$\operatorname{supp}(\nu_k)=\big(\operatorname{supp}(\theta_{n_{2k}})\cup\operatorname{supp}(\theta_{n_{2k+1}})\big)\cap\omega,$$
     thus $\operatorname{supp}(\nu_k)\cap\operatorname{supp}(\nu_l)=\emptyset$ for $k\neq l$ as θ_n's have disjoint supports on ω,

- we have $\|\nu_k\| = \|\theta_{n_{2k}} \upharpoonright \omega\| + \|\theta_{n_{2k+1}} \upharpoonright \omega\| \to \infty$, because $\|\theta_{n_l} \upharpoonright \omega\| \to \infty$ (which follows from $\|\theta_{n_l}\| \to \infty$ and $\|\theta_{n_l} \upharpoonright \{p_{\mathcal{F}}\}\| \to |\alpha|$),
- for all $A \in \operatorname{Clopen}(N_{\mathcal{F}})$ we have:

$$\nu_k(A) = \big(\theta_{n_{2k}} - \theta_{n_{2k+1}}\big)(A \cap \omega) =$$

$$= \big(\theta_{n_{2k}} - \theta_{n_{2k+1}}\big)(A) - \big(\theta_{n_{2k}} - \theta_{n_{2k+1}}\big)(A \cap \{p_{\mathcal{F}}\}) \to 0,$$

since $\langle \theta_{n_k} \colon\ k \in \omega \rangle$ is an fs-AN-sequence and it holds

$$\lim_{k\to\infty} \theta_{n_{2k}}(\{p_{\mathcal{F}}\}) - \theta_{n_{2k+1}}(\{p_{\mathcal{F}}\}) = \alpha - \alpha = 0.$$

b) $\lim_{n\to\infty} \big|\theta_n(\{p_{\mathcal{F}}\})\big| = \infty$:

Without losing of generality we may assume that the sequence $\langle \big|\theta_n(\{p_{\mathcal{F}}\})\big| \colon\ n \in \omega \rangle$ is strictly increasing. If we denote

$$\alpha_n = \frac{\theta_{2n}(\{p_{\mathcal{F}}\})}{\theta_{2n+1}(\{p_{\mathcal{F}}\})},$$

then we get that $|\alpha_n| < 1$ for all $n \in \omega$. We define a sequence $\langle \nu_n \colon\ n \in \omega \rangle$ of measures by setting

$$\nu_n = (\theta_{2n} - \alpha_n \cdot \theta_{2n+1}) \upharpoonright \omega$$

for every $n \in \omega$. Then, $\langle \nu_n \colon\ n \in \omega \rangle$ is a disjointly supported fs-AN-sequence on $N_{\mathcal{F}}$, as:

- for every $n \in \omega$ we have

$$\operatorname{supp}(\nu_n) = \big(\operatorname{supp}(\theta_{2n}) \cup \operatorname{supp}(\theta_{2n+1})\big) \cap \omega,$$

thus $\operatorname{supp}(\nu_k) \cap \operatorname{supp}(\nu_l) = \emptyset$ for $k \neq l$ as θ_n's have disjoint supports on ω,
- we have $\|\nu_n\| = \|\theta_{2n} \upharpoonright \omega\| + |\alpha_n| \cdot \|\theta_{2n+1} \upharpoonright \omega\| \to \infty$, because for every $k \in \omega$ it holds

$$\|\theta_k \upharpoonright \omega\| \geq |\theta_k(\omega)| = |\theta_k(N_{\mathcal{F}}) - \theta_k(\{p_{\mathcal{F}}\})| \to \infty,$$

which follows from $\theta_k(N_{\mathcal{F}}) \to 0$ and the assumption $\big|\theta_k(\{p_{\mathcal{F}}\})\big| \to \infty$,
- for all $A \in \operatorname{Clopen}(N_{\mathcal{F}})$ we have:

$$\nu_n(A) = \big(\theta_{2n} - \alpha_n \cdot \theta_{2n+1}\big)(A \cap \omega) =$$

$$= \big(\theta_{2n} - \alpha_n \cdot \theta_{2n+1}\big)(A) - \big(\theta_{2n} - \alpha_n \cdot \theta_{2n+1}\big)(A \cap \{p_{\mathcal{F}}\}) \to 0,$$

as $\theta_{2n}(\{p_{\mathcal{F}}\}) - \alpha_n \cdot \theta_{2n+1}(\{p_{\mathcal{F}}\}) = 0$ for each $n \in \omega$, $\langle \theta_n \colon\ n \in \omega \rangle$ is an fs-AN-sequence and $\langle \alpha_n \colon\ n \in \omega \rangle$ is bounded.

□

We will now characterize when a sequence of finitely supported measures on a space $N_{\mathcal{F}}$ is an fs-AN-sequence. We will need the following lemma about discrete spaces and the BJN property.

Lemma 3.1.9. *If X is a dicrete topological space, then X does not have BJNP.*

Proof. Let X be discrete. Assume, for the sake of contradiction, that $\langle \mu_n\colon\ n \in \omega\rangle$ is a BJN-sequence of measures on X. For every $n \in \omega$ we define a measure λ_n on the Čech-Stone compactification βX by setting $\lambda_n(A) = \mu_n(A \cap N_{\mathcal{F}})$ for each $A \in Bor(\beta X)$. By definition, $\langle \lambda_n\colon\ n \in \omega\rangle$ is a BJN-sequence on βX.

However, by [68, Theorem 8.9], βX is homeomorphic to the Stone space $St\big(\mathcal{P}\big(|X|\big)\big)$. Thus, by the Grothendieck's Theorem 2.5.8, $C(\beta X)$ is a Grothendieck space. This is a contradiction with the fact that $\langle \lambda_n\colon\ n \in \omega\rangle$ is a BJN-sequence on βX, by [64, Theorem 4.1]. □

Proposition 3.1.10. *Let $\mathcal{F}$ be a filter on ω and $\langle \mu_n\colon\ n \in \omega\rangle$ a sequence of finitely supported measures on $N_{\mathcal{F}}$. Then, $\langle \mu_n\colon\ n \in \omega\rangle$ is an fs-AN-sequence on $N_{\mathcal{F}}$ if and only if the following three conditions simultaneously hold:*

(1) $\lim_{n\to\infty} \|\mu_n\| = \infty$,

(2) $\lim_{n\to\infty} \mu_n(N_{\mathcal{F}}) = 0$,

(3) $\lim_{n\to\infty} \big\|\mu_n \restriction (\omega \setminus A)\big\| = 0$ *for every* $A \in \mathcal{F}$.

Proof. Let $\langle \mu_n\colon\ n \in \omega\rangle$ be an fs-AN-sequence on $N_{\mathcal{F}}$. Then, (1) and (2) hold by the definition. Assume, for the sake of contradiction, that for some $A \in \mathcal{F}$ there is a subsequence $\langle \mu_{n_k}\colon\ k \in \omega\rangle$ such that $\lim_{k\to\infty} \big\|\mu_{n_k} \restriction (\omega \setminus A)\big\| = C$ for some $C > 0$. We may additionally assume that $\big\|\mu_{n_k} \restriction (\omega \setminus A)\big\| \geq C/2$ for every $k \in \omega$. Then, by a similar reasoning as in the proof of Proposition 3.5.1,

$$\big\langle \mu_{n_k} \restriction (\omega \setminus A)/\big\|\mu_{n_k} \restriction (\omega \setminus A)\big\|\colon\ k \in \omega\big\rangle$$

is a BJN-sequence of measures on $\omega \setminus A$. But $\omega \setminus A$ is a discrete subspace of $N_{\mathcal{F}}$, and so we get a contradiction by Lemma 3.1.9.

Assume now that $\langle \mu_n\colon\ n \in \omega\rangle$ is a sequence of finitely supported measures on $N_{\mathcal{F}}$ satisfying conditions (1)–(3). If $C \in \mathrm{Clopen}(N_{\mathcal{F}})$ is such that $p_F \notin C$, then it must be of the form $\omega \setminus A$ for some $A \in \mathcal{F}$, and so $\mu_n(C) \to 0$ by (3). On the other hand, if $C \in \mathrm{Clopen}(N_{\mathcal{F}})$ is such that $p_{\mathcal{F}} \in C$, then we already proved that $\mu_n(N_{\mathcal{F}} \setminus C) \to 0$, thus $\mu_n(C) = \mu_n(N_{\mathcal{F}}) - \mu_n(N_{\mathcal{F}} \setminus C) \to 0$, by (2). □

Remark 3.1.11. We will use the forward implication of the previous theorem only in the case when an fs-AN sequence $\langle \mu_n\colon\ n \in \omega\rangle$ is disjointly supported. The proof of that implication in this case is easier and does not require Lemma 3.1.9: Assume that there is a subsequence $\langle \mu_{n_k}\colon\ k \in \omega\rangle$ and some $A \in \mathcal{F}$ such that $\big\|\mu_{n_k} \restriction (\omega \setminus A)\big\| \geq C/2 > 0$ for every $k \in \omega$, as in the previous proof. For each $k \in \omega$, there exists $F_k \subseteq \mathrm{supp}\,\big(\mu_{n_k} \restriction (\omega \setminus A)\big)$ satisfying $|\mu_{n_k}(F_k)| \geq C/4$. Since $\omega \setminus A$ is a discrete clopen subspace of $N_{\mathcal{F}}$, the set $F = \bigcup_{k\in\omega} F_k$ is clopen in $N_{\mathcal{F}}$. But $\lim_{k\to\infty} |\mu_{n_k}(F)| \geq C/4$, since F_k's are pairwise disjoint, contradicting the fact that $\langle \mu_{n_k}\colon\ k \in \omega\rangle$ is an fs-AN-sequence on $N_{\mathcal{F}}$.

3.2 The fs-Nikodym property and submeasures on ω

In this section we prove results about the correspondence between filters $\mathcal{F}$ without the fs-Nikodym property and density ideals. We will first characterize them using a sequences of non-negative measures on ω, and next using particular ideals on ω associated with submeasures. This gives similar characterizations to the ones for filters with the BJNP, proved by Marciszewski and Sobota [76, Theorem B and Theorem 6.2].

Theorem 3.2.1. *A filter $\mathcal{F}$ on ω has the fs-Nikodym property if and only if there is no (disjointly supported) sequence $\langle \mu_n \colon n \in \omega \rangle$ of finitely supported non-negative measures on $N_{\mathcal{F}}$ such that:*

(1) $\operatorname{supp}(\mu_n) \subseteq \omega$ *for every* $n \in \omega$,

(2) $\lim_{n\to\infty} \mu_n(\omega) = \infty$,

(3) $\lim_{n\to\infty} \mu_n(\omega \setminus A) = 0$ *for every* $A \in \mathcal{F}$.

Proof. Assume that the filter $\mathcal{F}$ does not have the fs-Nikodym property. By Proposition 3.1.8, there is an fs-AN-sequence $\langle \theta_n \colon n \in \omega \rangle$ of measures on $N_{\mathcal{F}}$ with pairwise disjoint supports contained entirely in ω. Let us define a measure $\mu_n = |\theta_n|$ for each $n \in \omega$. It follows that each μ_n is non-negative and finitely supported, $\mu_n(\omega) = \|\theta_n\| \to \infty$, and, by Proposition 3.1.10.(3), that

$$\lim_{n\to\infty} \mu_n(\omega \setminus A) = \lim_{n\to\infty} \big\|\theta_n \restriction (\omega \setminus A)\big\| = 0$$

for every $A \in \mathcal{F}$. Thus, conditions (1)–(3) are satisfied, and, moreover, the measures μ_n have pairwise disjoint supports.

Assume now that there is a sequence $\langle \mu_n \colon n \in \omega \rangle$ of finitely supported non-negative measures on $N_{\mathcal{F}}$ satisfying conditions (1)–(3). For each $n \in \omega$ let us define a measure:

$$\nu_n = \mu_n(\omega) \cdot \delta_{p_{\mathcal{F}}} - \mu_n.$$

By Proposition 3.1.10, it is immediate that $\langle \nu_n \colon n \in \omega \rangle$ is an fs-AN-sequence on $N_{\mathcal{F}}$. $\square$

Note that conditions (2) and (3) in the above theorem imply $\mu_n(A) \to \infty$ for $A \in \mathcal{F}$.

We are now ready to provide a characterization of the class $\mathcal{AN}$ in terms of exhaustive ideals.

Theorem 3.2.2. *Let $\mathcal{F}$ be a filter on ω. Then, the following are equivalent:*

(1) $\mathcal{F}$ does not have the fs-Nikodym property;

(2) there is a density submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{F} \subseteq \operatorname{Exh}(\varphi)^*$;

(3) there is a non-pathological lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{F} \subseteq \operatorname{Exh}(\varphi)^*$.

Proof. Let $\mathcal{I} = \mathcal{F}^*$.

(1) $\Rightarrow$ (2) Assume that $\mathcal{I}^*$ does not have the fs-Nikodym property. Let $\langle \mu_n \colon n \in \omega \rangle$ be a sequence of finitely supported non-negative measures on ω as in Theorem 3.2.1, where we may additionally assume that μ_n's have pairwise disjoint supports such that $\max\big(\operatorname{supp}(\mu_n)\big) < \min\big(\operatorname{supp}(\mu_{n+1})\big)$ for every $n \in \omega$. For each $A \subseteq \omega$ define:

$$\varphi(A) = \sup_{n\in\omega} \mu_n(A).$$

Then, φ is a density submeasure. We have $\varphi(\omega) = \sup_{n\in\omega} \mu_n(\omega) = \infty$ by the condition (2) from Theorem 3.2.1. By the condition (3) from Theorem 3.2.1 and Lemma 2.6.6, we get $\mathcal{I} \subseteq \operatorname{Exh}(\varphi)$.

(2) $\Rightarrow$ (3) Obvious, since every density submeasure is non-pathological.

(3) $\Rightarrow$ (1) Assume that there is a non-pathological lsc submeasure φ such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq \text{Exh}(\varphi)$. By Remark 3.1.6, it is enough to prove that $\text{Exh}(\varphi)^*$ does not have the fs-Nikodym property.

As φ is finite on finite sets and $\varphi(\omega) = \infty$, by the subadditivity of φ for every $n \in \omega$ we get:

$$\varphi\big(\omega \setminus [0, n]\big) = \infty. \tag{$*$}$$

Let $n_0 = 0$. Since φ is lower semi-continuous, there exists $n_1 > n_0$ such that $\varphi([n_0, n_1]) > 2$. By ($*$) we have $\varphi(\omega \setminus [0, n_1]) = \infty$, so again there is $n_2 > n_1$ such that $\varphi([n_1, n_2]) > 3$. We continue in this way until we get a strictly increasing sequence $\langle n_k \colon\ k \in \omega\rangle$ of natural numbers satisfying for every $k \in \omega$ the inequality

$$\varphi\big([n_k, n_{k+1}]\big) > k + 1.$$

The submeasure φ is non-pathological, so for each $k \in \omega$ there exists a non-negative measure μ_k on ω such that $\mu_k \le \varphi$, $\text{supp}(\mu_k) \subseteq [n_k, n_{k+1}]$, and

$$\mu_k\big([n_k, n_{k+1}]\big) > k.$$

Measures μ_k are finitely supported, and $\lim_{k\to\infty} \mu_k(\omega) \ge \lim_{k\to\infty} \mu_k\big([n_k, n_{k+1}]\big) = \infty$.

We claim that the sequence $\langle \mu_k \colon\ k \in \omega\rangle$ satisfies the equality $\lim_{k\to\infty} \mu_k(A) = 0$ for every $A \in \text{Exh}(\varphi)$, and thus, by Theorem 3.2.1, the filter $\text{Exh}(\varphi)^*$ does not have the fs-Nikodym property. Let $A \in \text{Exh}(\varphi)$ and $\varepsilon > 0$. Since $\lim_{n\to\infty} \varphi\big(A \setminus [0, n]\big) = 0$, there is $M \in \omega$ such that $\varphi\big(A \setminus [0, M]\big) < \varepsilon$. Let $k \in \omega$ be such that $n_k > M$. It holds:

$$\mu_k(A) = \mu_k\big(A \cap [n_k, n_{k+1}]\big) \le \varphi\big(A \cap [n_k, n_{k+1}]\big) \le \varphi\big(A \setminus [0, M]\big) < \varepsilon,$$

hence $\lim_{k\to\infty} \mu_k(A) = 0$. □

Corollary 3.2.3. *If $\mathcal{I}$ is a summable ideal, then $\mathcal{I} \in \mathcal{AN}$.*

Proof. Recall that for a summable ideal $\mathcal{I}$, there is $f \colon \omega \to [0, \infty)$ such that $\sum_{n\in\omega} f(n) = \infty$ and $\mathcal{I}_f = \text{Exh}\left(\mu_f\right)$, where μ_f is defined by

$$\mu_f(A) = \sum_{n\in A} f(n).$$

Since by the definition we have $\mu_f(\omega) = \infty$, and μ_f is a countably additive submeasure—so a non-pathological lsc submeasure, by Theorem 3.2.2 we get $\mathcal{I} \in \mathcal{AN}$. □

It follows now from [33, Theorem 3.3] by Filipów and Szuca, and the proof of their [33, Proposition 3.7], that every ideal $\mathcal{I} \in \mathcal{AN}$ is contained in some summable ideal, which gives us a simple characterization of the class $\mathcal{AN}$. We present here a short proof of this result in a more measure-theoretic language.

Theorem 3.2.4. *For every density submeasure φ on ω such that $\varphi(\omega) = \infty$, the ideal* $\text{Exh}(\varphi)$ *is contained in some summable ideal.*

Corollary 3.2.5. *For every ideal $\mathcal{I}$ on ω, we have $\mathcal{I} \in \mathcal{AN}$ if and only if there exists a summable ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$.*

Proof. By Theorems 3.2.2 and 3.2.4, Corollary 3.2.3 and Remark 3.1.6. □

For the proof of Theorem 3.2.4 we will need the following lemma.

Lemma 3.2.6. *For every density submeasure φ on ω such that $\varphi(\omega)=\infty$, there exists a finitely additive set function ρ on $\mathrm{Exh}(\varphi)$ and a sequence $\langle A_k\colon\ k\in\omega\rangle$ of pairwise disjoint finite subsets of ω such that $\rho(A_k)>k$ for every $k\in\omega$.*

Proof. Let $\langle \mu_n\colon\ n\in\omega\rangle$ be a sequence of disjointly supported finitely supported non-negative measures on ω, with supports contained in ω, and such that $\varphi=\sup_{n\in\omega}\mu_n$. Since $\varphi(\omega)=\infty$, there exists an increasing sequence $\langle n_k\colon\ k\in\omega\rangle$ such that

$$\left\|\mu_{n_k}\right\|=\mu_{n_k}(\omega)>k\cdot 2^k$$

for every $k\in\omega$. We define a finitely additive set function $\rho\colon\ \mathrm{Exh}(\varphi)\to[0,\infty)$ by setting

$$\rho(A)=\sum_{k=1}^{\infty}2^{-k}\cdot\mu_{n_k}(A)$$

for each $A\in\mathrm{Exh}(\varphi)$; note that the series in this definition converges because every $A\in\mathrm{Exh}(\varphi)$ satisfies $\lim_{k\to\infty}\mu_{n_k}(A)=0$. For every $k\in\omega$, setting $A_k=\operatorname{supp}\left(\mu_{n_k}\right)$, we have $\rho\big(A_k\big)>k$. □

Proof of Theorem 3.2.4. Let φ be a density submeasure on ω such that $\varphi(\omega)=\infty$, and let ρ and $\langle A_k\colon\ k\in\omega\rangle$ be as in Lemma 3.2.6. We define the function $f\colon\omega\to[0,\infty)$ by setting $f(n)=\rho(\{n\})$ for each $n\in\omega$. Since $\sum_{n\in A_k}f(n)=\rho(A_k)>k$ for every $k\in\omega$, we have $\sum_{n\in\omega}f(n)=\infty$. Moreover, since ρ is finitely additive and non-negative, by Remark 2.4.1 we have

$$\sum_{n\in A}f(n)\le\rho(A)<\infty$$

for every $A\in\mathrm{Exh}(\varphi)$. Therefore, the ideal

$$\mathcal{I}_f=\left\{A\subseteq\omega\colon\ \sum_{n\in A}f(n)<\infty\right\}$$

is a summable ideal such that $\mathrm{Exh}(\varphi)\subseteq\mathcal{I}_f$. □

Theorems 3.2.2 and 3.2.4 yield two more corollaries. Recall from Section 2.6 that $\mathrm{tr}(\mathcal{N})<_K\mathcal{Z}$.

Corollary 3.2.7. *For every $\mathcal{I}\in\mathcal{AN}$ we have $\mathcal{I}\le_K\mathrm{tr}(\mathcal{N})$ and so $\mathcal{I}<_K\mathcal{Z}$.*

Proof. By Hernández-Hernández and Hrušák [52, Proposition 3.5], for every summable ideal $\mathcal{J}$ we have $\mathcal{J}\le_K\mathrm{tr}(\mathcal{N})$. The conclusion now follows from Theorem 3.2.4 and $\mathrm{tr}(\mathcal{N})<_K\mathcal{Z}$. □

Recall that by a classical theorem of Sierpiński (see [11, Theorem 4.1.1]), every ideal of the form $\mathrm{Exh}(\varphi)$ for some lsc submeasure φ is meager and of measure zero (as it is a Borel subset of 2^ω), and every non-principal ultrafilter on ω is non-meager. Therefore, we get:

Corollary 3.2.8. *If $\mathcal{I}\in\mathcal{AN}$, then $\mathcal{I}$ is meager and of measure zero. In particular, if $\mathcal{U}$ is a non-principal ultrafilter on ω, then $\mathcal{U}$ has the fs-Nikodym property.*

It also appears that the notion of a totally bounded ideal is closely related to the class $\mathcal{AN}$. This will follow from Theorem 3.2.2 and the following equivalent, though formally stronger, condition for total boundedness.

Proposition 3.2.9. *For an analytic P-ideal $\mathcal{I}$ on ω, the following conditions are equivalent:*

(a) $\mathcal{I}$ is totally bounded,

(b) for every lsc submeasure φ on ω, if $\mathcal{I} \subseteq \mathrm{Exh}(\varphi)$, then $\varphi(\omega) < \infty$.

Proof. We only need to prove $(a) \Rightarrow (b)$. Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ for an lsc submeasure φ on ω. Assume that there is an lsc submeasure ψ on ω such that $\psi(\omega) = \infty$ and $\mathcal{I} \subseteq \mathrm{Exh}(\psi)$. Using Lemma 2.6.2, we define an lsc submeasure on ω by $\varphi' = \max(\psi, \varphi)$. Of course, we have $\varphi'(\omega) = \infty$ and $\mathrm{Exh}(\varphi') \subseteq \mathrm{Exh}(\varphi)$. Now, if $A \in \mathrm{Exh}(\varphi)$, then also $A \in \mathrm{Exh}(\psi)$, so

$$\lim_{n\to\infty} \varphi'(A \setminus [0,n]) = \max\Big(\lim_{n\to\infty} \varphi(A \setminus [0,n]), \lim_{n\to\infty} \psi(A \setminus [0,n])\Big) = 0,$$

and hence $\mathrm{Exh}(\varphi) \subseteq \mathrm{Exh}(\varphi')$. Thus we have $\mathrm{Exh}(\varphi) = \mathrm{Exh}(\varphi')$, and so $\mathcal{I}$ is not totally bounded. $\square$

Corollary 3.2.10. *Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ for an lsc submeasure φ on ω. If $\mathcal{I}$ is totally bounded, then $\mathcal{I} \notin \mathcal{AN}$.*

Remark 3.2.11. Let us note that the converse to Corollary 3.2.10 does not hold in general. Pathological ideals as counterexamples were given in [36, Theorem 4.12] by Filipów and Tryba (with the ideal defined in [33]) and in [76, Example 6.15]. Both of them are of the form $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ for some lsc submeasure φ on ω such that $\mathcal{I} \nsubseteq \mathrm{Exh}(\psi)$ for any non-pathological lsc submeasure ψ on ω. By Theorem 3.2.2, either ideal $\mathcal{I}$ is in the class $\mathcal{AN}$, and $\varphi(\omega) = \infty$ as $\omega \notin \mathrm{Fin}(\varphi)$, thus $\mathcal{I}$ is not totally bounded. We will give a non-pathological counterexample in Corollary 4.5.27.

3.3 The fs-Nikodym property and the Katětov order

We will focus in this section on the structure of the Katětov order (and its variant) on the class $\mathcal{AN}$. First, we show that $\mathcal{AN}$ is $\leq_K$-downward closed (this strengthens Remark 3.1.6).

Proposition 3.3.1. *If $\mathcal{F} \leq_K \mathcal{G}$ and $\mathcal{G}$ does not have the fs-Nikodym property, then also $\mathcal{F}$ does not have the fs-Nikodym property.*

Proof. Let $f\colon \omega \to \omega$ be a witness for $\mathcal{F} \leq_K \mathcal{G}$ and let $\langle \mu_n\colon\ n \in \omega \rangle$ be a sequence of finitely supported non-negative measures on $N_{\mathcal{G}}$ as in Theorem 3.2.1. For each $n \in \omega$ define a measure:

$$\mu_n' = \sum_{x \in \mathrm{supp}(\mu_n)} \mu_n(\{x\}) \cdot \delta_{f(x)}.$$

Then, we have $\|\mu_n'\| \to \infty$, $\mathrm{supp}(\mu_n') \subseteq \omega$, and $\mu_n' \geq 0$. We also claim that $\lim_{n\to\infty} \mu_n'(\omega \setminus A) = 0$ holds for any $A \in \mathcal{F}$, and thus, by Theorem 3.2.1, the filter $\mathcal{F}$ does not have the fs-Nikodym property. Indeed, if there was $A \in \mathcal{F}$ such that $\limsup_{n\to\infty} \mu_n'(\omega \setminus A) > 0$, then $f^{-1}[A] \in \mathcal{G}$ (as f is a Katětov reduction) and $\limsup_{n\to\infty} \mu_n\big(\omega \setminus f^{-1}[A]\big) > 0$, which would contradict condition (3) from Theorem 3.2.1. $\square$

Proposition 3.3.1 yields the following variant of Theorem 3.2.4.

Theorem 3.3.2. *For every ideal $\mathcal{I}$ on ω, we have $\mathcal{I} \in \mathcal{AN}$ if and only if there exists a summable ideal $\mathcal{J}$ such that $\mathcal{I} \leq_K \mathcal{J}$.*

Proof. Let $\mathcal{I} \in \mathcal{AN}$. By Theorem 3.2.2, there is a summable ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$, and so $\mathcal{I} \leq_K \mathcal{J}$.

On the other hand, let $\mathcal{J}$ be a summable ideal such that $\mathcal{I} \leq_K \mathcal{J}$. By Corollary 3.2.3, $\mathcal{J} \in \mathcal{AN}$, thus by Proposition 3.3.1 we get $\mathcal{I} \in \mathcal{AN}$. □

We will now prove that, in the case of density ideals, several known properties are equivalent with the fs-Nikodym property of the dual filter. The following theorem extends Tryba's [107, Theorem 3.16][2].

Theorem 3.3.3. *Let $\mathcal{I}$ be the density ideal on ω generated by a sequence of measures $\langle \mu_n \colon n \in \omega \rangle$. The following are equivalent:*

1. *$\mathcal{I}$ is isomorphic to an Erdős–Ulam ideal* [3].
2. *$\mathcal{I}$ does not satisfy the Bolzano–Weierstrass property.*
3. *The measures μ_n fulfill the following conditions:*
   a) $\sup_{n\in\omega} \|\mu_n\| < \infty$,
   b) $\lim_{n\to\infty} \mathrm{at}^+(\mu_n) = 0$.
4. *The Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$ contains a countable splitting family.*
5. *$\mathcal{I}$ is totally bounded.*
6. *$\mathcal{I} \equiv_K \mathcal{Z}$.*
7. *$\mathcal{I}^*$ has the fs-Nikodym property.*

Proof. The equivalence of the conditions (1), (2), and (3) was already shown in [107, Theorem 3.16].

(2) ⇔ (4) By [30, Theorem 5.1].

(4) ⇒ (5) By [52, Lemma 3.17].

(5) ⇒ (7) By Corollary 3.2.10.

(1) ⇒ (6) If there exists an Erdős–Ulam ideal $\mathcal{J}$ such that $\mathcal{I} \cong \mathcal{J}$, then in particular $\mathcal{I} \equiv_K \mathcal{J}$. By Farah's [27, Lemma 1.13.10] we have $\mathcal{J} \equiv_K \mathcal{Z}$, thus also $\mathcal{I} \equiv_K \mathcal{Z}$.

(6) ⇒ (7) Assume, for the sake of contradiction, that $\mathcal{I} \equiv_K \mathcal{Z}$ and $\mathcal{I}^*$ does not have the fs-Nikodym property. By Proposition 3.3.1 we get that $\mathcal{Z}$ does not have the fs-Nikodym property, but $\mathcal{Z}$ is totally bounded by [52, Proposition 3.18], which contradicts Corollary 3.2.10.

(7) ⇒ (3) Assume that condition *a)* or condition *b)* from (3) fails. If $\sup_{n\in\omega} \|\mu_n\| = \infty$, then for the density submeasure $\varphi = \sup_{n\in\omega} \mu_n$ we have $\mathcal{I} = \mathrm{Exh}(\varphi)$ and $\varphi(\omega) = \infty$, hence by Theorem 3.2.2 the filter $\mathcal{I}^*$ does not have the fs-Nikodym property. If

[2] We omit in (3) the condition $\limsup_{n\in\omega}(\omega) > 0$, as the definition of an ideal which we use in the thesis already implies this condition.

[3] Note that by Tryba [107, Corollary 3.9] an ideal isomorphic to an Erdős–Ulam ideal need not be itself an Erdős–Ulam ideal.

$\limsup_{n\to\infty} \mathrm{at}^+(\mu_n) > 0$, then $\limsup_{n\to\infty} \varphi(\{n\}) > 0$, and so by [52, Lemma 1.4] the ideal $\mathcal{I}$ is not tall. By Lemma 2.6.11, this implies that $\mathcal{I} \equiv_K \mathrm{Fin}$, and $\mathrm{Fin}^* = Fr$ does not have the fs-Nikodym property by the Example 3.1.5. Thus, by Proposition 3.3.1, the filter $\mathcal{I}^*$ does not have the fs-Nikodym property. In both cases we got a contradiction, hence (3) is satisfied. □

Corollary 3.3.4. *No Erdős–Ulam ideal is in the class $\mathcal{AN}$.*

By Corollary 3.2.7 and Theorem 3.3.3 we get the following corollary.

Corollary 3.3.5. *If $\mathcal{I}$ is a density ideal, then $\mathcal{I} \in \mathcal{AN}$ if and only if $I <_K \mathcal{Z}$.*

Let us note that the equivalence from the previous corollary does not hold even for ideals of the form $\mathcal{I} = \mathrm{Exh}(\varphi)$ for a non-pathological lsc submeasure φ. The ideal $\mathrm{tr}(\mathcal{N})$ is a counterexample, since $\mathrm{tr}(\mathcal{N}) <_K \mathcal{Z}$ (see Fact 2.6.17).

3.3.1 Cofinal structure of $(\mathcal{AN}, \leq_K)$

Now, we will define an operator Φ from ω^ω to $\mathcal{AN}$ such that its image is $\leq_K$-cofinal in $\mathcal{AN}$ and which preserves dominating families. This will enable us to obtain some properties of the order $(\mathcal{AN}, \leq_K)$.

Let $f \in \omega^\omega$ be such that $f(n) > 0$ for every $n \in \omega_+$. We define a sequence $\langle \mu_n^f \colon n \in \omega_+\rangle$ of measures satisfying:

- $\big\langle I_n \colon\ n \in \omega\big\rangle$ is an interval partition of ω, $\min(I_1) = 0$ and for every $n \in \omega_+$ we have $\max(I_n) + 1 = \min(I_{n+1})$,
- the measure μ_n^f is supported on the interval I_n,
- $\|\mu_n^f\| = n$,
- $\mathrm{at}^+(\mu_n^f) = \mathrm{at}^-(\mu_n^f) = 1/f(n)$.

Finally, let
$$\Phi(f) = \mathrm{Exh}\Big(\sup_{n\in\omega_+} \mu_n^f\Big).$$
Then, $\Phi(f)$ is a density ideal.

The following theorem presents the most important properties of the operator Φ.

Theorem 3.3.6.

1. *For every $\mathcal{I} \in \mathcal{AN}$ there exists $f \in \omega^\omega$ such that $\mathcal{I} \leq_K \Phi(f)$.*
2. *For every $g, h \in \omega^\omega$, if $2n^2 \cdot g(n) \leq h(n)$ for all but finitely many $n \in \omega$, then $\Phi(g) \leq_K \Phi(h)$.*

First, for the proof of Theorem 3.3.6 we will need the following measure-theoretic lemma.

Lemma 3.3.7. *Let λ be a non-negative measure on a finite non-empty set A and let $\varepsilon > 0$. If μ is a non-negative measure on a finite set B such that $\mu(B) = \lambda(A)$ and $\mathrm{at}^+(\mu) \leq \varepsilon/(2|A|)$, then there exists a function $f\colon B \to A$ such that for every $C \subseteq A$ we have:*
$$\big|\lambda(C) - \mu\big(f^{-1}[C]\big)\big| \leq \varepsilon.$$

Proof. As $\mathrm{at}^+(\mu) \le \varepsilon/(2|A|)$, we can construct a family $\langle X_a \colon a \in A\rangle$ of pairwise disjoint subsets of B satisfying

$$\lambda(\{a\}) - \varepsilon/(2|A|) < \mu(X_a) \le \lambda(\{a\})$$

for every $a \in A$. Let us define:

$$Y = B \setminus \bigcup\{X_a \colon a \in A\},$$

and note that above conditions together with $\mu(B) = \lambda(A)$ imply that $\mu(Y) < \varepsilon/2$.

Fix any $a_0 \in A$ and define the function $f \colon B \to A$ in the following way:

$$f(b) = \begin{cases} a, & \text{if } b \in X_a, \\ a_0, & \text{if } b \in Y. \end{cases}$$

It is not difficult to check that f has the desired property. $\square$

The next lemma provides a technical condition sufficient for the existence of a Katětov reduction between density ideals.

Lemma 3.3.8. *If $\langle \mu_n \colon n \in \omega_+\rangle$ and $\langle \lambda_n \colon n \in \omega_+\rangle$ are disjointly supported sequences of measures on ω with finite supports such that $\|\mu_n\| = \|\lambda_n\| = n$ for every $n \in \omega_+$, and $\mathrm{at}^+(\mu_n) \le \mathrm{at}^-(\lambda_n)/2n^2$ is satisfied for all but finitely many $n \in \omega_+$, then we have:*

$$\mathrm{Exh}\left(\sup_{n\in\omega_+} \lambda_n\right) \le_K \mathrm{Exh}\left(\sup_{n\in\omega_+} \mu_n\right).$$

Proof. For each $n \in \omega_+$ let $A_n = \mathrm{supp}(\lambda_n)$. We have $|A_n| \cdot \mathrm{at}^-(\lambda_n) \le \|\lambda_n\| = n$, thus $\mathrm{at}^-(\lambda_n) \le n/|A_n|$. Let $N \in \omega_+$ be such that $\mathrm{at}^+(\mu_n) \le \mathrm{at}^-(\lambda_n)/2n^2$ for every $n \ge N$. Then, for every $n \ge N$ we get:

$$\mathrm{at}^+(\mu_n) \le \mathrm{at}^-(\lambda_n)/2n^2 \le \frac{n}{2n^2 \cdot |A_n|} = \frac{1}{n} \cdot \frac{1}{2|A_n|}.$$

Therefore, for $n \ge N$ the assumptions of Lemma 3.3.7 are satisfied with $\lambda = \lambda_n$ on $A = A_n = \mathrm{supp}(\lambda_n)$ and $\mu = \mu_n$ on $B = \mathrm{supp}(\mu_n)$, and $\varepsilon = 1/n$. Thus, for every $n \ge N$ there exists a function $f_n \colon \mathrm{supp}(\mu_n) \to A_n$ such that for every $C \subseteq A_n$ we have:

$$\left|\lambda_n(C) - \mu_n\left(f_n^{-1}[C]\right)\right| \le 1/n. \tag{$*$}$$

Now we define the function $f \colon \omega \to \omega$ by the formula

$$f(l) = \begin{cases} f_n(l), & \text{if } l \in \mathrm{supp}(\mu_n) \text{ for some } n \ge N, \\ 0, & \text{otherwise}, \end{cases}$$

for every $l \in \omega$. We will show that f is a witness for $\mathrm{Exh}\left(\sup_{n\in\omega_+} \lambda_n\right) \le_K \mathrm{Exh}\left(\sup_{n\in\omega_+} \mu_n\right)$, i.e. $f^{-1}[X] \in \mathrm{Exh}\left(\sup_{n\in\omega_+} \mu_n\right)$ for any $X \in \mathrm{Exh}\left(\sup_{n\in\omega_+} \lambda_n\right)$. Let us fix $X \in \mathrm{Exh}\left(\sup_{n\in\omega_+} \lambda_n\right)$ and $p \ge 1$. There is $m \ge \max(N,p)$ such that for every $n \ge m$ we have $\lambda_n(X) < 1/p$.

Then, by ($*$) and the definition of f, for every $n \ge m$ we get:

$$\mu_n\left(f^{-1}[X]\right) \le 2/p,$$

which shows that $\lim_{n\to\infty} \mu_n\left(f^{-1}[X]\right) = 0$ and hence that $f^{-1}[X] \in \mathrm{Exh}\left(\sup_{n\in\omega_+} \mu_n\right)$. $\square$

We are in the position to prove Theorem 3.3.6.

Proof of Theorem 3.3.6. (1). Let $\mathcal{I} \in \mathcal{AN}$. By Theorem 3.2.2, there is a density ideal $\mathrm{Exh}\left(\sup_{n\in\omega} \lambda_n\right)$ for some sequence $\langle \lambda_n \colon n \in \omega \rangle$ of finitely supported measures with disjoint supports such that $\|\lambda_n\| \to \infty$ and $\mathcal{I} \subseteq \mathrm{Exh}\left(\sup_{n\in\omega} \lambda_n\right)$. Without losing of generality we may assume that $\|\lambda_n\| \geq n+1$ for all $n \in \omega$, as by taking a subsequence of the sequence $\langle \lambda_n \colon n \in \omega \rangle$ we can only enlarge the ideal $\mathrm{Exh}\left(\sup_{n\in\omega} \lambda_n\right)$.

We define another sequence $\langle \nu_n \colon n \in \omega_+ \rangle$ of measures by setting

$$\nu_n(A) = \lambda_{n-1}(A) \cdot n / \|\lambda_{n-1}\|$$

for each $A \subseteq \omega$ and $n \in \omega_+$. It follows that $\|\nu_n\| = n$ for every $n \in \omega_+$, and $\mathcal{I} \subseteq \mathrm{Exh}\left(\sup_{n\in\omega_+} \nu_n\right)$. Indeed, if $A \in \mathcal{I}$ then $\lambda_n(A) \to 0$, and so

$$\nu_n(A) = \lambda_{n-1}(A) \cdot n / \|\lambda_{n-1}\| \to 0,$$

since $n/\|\lambda_{n-1}\| \leq 1$ for every $n \in \omega$.

Let us define $f \in \omega^\omega$ by setting $f(n) = 2n^2 \cdot \lceil 1/\mathrm{at}^-(\nu_n) \rceil$ for each $n \in \omega_+$ and $f(0) = 0$. Then, for the sequence of measures $\langle \mu_n^f \colon n \in \omega_+ \rangle$ generating the density ideal $\Phi(f)$ we have $\mathrm{at}^+(\mu_n^f) \leq \mathrm{at}^-(\nu_n)/2n^2$ for every $n \in \omega_+$, and so by Lemma 3.3.8 we get $\mathrm{Exh}\left(\sup_{n\in\omega_+} \nu_n\right) \leq_K \Phi(f)$, thus also $\mathcal{I} \leq_K \Phi(f)$.

(2). Let $g, h \in \omega^\omega$ be such that $2n^2 \cdot g(n) \leq h(n)$ for all but finitely many $n \in \omega$. It follows immediately from Lemma 3.3.8 that $\Phi(g) \leq_K \Phi(h)$. □

Theorem 3.3.6 has several important consequences. The next corollary shows that Φ preserves dominating families.

Corollary 3.3.9. *If $\mathscr{F}$ is a dominating family in ω^ω, then $\Phi[\mathscr{F}]$ is $\leq_K$-dominating in $\mathcal{AN}$, i.e. for any $\mathcal{I} \in \mathcal{AN}$ there is $f \in \mathscr{F}$ such that $\mathcal{I} \leq_K \Phi(f)$.*

Proof. Let us fix a dominating family $\mathscr{F}$ in ω^ω and let $\mathcal{I} \in \mathcal{AN}$. By Theorem 3.3.6.(1) there is $f \in \omega^\omega$ such that $\mathcal{I} \leq_K \Phi(f)$. As $\mathscr{F}$ is dominating, there exists $g \in \mathscr{F}$ such that $g(n) \geq 2n^2 \cdot f(n)$ for all but finitely many $n \in \omega$. By Theorem 3.3.6.(2) we get $\Phi(f) \leq_K \Phi(g)$, and so $\mathcal{I} \leq_K \Phi(g)$. □

The following result follows immediately from Corollary 3.3.9, as by definition there is a dominating family of size $\mathfrak{d}$ in ω^ω.

Corollary 3.3.10. *There exists a family $\mathscr{F} \subseteq \mathcal{AN}$ of size $\mathfrak{d}$ consisting of density ideals and having the property that for any $\mathcal{I} \in \mathcal{AN}$ there is $\mathcal{J} \in \mathcal{F}$ such that $\mathcal{I} \leq_K \mathcal{J}$.*

Corollary 3.3.11. *Every subfamily of $\mathcal{AN}$ of size less than $\mathfrak{b}$ is bounded in the order $(\mathcal{AN}, \leq_K)$.*

Proof. Let $\{\mathcal{I}_\alpha \colon \alpha < \kappa\}$ be a subfamily of $\mathcal{AN}$ for some $\kappa < \mathfrak{b}$. By Theorem 3.3.6.(1), for every $\alpha < \kappa$ there is $f_\alpha \in \omega^\omega$ such that $\mathcal{I}_\alpha \leq_K \Phi(f_\alpha)$. As $\kappa < \mathfrak{b}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$ we have $g(n) \geq 2n^2 \cdot f_\alpha(n)$ for all but finitely many $n \in \omega$, so by Theorem 3.3.6.(2) we get $\mathcal{I}_\alpha \leq_K \Phi(g)$ for all $\alpha < \kappa$. □

Remark 3.3.12. By [53, Theorem 4.2] and Proposition 3.3.1, the order $(\mathcal{AN}, \leq_K)$ contains decreasing chains of cardinality $\mathfrak{c}^+$ and antichains of size $\mathfrak{c}$.

In Section 3.6 we will show that such a $\leq_K$-antichain of size $\mathfrak{c}$ in $\mathcal{AN}$ can consist of summable ideals, or of density ideals. Finally, we prove that the order $(\mathcal{AN}, \leq_K)$ does not have any maximal elements.

Theorem 3.3.13. *For every $\mathcal{I} \in \mathcal{AN}$ there is $\mathcal{J} \in \mathcal{AN}$ such that $\mathcal{I} <_K \mathcal{J}$.*

Proof. Let $\mathcal{I} \in \mathcal{AN}$. By Theorem 3.3.6.(1) there is $f \in \omega^\omega$ satisfying $\mathcal{I} \leq_K \Phi(f)$. Moreover, by Theorem 3.3.6.(2) we may assume that f is sufficiently big so that both $f(n) \geq n^4$ and $f(n) \geq \sum_{i<n} f(i)$ hold for every $n \in \omega$. Define $g \in \omega^\omega$ by setting $g(n) = n \cdot f(f(n))$ for each $n \in \omega$. We will show that $\Phi(f) <_K \Phi(g)$, and so $\mathcal{I} <_K \Phi(g)$.

First, we have $g(n) \geq n \cdot \big(f(n)\big)^4$ for every $n \in \omega$, by the assumption that $f(m) \geq m^4$ for $m = f(n)$. Next, for all $n \geq 2$ we have

$$n \cdot \big(f(n)\big)^4 \geq 2n^2 \cdot f(n), \text{ as } \big(f(n)\big)^3 \geq 2n.$$

Therefore, by Theorem 3.3.6.(2), we get $\Phi(f) \leq_K \Phi(g)$.

Let $\langle \lambda_n \colon n \in \omega_+ \rangle$ and $\langle \mu_n \colon n \in \omega_+ \rangle$ be the sequences of measures generating the ideals $\Phi(g)$ and $\Phi(f)$, respectively, as described in the definition of the operator Φ. Let $A_n = \operatorname{supp}(\lambda_n)$ and $B_n = \operatorname{supp}(\mu_n)$ for each $n \in \omega_+$, so that $|A_n| = n^2 \cdot f(f(n))$ and $|B_n| = n \cdot f(n)$. We have $\lambda_n(x) = 1/\big(n \cdot f(f(n))\big)$ for every $x \in A_n$ and $\mu_n(x) = 1/f(n)$ for every $x \in B_n$.

To show that $\Phi(g) \not\leq_K \Phi(f)$, assume for the sake of contradiction that there is a Katětov reduction $\varphi \colon \omega \to \omega$ witnessing $\Phi(g) \leq_K \Phi(f)$. Since every density ideal is a P-ideal, by Lemma 2.6.13 we may additionally assume that φ is finite-to-one. It is enough to find such a subset $X \subseteq \omega$ that

$$X \notin \operatorname{Exh}\big(\sup_{n\in\omega_+} \mu_n\big) = \Phi(f) \text{ and } \varphi[X] \in \operatorname{Exh}\big(\sup_{n\in\omega_+} \lambda_n\big) = \Phi(g)$$

(as $X \subseteq \varphi^{-1}\big[\varphi[X]\big]$, this will imply $\varphi^{-1}\big[\varphi[X]\big] \notin \operatorname{Exh}\big(\sup_{n\in\omega_+} \mu_n\big)$, contradicting that φ is a Katětov reduction). We proceed in two cases.

__**Case 1.**__ There is a strictly increasing sequence $\langle n_k \colon\ k \in \omega \rangle$ in ω_+ such that for each $k \in \omega$ there exists $F_k \subseteq B_{n_k}$ satisfying $|F_k| = f(n_k)$ and $\varphi[F_k] \subseteq \bigcup \big\{A_i \colon f(i) \geq n_k\big\}$.

Let $X = \bigcup_{k\in\omega} F_k$. Since for every $k \in \omega$ we have $\mu_{n_k}(X) = 1$, it follows that $X \notin \operatorname{Exh}\big(\sup_{n\in\omega_+} \mu_n\big)$. Next, for every $m \in \omega_+$ we have

$$\varphi[X] \cap A_m = \bigcup_{k\in\omega} \varphi[F_k] \cap A_m \subseteq \bigcup \big\{\varphi[F_k] \colon n_k \leq f(m)\big\},$$

and so

$$\big|\varphi[X] \cap A_m\big| \leq \sum \Big\{\big|\varphi[F_k]\big| \colon n_k \leq f(m)\Big\} \leq \sum_{i\leq f(m)} f(i) = f(f(m)) + \sum_{i<f(m)} f(i) \leq 2 \cdot f(f(m)),$$

where the last inequality follows from $f(f(m)) \geq \sum_{i<f(m)} f(i)$. Therefore, for every $m \in \omega_+$ we have:

$$\lambda_m\big(\varphi[X]\big) \leq \frac{2 \cdot f(f(m))}{m \cdot f(f(m))} = \frac{2}{m},$$

which shows that $\lim_{m\to\infty} \lambda_m\big(\varphi[X]\big) = 0$, and hence that $\varphi[X] \in \operatorname{Exh}\big(\sup_{n\in\omega_+} \lambda_n\big)$.

Case 2. If we are not in Case 1, then there exists $N \in \omega_+$ such that for each $n \geq N$ there is no subset $F_n \subseteq B_n$ satisfying $|F_n| = f(n)$ (i.e. $\mu_n(F_n) = 1$) and $\varphi[F_n] \subseteq \bigcup\{A_i \colon f(i) \geq n\}$. Then, for every $n \geq N$ there must exist $F_n \subseteq B_n$ satisfying $\mu_n(F_n) = n-1$ and $\varphi[F_n] \subseteq \bigcup\{A_i \colon f(i) < n\}$. Without losing generality we may assume that $N \geq 3$.

First, let us take any $n \geq N$. As $f(i) \geq i^4$ for every $i \in \omega$, we have

$$\left|\{i \in \omega_+ \colon f(i) < n\}\right| < \sqrt[4]{n} \leq \sqrt{n-1}.$$

Thus, from $F_n \subseteq \bigcup\{\varphi^{-1}[A_i] \colon f(i) < n\}$ it follows that there exists such a number $i(n) \in \omega_+$ that $f(i(n)) < n$ and

$$\mu_n\big(F_n \cap \varphi^{-1}[A_{i(n)}]\big) \geq (n-1)/\sqrt{n-1} = \sqrt{n-1}.$$

By $f(i(n)) < n$ and $f(i(n)) \geq i(n)^4$, it follows that $n \geq i(n)^4 + 1$ and so $\sqrt{n-1} \geq i(n)^2$. For each $n \geq N$ let us define

$$E_n = F_n \cap \varphi^{-1}[A_{i(n)}].$$

Then, $E_n \subseteq B_n$, $\mu_n(E_n) = \sqrt{n-1}$, and $\varphi[E_n] \subseteq A_{i(n)}$.

For each $n \in \omega_+$ let $A_n = \bigcup_{j=1}^{n^2} C_j^n$ be a division of A_n into n^2 disjoint parts of size $f(f(n))$, i.e. $\lambda_n(C_j^n) = 1/n$ for every $j = 1, \dots, n^2$ and $C_j^n \cap C_i^n = \emptyset$ for $i \neq j$. As we have

$$E_n \subseteq \bigcup_{j=1}^{n^2} \varphi^{-1}\left[C_j^{i(n)}\right],$$

for every $n \geq N$ there exists $j(n) \in \{1, \dots, n^2\}$ such that

$$\mu_n\left(E_n \cap \varphi^{-1}\left[C_{j(n)}^{i(n)}\right]\right) \geq \sqrt{n-1}/i(n)^2 \geq 1,$$

where the last inequality follows from $n \geq i(n)^4 + 1$. For each $n \geq N$ let us denote

$$D_n = E_n \cap \varphi^{-1}\left[C_{j(n)}^{i(n)}\right].$$

We have $D_n \subseteq B_n$, $\mu_n(D_n) \geq 1$, and $\varphi[D_n] \subseteq C_{j(n)}^{i(n)} \subseteq A_{i(n)}$.

As φ is finite-to-one, there is an increasing sequence $\langle n_k \colon\ k \in \omega\rangle$ of natural numbers such that $i(n_k) < i(n_j)$ for every $k < j$. Let

$$X = \bigcup_{k \in \omega} D_{n_k}.$$

Since for every $k \in \omega$ we have $\mu_{n_k}(X) \geq 1$, it follows that $X \notin \mathrm{Exh}\left(\sup_{n \in \omega_+} \mu_n\right)$. Finally, we have

$$\varphi[X] \subseteq \bigcup_{k \in \omega} C_{j(n_k)}^{i(n_k)},$$

where the union is disjoint, and so $\lambda_{i(n_k)}(\varphi[X]) \leq 1/i(n_k)$ for every $k \in \omega$ (and $\lambda_m(\varphi[X]) = 0$ for other $m \in \omega$). Thus, we get $\lim_{n\to\infty} \lambda_n\big(\varphi[X]\big) = 0$, due to the fact that $i(n_k) \to \infty$. Therefore, we have $\varphi[X] \in \mathrm{Exh}\left(\sup_{n \in \omega_+} \lambda_n\right)$.

□

Corollary 3.3.14. *For every family $\{\mathcal{I}_\alpha \colon \alpha < \kappa\} \subseteq \mathcal{AN}$, where $\kappa < \mathfrak{b}$, there is $\mathcal{J} \in \mathcal{AN}$ such that $\mathcal{I}_\alpha <_K \mathcal{J}$ for all $\alpha < \kappa$.*

Proof. If $\{\mathcal{I}_\alpha \colon \alpha < \kappa\} \subseteq \mathcal{AN}$ and $\kappa < \mathfrak{b}$, then, by Theorem 3.3.13, for each $\alpha < \kappa$ there is $\mathcal{J}_\alpha \in \mathcal{AN}$ such that $\mathcal{I}_\alpha <_K \mathcal{J}_\alpha$. By Corollary 3.3.11, there is $\mathcal{J} \in \mathcal{AN}$ satisfying $\mathcal{J}_\alpha \leq_K \mathcal{J}$ for every $\alpha < \kappa$, thus we have $\mathcal{I}_\alpha <_K \mathcal{J}$ for all $\alpha < \kappa$. □

3.4 The Tukey type of $(\mathcal{AN}, \leq_K)$

The analysis of the cofinal structure of the order $(\mathcal{AN}, \leq_K)$ shows that it is, in some way, similarly complicated as the cofinal structure of $(\omega^\omega, \leq^*)$. The standard way of comparing cofinal structures of two orders is through Tukey reductions, see e.g. Fremlin [38], and Solecki and Todorčević [104].

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partial orders. A function $f\colon P \to Q$ is said to be a *Tukey reduction* if for every $q_0 \in Q$ there exists $p_0 \in P$ such that, for every $p \in P$, the following implication holds:

$$f(p) \leq_Q q_0 \implies p \leq_P p_0,$$

i.e. if the preimages by f of bounded subsets of Q are bounded subsets of P. We write $P \preceq_T Q$ if there exists a Tukey reduction from P to Q. We say that P and Q are *Tukey equivalent*, which we denote by $P \equiv_T Q$, if $P \preceq_T Q$ and $Q \preceq_T P$. Recently, several mathematicians considered also the Galois–Tukey order $\preceq_{GT}$, see e.g. [83, page 885] where the authors follow the formulation and terminology of Blass [12, Section 4]. The Galois–Tukey order is stronger than $\preceq_T$ in the sense that $P \preceq_{GT} Q$ implies $P \preceq_T Q$. Again, we write $P \equiv_{GT} Q$, if $P \preceq_{GT} Q$ and $Q \preceq_{GT} P$.

An example of using Galois–Tukey reductions for classes of ideals ordered by $\leq_K$ is the following result about the class of all $\mathbb{F}_\sigma$ ideals on ω.

Theorem 3.4.1 (Minami–Sakai [83])**.**

$$(\mathbb{F}_\sigma\text{-}\textit{ideals}, \leq_K) \equiv_{GT} (\mathbb{F}_\sigma\text{-}\textit{ideals}, \leq_{KB}) \equiv_{GT} (\omega^\omega, \leq^*).$$

Using Theorem 3.3.6, it is not difficult to prove the following reduction.

Proposition 3.4.2. $(\mathcal{AN}, \leq_K) \preceq_T (\omega^\omega, \leq^*)$.

Proof. We define a Tukey reduction $\Psi\colon \mathcal{AN} \to \omega^\omega$ in the following way. Let $\mathcal{I} \in \mathcal{AN}$. By Theorem 3.3.6.(1) there exists $f \in \omega^\omega$ such that $\mathcal{I} \leq_K \Phi(f)$. Let us fix any such $f \in \omega^\omega$, and put $\Psi(\mathcal{I}) = g$, where $g(n) = 2n^2 \cdot f(n)$ for every $n \in \omega$.

To prove that Ψ is a Tukey reduction, let us take any $h \in \omega^\omega$. We need to show that the following implication holds for every $\mathcal{I} \in \mathcal{AN}$:

$$\Psi(\mathcal{I}) \leq^* h \implies \mathcal{I} \leq_K \Phi(h).$$

By the definition of Ψ, if $\Psi(\mathcal{I}) \leq^* h$ then there exists $f \in \omega^\omega$ such that $\mathcal{I} \leq_K \Phi(f)$ and $2n^2 \cdot f(n) \leq h(n)$ holds for all but finitely many $n \in \omega$. By Theorem 3.3.6.(2) we get $\Phi(f) \leq_K \Phi(h)$, and so $\mathcal{I} \leq_K \Phi(h)$. □

Note that Corollaries 3.3.10 and 3.3.11 both follow directly from Proposition 3.4.2 by [38, Theorem 1J].

Recently, He, Li and Zhang [51] proved that also the class of all summable ideals ordered by $\leq_K$ has the Tukey type of $(\omega^\omega, \leq^*)$.

Theorem 3.4.3 (He–Li–Zhang [51])**.**

$$(\textit{Summable ideals}, \leq_K) \equiv_{GT} (\textit{Summable ideals}, \leq_{KB}) \equiv_{GT} (\omega^\omega, \leq^*).$$

This result, combined with the characterization of the class $\mathcal{AN}$ from Theorem 3.2.4, gives us the following corollary, strengthening both Proposition 3.4.2 and Theorem 3.3.13.

Corollary 3.4.4. $(\mathcal{AN}, \leq_K) \equiv_T (\omega^\omega, \leq^*)$.

Proof. Let Σ denotes the class of all summable ideals on ω. By Theorem 3.2.4, the order $(\Sigma, \leq_K)$ is a cofinal subset of the order $(\mathcal{AN}, \leq_K)$, therefore we have $(\Sigma, \leq_K) \equiv_T (\mathcal{AN}, \leq_K)$. The result now follows from Theorem 3.4.3. □

3.5 $\mathcal{AN}$ vs $\mathcal{BJNP}$

We will now summarize differences between the classes of ideals $\mathcal{AN}$ and $\mathcal{BJNP}$. As we said in Section 3.1, these are connected with differences between the Nikodym property and the Grothendieck property of Boolean algebras. We start with the following result, stating that $\mathcal{AN}$ is a subclass of $\mathcal{BJNP}$. Note that it actually follows from Theorems 3.2.1 and 3.5.2, but for the convenience of the reader and the self-containment of the thesis we provide here a direct (and more analytic) argument.

Proposition 3.5.1. *If a filter $\mathcal{F}$ does not have the fs-Nikodym property, then it has the BJNP. Consequently, $\mathcal{AN} \subseteq \mathcal{BJNP}$.*

Proof. Let $\langle \mu_n \colon n \in \omega \rangle$ be a fs-anti-Nikodym sequence of measures on $N_{\mathcal{F}}$. As $\|\mu_n\| \to \infty$, by taking a suitable subsequence we may assume that $\|\mu_n\| \geq 1$ for each $n \in \omega$. We define a sequence $\langle \lambda_n \colon n \in \omega \rangle$ of measures by setting $\lambda_n = \mu_n / \|\mu_n\|$ for each $n \in \omega$. We have $\|\lambda_n\| = 1$ for all $n \in \omega$ and $|\lambda_n(A)| = |\mu_n(A)|/\|\mu_n\| \leq |\mu_n(A)|/1 \to 0$ for every $A \in \mathrm{Clopen}(N_{\mathcal{F}})$.

By Corollary 4.4.5, the Čech–Stone compactification $\beta N_{\mathcal{F}}$ is the Stone space of some Boolean algebra, thus in particular it is zero-dimensional. For every $n \in \omega$ we define a measure $\overline{\lambda_n}$ on $\beta N_{\mathcal{F}}$ by setting $\overline{\lambda_n}(A) = \lambda_n(A \cap N_{\mathcal{F}})$ for each $A \in Bor(\beta N_{\mathcal{F}})$. By the definition we have $\|\overline{\lambda_n}\| = 1$ for each $n \in \omega$ and $\overline{\lambda_n}(A) \to 0$ for every $A \in \mathrm{Clopen}(\beta N_{\mathcal{F}})$. We will show that, moreover, for every $f \in C(\beta N_{\mathcal{F}})$ we have $\int_{\beta N_{\mathcal{F}}} f d\overline{\lambda_n} \to 0$, i.e. $\langle \overline{\lambda_n} \colon n \in \omega \rangle$ is a BJN-sequence of measures on $\beta N_{\mathcal{F}}$.

Let us take arbitrary $f \in C(\beta N_{\mathcal{F}})$ and $\varepsilon > 0$. By the zero-dimensionality of $\beta N_{\mathcal{F}}$ and the Stone–Weierstrass theorem, the linear span of the set $\{\chi_A \colon A \in \mathrm{Clopen}(\beta N_{\mathcal{F}})\}$ is norm dense in $C(\beta N_{\mathcal{F}})$ with the supremum norm. Therefore, there exist $m \in \omega$, real numbers $\alpha_1, \ldots, \alpha_m$, and clopen subsets $A_1, \ldots, A_m$ of $\beta N_{\mathcal{F}}$ such that $\|f - \sum_{i=1}^m \alpha_i \chi_{A_i}\|_\infty < \varepsilon/2$. By the convergence of the sequence $\langle \overline{\lambda_n} \colon n \in \omega \rangle$ on clopen sets, there is $N \in \omega$ such that for every $n \geq N$ we have:

$$\sum_{i=1}^m |\alpha_i| \cdot |\overline{\lambda_n}(A_i)| < \varepsilon/2.$$

Then, for every $n \geq N$ it holds:

$$\Big| \int_{\beta N_{\mathcal{F}}} f d\overline{\lambda_n} \Big| \leq \int_{\beta N_{\mathcal{F}}} \Big| f - \sum_{i=1}^m \alpha_i \chi_{A_i} \Big| d\overline{\lambda_n} + \int_{\beta N_{\mathcal{F}}} \Big| \sum_{i=1}^m \alpha_i \chi_{A_i} \Big| d\overline{\lambda_n} \leq$$

$$\leq \Big\| f - \sum_{i=1}^m \alpha_i \chi_{A_i} \Big\|_\infty \cdot \|\overline{\lambda_n}\| + \sum_{i=1}^m |\alpha_i| \cdot |\overline{\lambda_n}(A_i)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which proves that $\langle \overline{\lambda_n} \colon n \in \omega \rangle$ is a BJN-sequence of measures on $\beta N_{\mathcal{F}}$.

To show that $\langle \lambda_n \colon n \in \omega \rangle$ is a BJN-sequence of measures on $N_{\mathcal{F}}$, let $f \in C_b(N_{\mathcal{F}})$. By the universal property of the Čech–Stone compactification (see [39, Theorem 6.5]), there is a continuous extension $\overline{f} \in C(\beta N_{\mathcal{F}})$ of f. We proved that $\int_{\beta N_{\mathcal{F}}} \overline{f} d\overline{\lambda_n} \to 0$, and by the definition of $\overline{\lambda_n}$'s we have $\int_{\beta N_{\mathcal{F}}} \overline{f} d\overline{\lambda_n} = \int_{N_{\mathcal{F}}} f d\lambda_n$ for every $n \in \omega$, thus $\int_{N_{\mathcal{F}}} f d\lambda_n \to 0$. $\square$

Let us recall that our characterization of the fs-Nikodym property via exhaustive ideals, Theorem 3.2.2, is similar to the following characterization of the BJNP.

Theorem 3.5.2 (Marciszewski–Sobota [76])**.** *Let $\mathcal{I}$ be an ideal on ω. Then, the following are equivalent:*

1. $\mathcal{I} \in \mathcal{BJNP}$,
2. there is a density submeasure φ such that $\mathcal{I} \subseteq \mathrm{Exh}(\varphi)$,
3. there is a non-pathological lsc submeasure φ such that $\mathcal{I} \subseteq \mathrm{Exh}(\varphi)$,
4. $\mathcal{I} \leq_K \mathcal{Z}$.

The only difference between conditions (2) and (3) of this theorem and the corresponding conditions of Theorem 3.2.2 is the requirement in Theorem 3.2.2 that the submeasure φ is not bounded, i.e. $\varphi(\omega) = \infty$. However, this apparently small difference changes significantly the properties of families of ideals, for example the cofinal structure of $\mathcal{BJNP}$ is very simple, as $(\mathcal{BJNP}, \leq_K)$ has a maximal element by Theorem 3.5.2, namely $\mathcal{Z}$.

By Theorem 3.5.2 and Corollary 3.2.10 we get:

Corollary 3.5.3. *If φ is a non-pathological lsc submeasure on ω and the ideal $\mathcal{I} = \mathrm{Exh}(\varphi)$ is totally bounded, then $\mathcal{I} \in \mathcal{BJNP} \setminus \mathcal{AN}$.*

In particular, all the density ideals satisfying any of the conditions from Theorem 3.3.3 are elements of $\mathcal{BJNP} \setminus \mathcal{AN}$.

Corollary 3.5.4. *$\mathcal{BJNP} \setminus \mathcal{AN}$ contains $\mathfrak{c}$ many Borel pairwise non-isomorphic ideals on ω.*

Proof. By Kwela [72, Propositions 5 and 6], there is a family of pairwise non-isomorphic Erdős–Ulam ideals of size $\mathfrak{c}$. Every Erdős–Ulam as a density ideal is Borel. The result follows then from Corollary 3.3.4. □

3.6 Examples

In this section we present some examples of classes of ideals on ω which are connected with the class $\mathcal{AN}$.

3.6.1 Summable ideals and large families of non-isomorphic filters

Let us denote by Σ the family of all summable ideals. It appears that $(\Sigma, \leq_K)$—and so, by Corollary 3.2.3, $(\mathcal{AN}, \leq_K)$—has a rich structure. Recall that $\mathcal{P}(\omega)/\mathrm{Fin}$ is the quotient algebra of subsets of ω modulo finite sets, endowed with the ordering induced by $\subseteq^*$. It is well-known that $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq^*)$ contains increasing chains of size $\mathfrak{b}$ and antichains of size $\mathfrak{c}$. The following result connects $(\mathcal{P}(\omega)/Fin, \subseteq^*)$ and $(\Sigma, \leq_K)$.

Theorem 3.6.1 (Guzmán-González and Meza-Alcántara [47])**.** *There is an order embedding of $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq^*)$ into $(\Sigma, \leq_K)$.*

Corollary 3.6.2. *There is an order embedding of $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq^*)$ into $(\mathcal{AN}, \leq_K)$. In particular, $(\mathcal{AN}, \leq_K)$ contains increasing chains of size $\mathfrak{b}$ and antichains of size $\mathfrak{c}$, consisting of summable ideals.*

Recall that in Theorem 3.3.10 and Corollary 3.3.12 we have shown that $(\mathcal{AN}, \leq_K)$ contains dominating families of size $\mathfrak{d}$ and decreasing chains of size $\mathfrak{c}^+$.

We now present the construction of large families of non-isomorphic filters with and without the fs-Nikodym property, following the ideas from [76, Section 6.1]).

Corollary 3.6.3. *There exist families $\mathscr{F}_1$ and $\mathscr{F}_2$ of size $2^{\mathfrak{c}}$, each consisting of pairwise non-isomorphic filters on ω whose dual ideals are tall, such that:*

1. *every $\mathcal{F} \in \mathscr{F}_1$ does not have the fs-Nikodym property,*
2. *every $\mathcal{F} \in \mathscr{F}_2$ has the fs-Nikodym property.*

Proof. Let $\mathscr{G}_2$ be the family of all ultrafilters on ω. We put:

$$\mathscr{G}_1 = \{\mathcal{G} \oplus \mathcal{I}^*_{1/n} \colon \mathcal{G} \in \mathscr{G}_2\}.$$

By Corollary 3.2.8 all members of $\mathscr{G}_2$ do have the fs-Nikodym property. By Corollary 3.2.3, $\mathcal{I}^*_{1/n}$ does not have the fs-Nikodym property, and, by Remark 2.6.12, $\mathcal{G} \oplus \mathcal{I}^*_{1/n} \leq_K \mathcal{I}^*_{1/n}$ for every $\mathcal{G} \in \mathscr{G}_2$. Thus, by Proposition 3.3.1, all members of $\mathscr{G}_1$ do not have the fs-Nikodym property. Moreover, $\big(\mathcal{G} \oplus \mathcal{I}_{1/n}\big)^* = \mathcal{G}^* \oplus \mathcal{I}^*_{1/n}$ is tall for each $\mathcal{G} \in \mathscr{G}_2$, as $\mathcal{G}^*$ and $\mathcal{I}^*_{1/n}$ are tall. As $|\mathscr{G}_2| = |\mathscr{G}_1| = 2^{\mathfrak{c}}$, and each family of pairwise isomorphic filters has cardinality $\leq \mathfrak{c}$, for each $i = 1, 2$ we can select a subfamily $\mathscr{F}_i \subseteq \mathscr{G}_i$ consisting of $2^{\mathfrak{c}}$ many pairwise non-isomorphic filters. □

3.6.2 Simple density ideals

The class of simple density ideals was introduced in Balcerzak *et al.* [4] (cf. also Kwela *et al.* [73]). For every function $g \colon \omega \to [0, \infty)$ satisfying $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$ we define the *simple density ideal* corresponding to g as:

$$\mathcal{Z}_g = \Big\{A \subseteq \omega \colon \limsup_{n\to\infty} \frac{|A \cap n|}{g(n)} = 0\Big\}.$$

It is proved in [4, Theorem 3.2] that every $\mathcal{Z}_g$ is a density ideal. [73, Proposition 10] presents a sufficient condition for a density ideal generated by a sequence of measures $\langle \mu_n \colon\ n \in \omega\rangle$ to be a simple density ideal. Since the statement of this result is rather long and technical, we omit it here. The ideal $\Phi(f)$, defined in Section 3.3.1, satisfies conditions $(i), (ii)$ and (v) from [73, Proposition 10] for any $f \in \omega^\omega$. Conditions (iv) and (vi) are also satisfied for $\Phi(f)$ when f is non-decreasing and $f(n) \to \infty$, as using the notation from that proposition we have $a_n = 1/f(n)$. By the definition of $\Phi(f)$ we have also $\min\big(\operatorname{supp}(\mu^f_n)\big) = \sum_{i<n} f(i)$ for every $n \in \omega$, where $\langle \mu^f_n \colon\ n \in \omega\rangle$ is the sequence of measures generating $\Phi(f)$. Thus, for condition (iii) to be satisfied, we need that

$$\frac{1}{f(n)} \cdot \sum_{i<n} f(i) \to 0.$$

Therefore, by [73, Proposition 10] we have the following corollary.

Corollary 3.6.4. *Let $f \in \omega^\omega$ be non-decreasing and satisfy the condition $f(n)/\sum_{i<n} f(i) \to \infty$. Then, the ideal $\Phi(f)$ is equal to the simple density ideal $\mathcal{Z}_g$, where $g \colon \omega \to [0, \infty)$ is defined by $g(k) = f(n)$ for $k \in \omega$ when $k \in \operatorname{supp}(\mu_n)$.*

It is proved in [73, Theorem 3] that there is a family $\{\mathcal{Z}_{g_\alpha} \colon \alpha < \mathfrak{c}\}$ such that $\mathcal{Z}_{g_\alpha}$ is not $\leq_K$-comparable to $\mathcal{Z}_{g_\beta}$ for any $\alpha \neq \beta$. By Theorem 3.3.3, for every density ideal $\mathcal{I}$ such that $\mathcal{I} \notin \mathcal{AN}$ we have $\mathcal{I} \equiv_K \mathcal{Z}$, and so the $\leq_K$-antichain must lie inside the $\mathcal{AN}$ family:

Corollary 3.6.5. $(\mathcal{AN}, \leq_K)$ *contains antichains of size $\mathfrak{c}$ consisting of density ideals.*

Chapter 4

The Nikodym and Grothendieck properties of Boolean algebras and rings related to ideals

4.1 Concentration points and modifications of anti-Nikodym sequences of measures on Boolean algebras

The goal of this section is to generalize the fact that $\mathcal{A}$ has neither the Nikodym property, nor the Grothendieck property when $St(\mathcal{A})$ contains a non-trivial convergent sequence (cf. Example 4.1.8 below). We will study properties of special sequences of measures on Boolean algebras, which witness the lack of the Nikodym property, and show that their existence implies the lack of the Grothendieck property, too.

For the rest of this section $\mathcal{A}$ will always stand for an infinite Boolean algebra. We start with the following alike definitions.

Definition 4.1.1. A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $St(\mathcal{A})$ is *anti-Nikodym* if we have $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathrm{Clopen}(St(\mathcal{A}))$ and $\sup_{n\in\omega} \|\mu_n\| = \infty$.

A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ is *anti-Nikodym* if the sequence $\langle \widehat{\mu}_n\colon\ n \in \omega\rangle$ of the corresponding Radon extensions on $St(\mathcal{A})$ is anti-Nikodym.

Definition 4.1.2. A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ is *anti-Grothendieck* if it is weak* convergent to 0 but not weakly convergent to 0.

Trivially by definition, $\mathcal{A}$ has the Nikodym property if and only if there are no anti-Nikodym sequences on $\mathcal{A}$, and, similarly, $\mathcal{A}$ has the Grothendieck property if and only if there are no anti-Grothendieck sequences on $\mathcal{A}$ (see Sections 2.4 and 2.5).

Anti-Nikodym sequences of measures have a crucial property, namely, they "concentrate" around points in the Stone spaces in a sense explained by the following lemma and definition.

Lemma 4.1.3. *Let $A \in \mathcal{A}$. If $\langle \mu_n\colon\ n \in \omega\rangle$ is a sequence in $\mathrm{ba}(\mathcal{A})$ such that $\sup_{n\in\omega} \|\mu_n \restriction A\| = \infty$, then there is $t \in [A]_{\mathcal{A}}$ such that for every $B \in \mathcal{A}$ with $t \in [B]_{\mathcal{A}}$ we have $\sup_{n\in\omega} \|\mu_n \restriction B\| = \infty$.*

Proof. Assume that for every $t \in [A]_{\mathcal{A}}$ there is $B_t \in \mathcal{A}$ such that $t \in [B_t]_{\mathcal{A}}$ and $\sup_{n\in\omega} \left\|\mu_n \restriction B_t\right\| < \infty$. By the compactness of $[A]_{\mathcal{A}}$ there are $t_1, \ldots, t_k \in [A]_{\mathcal{A}}$ such that $[A]_{\mathcal{A}} \subseteq \bigcup_{i=1}^{k} [B_{t_i}]_{\mathcal{A}}$. We then have

$$\infty = \sup_{n\in\omega} \left\|\mu_n \restriction A\right\| \leq \sup_{n\in\omega} \sum_{i=1}^{k} \left\|\mu_n \restriction B_{t_i}\right\| < \infty,$$

which is a contradiction. $\square$

Lemma 4.1.3 justifies introducing the following notion, already studied by Sobota [97] and [98] (see also Aizpuru [1]).

Definition 4.1.4. If $\langle \mu_n \colon\ n \in \omega\rangle$ is an anti-Nikodym sequence of measures on $\mathcal{A}$, then a point $t \in St(\mathcal{A})$ is its *Nikodym concentration point* if for every $A \in \mathcal{A}$ with $t \in [A]_{\mathcal{A}}$ we have $\sup_{n\in\omega} \left\|\mu_n \restriction A\right\| = \infty$.

Consequently, Lemma 4.1.3 implies that every anti-Nikodym sequence of measures has a Nikodym concentration point. The following lemma gives as a corollary another useful property of anti-Nikodym sequences.

Lemma 4.1.5. *If $\langle \mu_n \colon\ n \in \omega\rangle$ is an anti-Nikodym sequence of measures on $\mathcal{A}$ and $t \in St(\mathcal{A})$ is its Nikodym concentration point, then for every $\alpha > 0$ and $A \in \mathcal{A}$ with $t \in [A]_{\mathcal{A}}$ there are $n \in \omega$ and $B \leq A$ such that $t \notin [B]_{\mathcal{A}}$ and $\left|\mu_n(B)\right| > \alpha$.*

Proof. Since $\sup_{n\in\omega} \left\|\mu_n \restriction A\right\| = \infty$, there are $C \leq A$ and $n \in \omega$ such that

$$\left|\mu_n(C)\right| > \alpha + \sup_{m\in\omega} \left|\mu_m(A)\right|.$$

If $t \notin [C]_{\mathcal{A}}$, then let $B = C$. Otherwise, $t \notin [A \setminus C]_{\mathcal{A}}$ and we have

$$\left|\mu_n(A \setminus C)\right| = \left|\mu_n(A) - \mu_n(C)\right| \geq \left|\mu_n(C)\right| - \left|\mu_n(A)\right| >$$

$$> \alpha + \sup_{m\in\omega} \left|\mu_m(A)\right| - \left|\mu_n(A)\right| \geq \alpha,$$

so set $B = A \setminus C$. $\square$

An inductive application of Lemma 4.1.5 yields the following standard corollary.

Corollary 4.1.6. *If $\langle \mu_n \colon\ n \in \omega\rangle$ is an anti-Nikodym sequence of measures on $\mathcal{A}$, then there exists an antichain $\langle A_k \colon\ k \in \omega\rangle$ in $\mathcal{A}$ and a strictly increasing sequence $\langle n_k \colon\ k \in \omega\rangle$ such that $\left|\mu_{n_k}(A_k)\right| > k$ for every $k \in \omega$.*

To establish a new criterion for a Boolean algebra without the Nikodym property to not have the Grothendieck property either, we will need however the following stronger notion of Nikodym concentration points.

Definition 4.1.7. Let $\langle \mu_n \colon\ n \in \omega\rangle$ be an anti-Nikodym sequence of measures on $\mathcal{A}$. If $t \in St(\mathcal{A})$ has the property that for every $N \in \omega$ and $A \in \mathcal{A}$ with $t \in [A]_{\mathcal{A}}$ there are a clopen set $U \subseteq [A]_{\mathcal{A}} \setminus \{t\}$ and $n \in \omega$ such that

$$\left|\widehat{\mu}_n(U)\right| > N \cdot \Big(\left\|\mu_n \restriction A^c\right\| + 1\Big),$$

then t is a *strong Nikodym concentration point* of $\langle \mu_n \colon\ n \in \omega\rangle$.

It is easy to see that every strong Nikodym concentration point is a Nikodym concentration point. The converse is however not true, see Example 4.1.10 below.

Example 4.1.8. Assume that $\langle x_n\colon\ n\in\omega\rangle$ is a non-trivial sequence in the Stone space $St(\mathcal{A})$ which converges to some $x\in St(\mathcal{A})$. We showed in Example 2.5.5 that the sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on $\mathcal{A}$, defined for every $n\in\omega$ via their Radon extensions as follows:

$$\widehat{\mu}_n = n\big(\delta_{x_n}-\delta_x\big),$$

is anti-Nikodym. The point x is a strong Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$. To see this, let $N\in\omega$ be arbitrary, and let $A\in\mathcal{A}$ satisfy $x\in[A]_{\mathcal{A}}$. Let $n>N$ be sufficiently large such that $x_n\in[A]_{\mathcal{A}}$, and let $U\subseteq[A]_{\mathcal{A}}\setminus\{x\}$ be a clopen neighbourhood of x_n. We have $\|\mu_n\restriction A^c\|=0$, and so

$$\big|\widehat{\mu}_n(U)\big| = n > N = N\cdot\Big(\big\|\mu_n\restriction A^c\big\|+1\Big).$$

Note that the sequence $\langle\mu_n/n\colon\ n\in\omega\rangle$ is anti-Grothendieck on $\mathcal{A}$, as it was showed in Example 2.5.9. This example will be generalized in Corollaries 4.2.2 and 4.2.5.

Example 4.1.9. Expanding Example 4.1.8, we can easily get an example of an anti-Nikodym sequence of measures with infinitely many strong Nikodym concentration points. Indeed, assume that $\mathcal{A}$ contains an antichain $\langle A_k\colon\ k\in\omega\rangle$ such that for each $k\in\omega$ there is a point $t_k\in[A_k]_{\mathcal{A}}$ being the limit of a non-trivial sequence $\langle x_{k,n}\colon\ n\in\omega\rangle$ which is entirely contained in $[A_k]_{\mathcal{A}}$. Let $a\colon\omega\to\omega\times\omega$ be a bijection; write $a=(b,c)$, where $b,c\colon\omega\to\omega$. For each $n\in\omega$ define the measure μ_n on $\mathcal{A}$ via its Radon extension as follows:

$$\widehat{\mu}_n = n\Big(\delta_{x_{b(n),c(n)}}-\delta_{t_{b(n)}}\Big).$$

It follows, similarly as in the previous Example, that each point t_k is a strong Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$.

Example 4.1.10. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be the sequence of measures on Clopen(2^ω) defined in the proof of [97, Proposition 4.6], i.e. for every $s\in 2^{<\omega}$ let $x(s)$ and $y(s)$ be two arbitrary distinct points from the clopen $[s]$, and we put:

$$\mu_n = \sum_{s\in 2^n}\big(\delta_{x(s)}-\delta_{y(s)}\big).$$

Then, every $t\in 2^\omega$ is a Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$ (see [97, Proposition 4.6]), but $\langle\mu_n\colon\ n\in\omega\rangle$ has no strong Nikodym concentration points. To see this, let $t\in 2^\omega$ be arbitrary and set

$$V=\{x\in 2^\omega\colon x(0)=t(0)\}$$

so that $t\in V$. For every $n\in\omega$ we have

$$\big\|\mu_n\restriction V\big\| = \big\|\mu_n\restriction V^c\big\| = 2^n.$$

Therefore, for every clopen set $U\subseteq V\setminus\{t\}$ and $n\in\omega$ it holds

$$\big|\mu_n(U)\big|\le\big\|\mu_n\restriction V\big\| < \big\|\mu_n\restriction V^c\big\|+1,$$

and so t is not a strong Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$.

Example 4.1.11. Lemma 4.1.13 below asserts that if an anti-Nikodym sequence $\langle\mu_n\colon\ n\in\omega\rangle$ on a Boolean algebra $\mathcal{A}$ has a unique Nikodym concentration point $t\in St(\mathcal{A})$, then t must be a strong Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$. In Example 4.4.24 we describe a large class $\mathscr{C}$ of Boolean subalgebras of $\mathcal{P}(\omega)$ carrying only anti-Nikodym sequences with single Nikodym concentration points and therefore carrying only anti-Nikodym sequences with strong Nikodym concentration points. The simplest element of $\mathscr{C}$ is the Boolean algebra

$$\big\{A\in\mathcal{P}(\omega)\colon\ |A|<\infty \text{ or } |A^c|<\infty\big\},$$

generated by the ideal Fin of finite subsets of ω, whose Stone space is homeomorphic to the space $\{1/(n+1)\colon n\in\omega\}\cup\{0\}$ endowed with the standard Euclidean topology.

Example 4.1.12. Examples 4.1.8 and 4.1.10 show that the Boolean algebra Clopen(2^ω) carries anti-Nikodym sequences with strong Nikodym concentration points as well as anti-Nikodym sequences without such points. Theorem 4.1.20, on the other hand, implies that if a Boolean algebra $\mathcal{A}$ has the Grothendieck property but not the Nikodym property, then no anti-Nikodym sequence of measures on $\mathcal{A}$ can have a strong Nikodym concentration point. Such Boolean algebras have been however so far constructed only consistently, e.g. under Continuum Hypothesis (Talagrand [106]) or under Martin's Axiom (Sobota and Zdomskyy [101]); cf. Question 4.6.2.

The following lemma (aforementioned in Example 4.1.11) provides a handy criterion for Nikodym concentration points to be strong.

Lemma 4.1.13. *Let $\langle\mu_n\colon\ n\in\omega\rangle$ be an anti-Nikodym sequence of measures on $\mathcal{A}$. Let $t\in St(\mathcal{A})$ be the only Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$. Then, t is a strong Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$.*

Proof. Fix $N\in\omega$ and $A\in\mathcal{A}$ such that $t\in[A]_{\mathcal{A}}$. Since $t\notin[A^c]_{\mathcal{A}}$ and $\langle\mu_n\colon\ n\in\omega\rangle$ has only t as its Nikodym concentration point, Lemma 4.1.3 implies that $\sup_{m\in\omega}\left\|\mu_m\restriction A^c\right\|<\infty$. As $\sup_{n\in\omega}\left\|\mu_n\restriction A\right\|=\infty$, by Lemma 4.1.5 we get a clopen $U\subseteq[A]_{\mathcal{A}}\setminus\{t\}$ and $n\in\omega$ such that

$$\left|\widehat{\mu}_n(U)\right|>N\cdot\Big(\sup_{m\in\omega}\left\|\mu_m\restriction A^c\right\|+1\Big)\geq N\cdot\Big(\left\|\mu_n\restriction A^c\right\|+1\Big),$$

which proves that t is a strong Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$. □

Corollary 4.1.14. *Assume that $\langle\mu_n\colon\ n\in\omega\rangle$ is an anti-Nikodym sequence of measures on $\mathcal{A}$ and $t\in St(\mathcal{A})$ is its Nikodym concentration point such that for some $A\in t$ there is no Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$ in $[A]_{\mathcal{A}}\setminus\{t\}$. Then, $\langle\mu_n\restriction A\colon\ n\in\omega\rangle$ is an anti-Nikodym sequence on $\mathcal{A}$ with the unique strong Nikodym concentration point t.*

In the following series of lemmas we will show that if a Boolean algebra $\mathcal{A}$ carries an anti-Nikodym sequence of measures with a strong Nikodym concentration point, then $\mathcal{A}$ carries an anti-Nikodym sequences of measures with pairwise disjoint supports (see Theorem 4.1.18). We start with the following standard result.

Lemma 4.1.15. *Let μ be a measure on $\mathcal{A}$. For every $\varepsilon>0$ and $t\in St(\mathcal{A})$ there is $A\in\mathcal{A}$ such that $t\in[A]_{\mathcal{A}}$ and $|\widehat{\mu}|\big([A]_{\mathcal{A}}\setminus\{t\}\big)<\varepsilon$.*

Proof. Assume not, that is, there are $\varepsilon > 0$ and $t \in St(\mathcal{A})$ such that for every $A \in \mathcal{A}$ with $t \in [A]_{\mathcal{A}}$ we have $|\widehat{\mu}|\big([A]_{\mathcal{A}} \setminus \{t\}\big) \geq \varepsilon$. Let $N \in \omega$ be such that $N \cdot \varepsilon/2 > \|\widehat{\mu}\|$. By the regularity of $|\widehat{\mu}|$, there is an antichain $A_1, \ldots, A_N \in \mathcal{A}$ such that $t \notin [A_i]_{\mathcal{A}}$ and $|\widehat{\mu}|\big([A_i]_{\mathcal{A}}\big) > \varepsilon/2$ for $i = 1, \ldots, N$. Then,

$$\|\widehat{\mu}\| \geq \sum_{i=1}^{N} |\widehat{\mu}|\big([A_i]_{\mathcal{A}}\big) > N \cdot \varepsilon/2 > \|\widehat{\mu}\|,$$

a contradiction. $\square$

Lemma 4.1.16. *Let $\langle \mu_n\colon\ n \in \omega\rangle$ be an anti-Nikodym sequence of measures on $\mathcal{A}$ and $t \in St(\mathcal{A})$ its strong Nikodym concentration point. Let $\langle a_n\colon\ n \in \omega\rangle$ be a sequence of positive real numbers. Then, there are an anti-Nikodym sequence $\langle \nu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$ and an antichain $\langle B_n\colon\ n \in \omega\rangle$ in $\mathcal{A}$ such that for every $n \in \omega$ we have:*

- $t \notin [B_n]_{\mathcal{A}}$,
- $\operatorname{supp}(\widehat{\nu}_n) \subseteq [B_n]_{\mathcal{A}} \cup \{t\}$,
- $\big\|\widehat{\nu}_n \restriction St(\mathcal{A}) \setminus \{t\}\big\| > a_n$.

Proof. Let $\langle M_n\colon\ n \in \omega\rangle$ be an increasing sequence of positive real numbers such that $M_0 > 1$ and

$$\lim_{n\to\infty} a_n \cdot M_n = \lim_{n\to\infty} M_n = \infty. \tag{$*$}$$

Put $A_0 = 1_{\mathcal{A}}$. Since t is a strong Nikodym concentration point of $\langle \mu_n\colon\ n \in \omega\rangle$, there are $n_0 \in \omega$ and a clopen set $U_0 \subseteq [A_0]_{\mathcal{A}} \setminus \{t\}$ such that

$$\big|\widehat{\mu}_{n_0}(U_0)\big| > a_0 \cdot M_0^2.$$

By Lemma 4.1.15, there is $A_1 \in \mathcal{A}$ with $t \in [A_1]_{\mathcal{A}} \subseteq [A_0]_{\mathcal{A}} \setminus U_0$ and such that

$$\big|\widehat{\mu}_{n_0}\big|\big([A_1]_{\mathcal{A}} \setminus \{t\}\big) < 1.$$

Similarly, there are $n_1 > n_0$ and a clopen set $U_1 \subseteq [A_1]_{\mathcal{A}} \setminus \{t\}$ such that

$$\big|\widehat{\mu}_{n_1}(U_1)\big| > a_1 \cdot M_1^2 \cdot \Big(\big\|\mu_{n_1} \restriction A_1^c\big\| + 1\Big).$$

By Lemma 4.1.15, there is $A_2 \in \mathcal{A}$ with $t \in [A_2]_{\mathcal{A}} \subseteq [A_1]_{\mathcal{A}} \setminus U_1$ and such that

$$\big|\widehat{\mu}_{n_1}\big|\big([A_2]_{\mathcal{A}} \setminus \{t\}\big) < 1/2.$$

Again, there are $n_2 > n_1$ and a clopen set $U_2 \subseteq [A_2]_{\mathcal{A}} \setminus \{t\}$ such that

$$\big|\widehat{\mu}_{n_2}(U_2)\big| > a_2 \cdot M_2^2 \cdot \Big(\big\|\mu_{n_2} \restriction A_2^c\big\| + 1\Big).$$

By Lemma 4.1.15, there is $A_3 \in \mathcal{A}$ with $t \in [A_3]_{\mathcal{A}} \subseteq [A_2]_{\mathcal{A}} \setminus U_2$ and such that

$$\big|\widehat{\mu}_{n_2}\big|\big([A_3]_{\mathcal{A}} \setminus \{t\}\big) < 1/3.$$

We continue in this manner until we get a descending sequence $\langle A_k\colon\ k \in \omega\rangle$ in $\mathcal{A}$, a sequence $\langle U_k\colon\ k \in \omega\rangle$ of pairwise disjoint clopen subsets of $St(\mathcal{A})$, and a strictly increasing sequence $\langle n_k\colon\ k \in \omega\rangle$ of natural numbers such that for every $k \in \omega$ we have:

- $t \in [A_k]_{\mathcal{A}}$,
- $U_k \subseteq [A_k \setminus A_{k+1}]_{\mathcal{A}}$,
- $\left|\widehat{\mu}_{n_k}(U_k)\right| > a_k \cdot M_k^2 \cdot \Big(\left\|\mu_{n_k} \restriction A_k^c\right\| + 1\Big)$,
- $\left\|\widehat{\mu}_{n_k} \restriction \big([A_{k+1}]_{\mathcal{A}} \setminus \{t\}\big)\right\| < 1/(k+1)$.

We will define the desired sequence $\langle \nu_k \colon\ k \in \omega \rangle$ in two steps. First, for every $k \in \omega$ put

$$\theta_k = \mu_{n_k} / \big(M_k \cdot \left\|\mu_{n_k} \restriction A_k^c\right\| + M_k\big).$$

We claim that $\langle \theta_k \colon\ k \in \omega \rangle$ is anti-Nikodym. Indeed, for every $A \in \mathcal{A}$ and $k \in \omega$ we have

$$\left|\theta_k(A)\right| = \left|\mu_{n_k}(A)\right| / \big(M_k \cdot \left\|\mu_{n_k} \restriction A_k^c\right\| + M_k\big) \le \left|\mu_{n_k}(A)\right|,$$

as $M_k > 1$, and hence $\lim_{k\to\infty} \left|\theta_k(A)\right| = 0$, which proves that $\langle \theta_k \colon\ k \in \omega \rangle$ is pointwise convergent to 0. For every $k \in \omega$ it also holds

$$\begin{aligned} \left\|\theta_k\right\| \ge \left\|\theta_k \restriction (A_k \setminus A_{k+1})\right\| \ge \left|\widehat{\theta}_k(U_k)\right| = & \qquad (**) \\ = \left|\widehat{\mu}_{n_k}(U_k)\right| / \big(M_k \cdot \left\|\mu_{n_k} \restriction A_k^c\right\| + M_k\big) > a_k \cdot M_k, & \end{aligned}$$

so $\langle \theta_k \colon\ k \in \omega \rangle$ is not norm bounded by $(*)$. Similarly as in $(**)$, for every $k \in \omega$ we also have

$$\begin{aligned} \left|\theta_k(A \wedge A_k^c)\right| \le \left|\mu_{n_k}(A \wedge A_k^c)\right| / \big(M_k \cdot \left\|\mu_{n_k} \restriction A_k^c\right\| + M_k\big) \le & \qquad (***) \\ \le \left\|\mu_{n_k} \restriction A_k^c\right\| / \big(M_k \cdot \left\|\mu_{n_k} \restriction A_k^c\right\| + M_k\big) \le 1/M_k. & \end{aligned}$$

Second, for every $k \in \omega$ define ν_k via its Radon extension $\widehat{\nu}_k$ on $St(\mathcal{A})$ as follows:

$$\widehat{\nu}_k = \widehat{\theta}_k \restriction \big([A_k \setminus A_{k+1}]_{\mathcal{A}} \cup \{t\}\big).$$

It follows immediately that for every $k \neq l \in \omega$ we have

$$\operatorname{supp}(\nu_k) \cap \operatorname{supp}(\nu_l) \subseteq \{t\},$$

so for every $k \in \omega$ put $B_k = A_k \setminus A_{k+1}$. We need to show that $\langle \nu_k \colon\ k \in \omega \rangle$ is anti-Nikodym. Let $A \in \mathcal{A}$ and $k \in \omega$. Having in mind that $A_{k+1} \le A_k$ and $t \in [A_{k+1}]_{\mathcal{A}}$, we get

$$\begin{aligned} \left|\nu_k(A)\right| &= \left|\theta_k\big(A \wedge (A_k \setminus A_{k+1})\big) + \widehat{\theta}_k\big([A]_{\mathcal{A}} \cap \{t\}\big)\right| = \\ &= \left|\theta_k(A) - \theta_k\big(A \setminus (A_k \setminus A_{k+1})\big) + \widehat{\theta}_k\big([A]_{\mathcal{A}} \cap \{t\}\big)\right| = \\ &= \left|\theta_k(A) - \theta_k(A \setminus A_k) - \theta_k(A \wedge A_{k+1}) + \widehat{\theta}_k\big([A]_{\mathcal{A}} \cap \{t\}\big)\right| = \\ &= \left|\theta_k(A) - \theta_k(A \wedge A_k^c) - \widehat{\theta}_k\big([A]_{\mathcal{A}} \cap [A_{k+1}]_{\mathcal{A}}\big) + \widehat{\theta}_k\big([A]_{\mathcal{A}} \cap \{t\}\big)\right| = \\ &= \left|\theta_k(A) - \theta_k(A \wedge A_k^c) - \widehat{\theta}_k\Big([A]_{\mathcal{A}} \cap \big([A_{k+1}]_{\mathcal{A}} \setminus \{t\}\big)\Big)\right| \le \\ &\le \left|\theta_k(A)\right| + \left|\theta_k(A \wedge A_k^c)\right| + \left|\widehat{\theta}_k\Big([A]_{\mathcal{A}} \cap \big([A_{k+1}]_{\mathcal{A}} \setminus \{t\}\big)\Big)\right| \le \\ &\le \left|\theta_k(A)\right| + \left|\theta_k(A \wedge A_k^c)\right| + \left\|\widehat{\mu}_{n_k} \restriction \big([A_{k+1}]_{\mathcal{A}} \setminus \{t\}\big)\right\| < \\ &< \left|\theta_k(A)\right| + 1/M_k + 1/(k+1), \end{aligned}$$

where the last inequality follows from $(***)$ and the properties of the measure $\widehat{\mu}_{n_k}$. Consequently, $\lim_{k\to\infty}\left|\nu_k(A)\right| = 0$, so $\langle \nu_k\colon\ k\in\omega\rangle$ is pointwise convergent to 0. To prove that $\langle \nu_n\colon\ n\in\omega\rangle$ is not norm bounded, take $k\in\omega$ and note that by $(**)$ we have

$$\left\|\nu_k\right\| = \left\|\widehat{\theta}_k \restriction \left([A_k\setminus A_{k+1}]_{\mathcal{A}}\cup\{t\}\right)\right\| \geq \left\|\theta_k\restriction\left(A_k\setminus A_{k+1}\right)\right\| > a_k\cdot M_k,$$

which implies that $\sup_{k\in\omega}\left\|\nu_k\right\| = \infty$. Besides, for every $k\in\omega$ we have

$$\left\|\widehat{\nu}_k\restriction St(\mathcal{A})\setminus\{t\}\right\| = \left\|\nu_k\restriction\left(A_k\setminus A_{k+1}\right)\right\| > a_k,$$

which finishes the proof. □

The proof of the next lemma follows closely Step 2 of the proof of Proposition 3.1.8.

Lemma 4.1.17. *Let $\langle a_n\colon\ n\in\omega\rangle$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = \infty$. Let $\langle \rho_n\colon\ n\in\omega\rangle$ be an anti-Nikodym sequence of measures on $\mathcal{A}$, $t\in St(\mathcal{A})$, and $\langle B_n\colon\ n\in\omega\rangle$ an antichain in $\mathcal{A}$ such that for every $n\in\omega$ we have:*

(a) $t\notin[B_n]_{\mathcal{A}}$,

(b) $\operatorname{supp}(\widehat{\rho}_n)\subseteq[B_n]_{\mathcal{A}}\cup\{t\}$,

(c) $\left\|\widehat{\rho}_n\restriction St(\mathcal{A})\setminus\{t\}\right\| > a_n$.

Then, there are an anti-Nikodym sequence $\langle \nu_k\colon\ k\in\omega\rangle$ of measures on $\mathcal{A}$ and a strictly increasing sequence $\langle n_k\colon\ k\in\omega\rangle$ such that for every $k\in\omega$ we have:

(A) $\operatorname{supp}(\widehat{\nu}_k)\subseteq\left[B_{n_{2k}}\vee B_{n_{2k+1}}\right]_{\mathcal{A}}$,

(B) $\left\|\nu_k\right\| > a_{n_{2k}}$.

Proof. We divide the proof into two cases.

Case 1. We have $\liminf_{n\to\infty}\left|\widehat{\rho}_n(\{t\})\right| < \infty$.

We find a subsequence $\langle \rho_{n_k}\colon\ k\in\omega\rangle$ and $\alpha\in\mathbb{R}$ such that $\lim_{k\to\infty}\widehat{\rho}_{n_k}(\{t\}) = \alpha$. By (c) and $\lim_{n\to\infty} a_n = \infty$ the sequence $\langle \rho_{n_k}\colon\ k\in\omega\rangle$, is anti-Nikodym on $\mathcal{A}$. For every $k\in\omega$ we define ν_k via its Radon extension $\widehat{\nu}_k$ on $St(\mathcal{A})$ as follows:

$$\widehat{\nu}_k = \left(\widehat{\rho}_{n_{2k}} - \widehat{\rho}_{n_{2k+1}}\right)\restriction\left(St(\mathcal{A})\setminus\{t\}\right).$$

For every $k\in\omega$ by (b) we have

$$\operatorname{supp}(\widehat{\nu}_k)\subseteq\left(\operatorname{supp}(\widehat{\rho}_{n_{2k}})\cup\operatorname{supp}(\widehat{\rho}_{n_{2k+1}})\right)\setminus\{t\}\subseteq\left[B_{n_{2k}}\vee B_{n_{2k+1}}\right]_{\mathcal{A}},$$

so (A) holds. Since $\left\|\widehat{\rho}_n\restriction St(\mathcal{A})\setminus\{t\}\right\| > a_n$ for every $n\in\omega$ and $\langle B_n\colon\ n\in\omega\rangle$ is an antichain in $\mathcal{A}$, we get that

$$\left\|\nu_k\right\| = \left\|\widehat{\nu}_k\right\| > a_{n_{2k}} + a_{n_{2k+1}} > a_{n_{2k}}$$

for all $k\in\omega$, so (B) holds as well. In particular, $\langle \nu_k\colon\ k\in\omega\rangle$ is not norm bounded.

We need to prove that $\langle \nu_k\colon\ k\in\omega\rangle$ is anti-Nikodym on $\mathcal{A}$. Let $A\in\mathcal{A}$. For any $l\in\omega$ we have

$$\begin{aligned}\nu_l(A) = \widehat{\nu}_l\left([A]_{\mathcal{A}}\right) &= \left(\widehat{\rho}_{n_{2l}} - \widehat{\rho}_{n_{2l+1}}\right)\left([A]_{\mathcal{A}}\setminus\{t\}\right) = \\ = \left(\widehat{\rho}_{n_{2l}} - \widehat{\rho}_{n_{2l+1}}\right)\left([A]_{\mathcal{A}}\right) &- \left(\widehat{\rho}_{n_{2l}} - \widehat{\rho}_{n_{2l+1}}\right)\left([A]_{\mathcal{A}}\cap\{t\}\right) =\end{aligned}$$

$$= \big(\rho_{n_{2l}}(A) - \rho_{n_{2l+1}}(A)\big) - \Big(\widehat{\rho}_{n_{2l}}\big([A]_{\mathcal{A}} \cap \{t\}\big) - \widehat{\rho}_{n_{2l+1}}\big([A]_{\mathcal{A}} \cap \{t\}\big)\Big).$$

As $\langle \rho_{n_k}\colon\ k \in \omega \rangle$ is pointwise convergent to 0 and it also holds

$$\lim_{k\to\infty} \Big(\widehat{\rho}_{n_{2k}}(\{t\}) - \widehat{\rho}_{n_{2k+1}}(\{t\})\Big) = \alpha - \alpha = 0,$$

we get that $\lim_{k\to\infty} \nu_k(A) = 0$. It follows that $\langle \nu_k\colon\ k \in \omega \rangle$ is pointwise convergent to 0, and so is anti-Nikodym.

Case 2. We have $\lim_{n\to\infty} \big|\widehat{\rho}_n(\{t\})\big| = \infty$.

We find a subsequence $\langle \widehat{\rho}_{n_k}\colon\ k \in \omega \rangle$ such that $\big\langle \big|\widehat{\rho}_{n_k}(\{t\})\big|\colon\ k \in \omega \big\rangle$ is strictly increasing and $\big|\widehat{\rho}_{n_0}(\{t\})\big| > 0$. By (c) the sequence $\langle \rho_{n_{2k}}\colon\ k \in \omega \rangle$ is still anti-Nikodym. For each $k \in \omega$ we denote

$$\alpha_k = \frac{\widehat{\rho}_{n_{2k}}(\{t\})}{\widehat{\rho}_{n_{2k+1}}(\{t\})},$$

so $0 < |\alpha_k| < 1$. For every $k \in \omega$ we define ν_k via its Radon extension $\widehat{\nu}_k$ on $St(\mathcal{A})$ as follows:

$$\widehat{\nu}_k = \big(\widehat{\rho}_{n_{2k}} - \alpha_k \cdot \widehat{\rho}_{n_{2k+1}}\big) \restriction \big(St(\mathcal{A}) \setminus \{t\}\big).$$

Again, as previously, for every $k \in \omega$ we have

$$\operatorname{supp}(\widehat{\nu}_k) \subseteq \big(\operatorname{supp}(\widehat{\rho}_{n_{2k}}) \cup \operatorname{supp}(\widehat{\rho}_{n_{2k+1}})\big) \setminus \{t\} \subseteq \big[B_{n_{2k}} \vee B_{n_{2k+1}}\big]_{\mathcal{A}},$$

so (A) holds.

Since $\big\|\widehat{\rho}_n \restriction St(\mathcal{A}) \setminus \{t\}\big\| > a_n$ for every $n \in \omega$ and $\langle B_n\colon\ n \in \omega \rangle$ is an antichain in $\mathcal{A}$, we get that

$$\big\|\nu_k\big\| = \big\|\widehat{\nu}_k\big\| > a_{n_{2k}} + |\alpha_k| \cdot a_{n_{2k+1}} \geq a_{n_{2k}}$$

for all $k \in \omega$, so (B) holds, too. Again, $\langle \nu_k\colon\ k \in \omega \rangle$ is also not norm bounded.

We again need to prove that $\langle \nu_k\colon\ k \in \omega \rangle$ is pointwise convergent to 0. Let $A \in \mathcal{A}$. For every $l \in \omega$ we have

$$\nu_l(A) = \widehat{\nu}_l\big([A]_{\mathcal{A}}\big) = \big(\widehat{\rho}_{n_{2l}} - \alpha_l \cdot \widehat{\rho}_{n_{2l+1}}\big)\big([A]_{\mathcal{A}} \setminus \{t\}\big) =$$

$$= \big(\widehat{\rho}_{n_{2l}} - \alpha_l \cdot \widehat{\rho}_{n_{2l+1}}\big)\big([A]_{\mathcal{A}}\big) - \big(\widehat{\rho}_{n_{2l}} - \alpha_l \cdot \widehat{\rho}_{n_{2l+1}}\big)\big([A]_{\mathcal{A}} \cap \{t\}\big).$$

As

$$\widehat{\rho}_{n_{2l}}(\{t\}) - \alpha_l \cdot \widehat{\rho}_{n_{2l+1}}(\{t\}) = 0$$

for every $l \in \omega$, $\langle \rho_{n_k}\colon\ k \in \omega \rangle$ is an anti-Nikodym sequence, and $\langle \alpha_k\colon\ k \in \omega \rangle$ is bounded by 1, we get that $\lim_{k\to\infty} \nu_k(A) = 0$, as required. □

The following theorem concludes the above lemmas.

Theorem 4.1.18. *If $\mathcal{A}$ carries an anti-Nikodym sequence of measures which has a strong Nikodym concentration point in $St(\mathcal{A})$, then there are an anti-Nikodym sequence $\langle \nu_k\colon\ k \in \omega \rangle$ of measures on $\mathcal{A}$ and an antichain $\langle B_k\colon\ k \in \omega \rangle$ in $\mathcal{A}$ such that for every $k \in \omega$ we have:*

- $\operatorname{supp}\big(\widehat{\nu}_k\big) \subseteq [B_k]_{\mathcal{A}}$,
- $\big\|\nu_k\big\| > k$.

Proof. Combine Lemmas 4.1.16 and 4.1.17 (with $a_k = k$ for each $k \in \omega$). □

4.1.1 Strong Nikodym concentration points and the Grothendieck property

In this short subsection we will quickly show how the existence of anti-Nikodym sequences of measures on a Boolean algebra $\mathcal{A}$ with strong Nikodym concentration points implies that $\mathcal{A}$ does not have the Grothendieck property. We start with the following folklore proposition (cf. Kąkol *et al.* [63, Theorem 19.3.5]).

Proposition 4.1.19. *Let $\langle \mu_n \colon n \in \omega \rangle$ be an anti-Nikodym sequence of non-zero measures on $\mathcal{A}$ and $\langle B_n \colon n \in \omega \rangle$ such an antichain in $\mathcal{A}$ that $\operatorname{supp}(\widehat{\mu}_n) \subseteq [B_n]_{\mathcal{A}}$ and $\|\mu_n\| > n$ for every $n \in \omega$. Then, the sequence $\langle \mu_n / \|\mu_n\| \colon n \in \omega \rangle$ of normalized measures is an anti-Grothendieck sequence on $\mathcal{A}$.*

Proof. For every $n \in \omega$ let $\nu_n = \mu_n / \|\mu_n\|$, so $\|\nu_n\| = 1$. For every $A \in \mathcal{A}$ and $n \in \omega$ we have

$$\left|\nu_n(A)\right| = \left|\mu_n(A)\right| / \|\mu_n\| \leq \left|\mu_n(A)\right|,$$

so $\lim_{n\to\infty} \nu_n(A) = 0$. Consequently, $\langle \nu_n \colon n \in \omega \rangle$ is also pointwise convergent to 0 and hence weak* convergent to 0.

We claim that $\langle \nu_n \colon n \in \omega \rangle$ is not weakly convergent to 0. For every $n \in \omega$, since $\|\nu_n\| = 1$ and $\operatorname{supp}(\widehat{\nu}_n) \subseteq [B_n]_{\mathcal{A}}$, there is a clopen set $U_n \subseteq [B_n]_{\mathcal{A}}$ such that

$$\left|\widehat{\nu}_n(U_n)\right| > 1/3.$$

Set

$$B = \bigcup_{n \in \omega} U_n.$$

Then, B is an open set, so Borel. For each $n \in \omega$, since

$$B \cap \operatorname{supp}(\widehat{\nu}_n) = B \cap B_n \cap \operatorname{supp}(\widehat{\nu}_n) = U_n \cap \operatorname{supp}(\widehat{\nu}_n),$$

we have $\left|\widehat{\nu}_n(B)\right| > 1/3$, and hence $\langle \nu_n \colon n \in \omega \rangle$ is not weakly convergent to 0 (by the characterization of weak convergence given at the end of Section 2.4). It follows that $\langle \nu_n \colon n \in \omega \rangle$ is an anti-Grothendieck sequence on $\mathcal{A}$. □

We now prove the theorem providing a criterion for a Boolean algebra without the Nikodym property to not have the Grothendieck property. This generalizes the aforementioned fact asserting that if a zero-dimensional compact space K contains a non-trivial convergent sequence, then $\operatorname{Clopen}(K)$ does not have the Nikodym property nor the Grothendieck property (cf. Example 4.1.8).

Theorem 4.1.20. *If a Boolean algebra $\mathcal{A}$ does not have the Nikodym property and this is witnessed by an anti-Nikodym sequence with a strong Nikodym concentration point in the Stone space $St(\mathcal{A})$, then $\mathcal{A}$ does not have the Grothendieck property.*

Proof. Let $\langle \mu_n \colon n \in \omega \rangle$ be an anti-Nikodym sequence on $\mathcal{A}$ with a strong Nikodym concentration point in $St(\mathcal{A})$. From $\langle \mu_n \colon n \in \omega \rangle$, with an aid of Theorem 4.1.18, we obtain a new anti-Nikodym sequence $\langle \nu_k \colon k \in \omega \rangle$ on $\mathcal{A}$ and an antichain $\langle B_k \colon k \in \omega \rangle$ in $\mathcal{A}$ such that for every $k \in \omega$ we have $\operatorname{supp}(\widehat{\nu}_k) \subseteq [B_k]_{\mathcal{A}}$ and $\|\nu_k\| > k$. Proposition 4.1.19 implies that the sequence $\langle \nu_k / \|\nu_k\| \colon k \in \omega \rangle$ is anti-Grothendieck and so that $\mathcal{A}$ does not have the Grothendieck property. □

We finish this section with the following important consequence of Theorem 4.1.20 and Lemma 4.1.13; cf. Corollary 4.3.13 below.

Corollary 4.1.21. *If $\mathcal{A}$ admits an anti-Nikodym sequence of measures having a unique Nikodym concentration point, then $\mathcal{A}$ does not have the Grothendieck property.*

4.2 The Nikodym property and the spaces $N_{\mathcal{F}}$

This section is devoted to an analysis of the connection between embedding of a space $N_{\mathcal{F}}$ without the finitely supported Nikodym property into the Stone space of a given Boolean algebra $\mathcal{A}$ and the lack of the Nikodym property of $\mathcal{A}$. As we stated in Example 4.1.8, if the Stone space $St(\mathcal{A})$ of a Boolean algebra $\mathcal{A}$ contains a non-trivial convergent sequence, then $\mathcal{A}$ admits an anti-Nikodym sequence of measures with a strong Nikodym concentration point. Since the space N_{Fr} is clearly homeomorphic to any space consisting of a non-trivial convergent sequence together with its limit, the example may be rephrased as follows: if N_{Fr} homeomorphically embeds into $St(\mathcal{A})$, then $\mathcal{A}$ admits an anti-Nikodym sequence of measures with a strong Nikodym concentration point. The next theorem asserts that actually the latter statement holds true for any filter $\mathcal{F}$ with $\mathcal{F}^* \in \mathcal{AN}$ and the space $N_{\mathcal{F}}$.

Theorem 4.2.1. *Let $\mathcal{A}$ be a Boolean algebra and let $\mathcal{I} \in \mathcal{AN}$. Suppose that there is a continuous function $f\colon N^*_{\mathcal{I}} \to St(\mathcal{A})$ satisying $f^{-1}\big(f(p_{\mathcal{I}^*})\big) = \{p_{\mathcal{I}^*}\}$ (e.g. f is an injection). Then, $\mathcal{A}$ admits an anti-Nikodym sequence of measures with a unique strong Nikodym concentration point.*

Proof. Let $f\colon N_{\mathcal{I}^*} \to St(\mathcal{A})$ be continuous such that $f^{-1}\big(f(p_{\mathcal{I}^*})\big) = \{p_{\mathcal{I}^*}\}$. Set $x = h\big(p_{\mathcal{I}^*}\big)$ and $Y = h[\omega]$. By the assumption, we have $x \notin Y$. Let $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ be the sequence non-negative measures on $N^*_{\mathcal{I}}$ from Theorem 3.2.1. We define a sequence of non-negative measures on $St(\mathcal{A})$ by setting

$$\nu_n(X) = \mu_n\big(h^{-1}[X]\big)$$

for each $X \subseteq St(\mathcal{A})$. It satisfies the following properties:

(a) $\operatorname{supp}\big(\nu_n\big) \subseteq Y$ for every $n \in \omega$,

(b) $\lim_{n\to\infty} \nu_n(St(\mathcal{A})) = \infty$, and

(c) $\lim_{n\to\infty} \nu_n(St(\mathcal{A}) \setminus U) = 0$ for every open neighborhood U of x,

since $(a) - (b)$ are immediate consequences of the properties $(1) - (2)$ from Theorem 3.2.1, and if U is an open neighbourhood of x then $h^{-1}[U]$ is an open neighbourhood of $p_{\mathcal{F}}$, thus there exists $A \in \mathcal{F}$ such that

$$h^{-1}\big[St(\mathcal{A}) \setminus U\big] = N_{\mathcal{F}} \setminus h^{-1}[U] = \omega \setminus A,$$

and so (c) follows from the property (3) of $\big\langle \mu_n\colon\ n \in \omega\big\rangle$.

For each $n \in \omega$ define the measure λ_n on $\mathcal{A}$ via its Radon extension as

$$\widehat{\lambda}_n = \nu_n(St(\mathcal{A})) \cdot \delta_x - \nu_n.$$

We claim that the sequence $\langle\lambda_n\colon\ n\in\omega\rangle$ is anti-Nikodym on $\mathcal{A}$ and x is its unique (strong) Nikodym concentration point. First, note that by (a), (b) and $x\notin Y$ we have

$$\sup_{n\in\omega}\|\lambda_n\|\ge\sup_{n\in\omega}|\nu_n(St(\mathcal{A}))|=\infty.$$

Second, by (c), for every $A\in\mathcal{A}$, if $A\notin x$, then we have

$$\lim_{n\to\infty}\widehat{\lambda}_n\big([A]_{\mathcal{A}}\big)=0-\lim_{n\to\infty}\nu_n\big([A]_{\mathcal{A}}\big)=0,$$

and if $A\in x$, then it holds

$$\lim_{n\to\infty}\widehat{\lambda}_n\big([A]_{\mathcal{A}}\big)=\lim_{n\to\infty}\Big(\nu_n(St(\mathcal{A}))-\nu_n\big([A]_{\mathcal{A}}\big)\Big)=\lim_{n\to\infty}\mu_n\big(St(\mathcal{A})\setminus[A]_{\mathcal{A}}\big)=0.$$

Consequently, $\langle\lambda_n\colon\ n\in\omega\rangle$ is indeed anti-Nikodym.

We now show that x is a unique strong Nikodym concentration point of $\langle\lambda_n\colon\ n\in\omega\rangle$. To achieve this, we prove that x is the only Nikodym concentration point of $\langle\nu_n\colon\ n\in\omega\rangle$ and then appeal to Lemma 4.1.13. By (a), (b) and $x\notin Y$, for any $A\in x$ we have

$$\sup_{n\in\omega}\|\lambda_n\restriction A\|\ge\sup_{n\in\omega}|\nu_n(St(\mathcal{A}))|=\infty,$$

so x is a Nikodym concentration point of $\langle\nu_n\colon\ n\in\omega\rangle$. If, on the other hand, $y\in St(\mathcal{A})\setminus\{x\}$, then there is $B\in y$ such that $B\notin x$. By (c), we then have

$$\sup_{n\in\omega}\|\lambda_n\restriction B\|=\sup_{n\in\omega}\nu_n\big([B]_{\mathcal{A}}\big)<\infty,$$

so y is not a Nikodym concentration point of $\langle\lambda_n\colon\ n\in\omega\rangle$. □

Corollary 4.2.2. *Let $\mathcal{A}$ be a Boolean algebra and let $\mathcal{I}\in\mathcal{AN}$. If $N_{\mathcal{I}^*}$ homeomorphically embeds into $St(\mathcal{A})$, then $\mathcal{A}$ does not have the Nikodym property and, moreover, $\mathcal{A}$ admits an anti-Nikodym sequence of measures with a unique strong Nikodym concentration point.*

The following corollary follows now from Theorem 3.3.3 and Corollary 3.2.3.

Corollary 4.2.3. *Let $\mathcal{A}$ be a Boolean algebra, and let $\mathcal{I}$ be either a summable ideal on ω or a density ideal which is not totally bounded. If $N_{\mathcal{I}^*}$ homeomorphically embeds into $St(\mathcal{A})$, then $\mathcal{A}$ does not have the Nikodym property.*

Remark 4.2.4. It turns out that embedding a suitable space $N_{\mathcal{F}}$ into the Stone space is not the only reason for a Boolean algebra to lack the Nikodym property. There exists an algebra $\mathcal{S}$ which does not have the Nikodym property and such that $St(\mathcal{S})$ does not contain any homeomorphic copy of a space $N_{\mathcal{F}}$ for which $\mathcal{F}\in\mathcal{BJNP}$ (see Schachermayer [91, Example 4.10] and Marciszewski, Sobota [76, Example 7.8]).

The family of filters $\mathscr{F}_1$ constructed in Corollary 3.6.3 gives us a rich family of pairwise non-homeomorphic spaces $N_{\mathcal{F}}$ which forbid the Nikodym property of Boolean algebras. By [76, Proposition 4.3], for each $\mathcal{F}\in\mathscr{F}_1$ the space $N_{\mathcal{F}}$ contains no non-trivial convergent sequence. Therefore, by Proposition 3.1.3 and Corollary 4.2.2, we get the following result. Note that the cardinal number $2^{\mathfrak{c}}$ is here maximal possible.

Corollary 4.2.5. *There exists a family $\mathcal{X}$ of $2^{\mathfrak{c}}$ many pairwise non-homeomorphic countable spaces with exactly one non-isolated point and without non-trivial convergent sequences such that, for any $X\in\mathcal{X}$, if X homeomorphically embeds into the Stone space $St(\mathcal{A})$ of a Boolean algebra $\mathcal{A}$, then $\mathcal{A}$ admits an anti-Nikodym sequence of measures with a unique strong Nikodym concentration point.*

4.3 Boolean algebras generated by ideals

We start with the following definitions, analogous to Definitions 2.5.2 and 4.1.1 for Boolean algebras.

Definition 4.3.1. A ring $\mathcal{R}$ of subsets of a set X has the *Nikodym property* if every sequence $\langle \mu_n\colon\ n \in \omega \rangle$ in $\mathrm{ba}(\mathcal{R})$ such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{R}$ is norm bounded.

Equivalently, a ring $\mathcal{R}$ of subsets of a set X has the Nikodym property if every sequence $\langle \mu_n\colon\ n \in \omega \rangle$ in $\mathrm{ba}(\mathcal{R})$ such that $\sup_{n\in\omega} \left|\mu_n(A)\right| < \infty$ for every $A \in \mathcal{R}$ is norm bounded.

Definition 4.3.2. Let $\mathcal{R}$ be a ring of subsets of a set X. A sequence $\langle \mu_n\colon\ n \in \omega \rangle$ of measures on $\mathcal{R}$ is *anti-Nikodym* if $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{R}$ and $\sup_{n\in\omega} \|\mu_n\| = \infty$.

It immediately follows that a Boolean algebra $\mathcal{A}$ has the Nikodym property if and only if the ring $\mathrm{Clopen}(St(\mathcal{A}))$ of clopen subsets of $St(\mathcal{A})$ has the Nikodym property. Also, a ring $\mathcal{R}$ has the Nikodym property if and only if there are no anti-Nikodym sequences on $\mathcal{R}$.

Drewnowski and Paúl [25, Remark 2.2.(a)] observed that a ring $\mathcal{R} \subseteq \mathcal{P}(X)$ of subsets of a set X has the Nikodym property if and only if the Boolean subalgebra $\mathcal{P}(X)\langle\mathcal{R}\rangle$ of $\mathcal{P}(X)$ generated by $\mathcal{R}$ has the Nikodym property. For the "if" part they briefly noted that if $X \notin \mathcal{R}$, then every measure $\mu \in \mathrm{ba}(\mathcal{R})$ can be easily extended to a measure $\nu \in \mathrm{ba}\left(\mathcal{P}(X)\langle\mathcal{R}\rangle\right)$, simply by setting $\nu(X \setminus A) = -\mu(A)$ for every $A \in \mathcal{R}$.

We will expand the above observations in the next subsection by describing, for a Boolean algebra $\mathcal{A}$ and its ideal $\mathcal{I}$, an isomorphism between the Banach spaces $\mathrm{ba}(\mathcal{I}) \oplus \mathbb{R}$ and $\mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$ (Proposition 4.3.7 and Theorem 4.3.8). Our approach is topological, that is, we study the values of the Radon extensions of measures from the space $\mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$ on the open subset of the Stone space $St(\mathcal{A}\langle\mathcal{I}\rangle)$ corresponding to the ideal $\mathcal{I}$.

In Subsection 4.3.2 we will use results obtained in the previous sections to show that if a Boolean algebra $\mathcal{A}$ has the Nikodym property, but its subalgebra $\mathcal{A}\langle\mathcal{I}\rangle$ generated by an ideal $\mathcal{I}$ does not, then $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Grothendieck property either (Proposition 4.3.12 and Corollary 4.3.13). We will also obtain a characterization of the Nikodym property for Boolean algebras of the form $\mathcal{A}\langle\mathcal{I}\rangle$ in terms of sequences of non-negative measures (Theorem 4.3.15).

4.3.1 Extending measures from ideals to Boolean algebras

Let $\mathcal{A}$ be a Boolean algebra. Note that if $\mathcal{F}$ is a filter in $\mathcal{A}$, then $\mathcal{A}\langle\mathcal{F}\rangle = \mathcal{F} \cup \mathcal{F}^*$, where $\mathcal{F}^* = \{1_\mathcal{A} \setminus A\colon A \in \mathcal{F}\}$ is the ideal in $\mathcal{A}$ dual to $\mathcal{F}$, and $\mathcal{F}$ is an ultrafilter in $\mathcal{A}\langle\mathcal{F}\rangle$. It follows that $\mathcal{F} \in St(\mathcal{A}\langle\mathcal{F}\rangle)$, but in order to distinguish $\mathcal{F}$ as a *filter* in $\mathcal{A}$ from the *point* in $St(\mathcal{A}\langle\mathcal{F}\rangle)$ we will denote the latter by $p_\mathcal{F}$. Of course, formally $\mathcal{F} = p_\mathcal{F}$. Note that $p_\mathcal{F} \notin [A]_{\mathcal{A}\langle\mathcal{F}\rangle}$ for every $A \in \mathcal{F}^*$ and that

$$St(\mathcal{A}\langle\mathcal{F}\rangle) = \bigcup \left\{ [A]_{\mathcal{A}\langle\mathcal{F}\rangle}\colon\ A \in \mathcal{F}^* \right\} \cup \left\{p_\mathcal{F}\right\}. \tag{$\star$}$$

Consequently, since $\left\{ [A]_{\mathcal{A}\langle\mathcal{F}\rangle}\colon\ A \in \mathcal{F}^* \right\}$ is an increasing net of open sets (indexed by the directed set $(\mathcal{F}^*, \leq)$, where $\leq$ is the standard order on $\mathcal{A}$) and by the τ-additivity of

Radon measures (see [13, Section 7.2]), for every measure $\mu \in \mathrm{ba}(\mathcal{A}\langle\mathcal{F}\rangle)$ we have

$$\|\mu\| = \|\widehat{\mu}\| = \Big\|\widehat{\mu} \restriction \bigcup_{A\in\mathcal{F}^*} [A]_{\mathcal{A}\langle\mathcal{F}\rangle}\Big\| + \big|\widehat{\mu}\big(\{p_\mathcal{F}\}\big)\big| = |\widehat{\mu}|\Big(\bigcup_{A\in\mathcal{F}^*} [A]_{\mathcal{A}\langle\mathcal{F}\rangle}\Big) + \big|\widehat{\mu}\big(\{p_\mathcal{F}\}\big)\big| = (+)$$
$$= \lim_{A\in\mathcal{F}^*} |\widehat{\mu}|\big([A]_{\mathcal{A}\langle\mathcal{F}\rangle}\big) + \big|\widehat{\mu}\big(\{p_\mathcal{F}\}\big)\big| = \lim_{A\in\mathcal{F}^*} |\mu|(A) + \big|\widehat{\mu}\big(\{p_\mathcal{F}\}\big)\big|$$

and so, in the case of non-negative μ,

$$\mu\big(1_\mathcal{A}\big) = \lim_{A\in\mathcal{F}^*} \mu(A) + \widehat{\mu}\big(\{p_\mathcal{F}\}\big)$$

and

$$\widehat{\mu}\big(\{p_\mathcal{F}\}\big) = \lim_{A\in\mathcal{F}} \mu(A).$$

The following lemma is a standard application of the regularity of measures.

Lemma 4.3.3. *Let $\mathcal{A}$ be a Boolean algebra and $\mu \in \mathrm{ba}(\mathcal{A})$. For every Borel set B in $St(\mathcal{A})$, open set U in $St(\mathcal{A})$ such that $B \subseteq U$, and $\varepsilon > 0$, there is $C \in \mathcal{A}$ such that $[C]_\mathcal{A} \subseteq U$ and $|\widehat{\mu}(B)| \le |\mu(C)| + \varepsilon$.*

Proof. Let $St(\mathcal{A}) = P \cup N$ be a Hahn decomposition of the space $St(\mathcal{A})$ for the measure $\widehat{\mu}$, i.e. partition such that $\widehat{\mu}(P \cap B) \ge 0$ and $\widehat{\mu}(N \cap B) \le 0$ for every $B \in Bor(St(\mathcal{A}))$. Let $\widehat{\mu} = \mu^+ - \mu^-$ be the Jordan decomposition of $\widehat{\mu}$ into two non-negative measures on $St(\mathcal{A})$ (see [49, Section 29] for the constructions of Hahn and Jordan decompositions).

Fix a Borel set B and an open set U such that $B \subseteq U \subseteq St(\mathcal{A})$. Fix also $\varepsilon > 0$. We may assume that $\mu^+(B) \ge \mu^-(B)$ as the other case is symmetric. By the regularity of μ^+, there is a compact set $K \subseteq B \cap P$ such that

$$\mu^+(B) \le \mu^+(K) + \varepsilon/2. \tag{$*$}$$

By the compactness of K, the outer regularity of the variation $|\widehat{\mu}|$, and the zero-dimensionality of $St(\mathcal{A})$, there is $C \in \mathcal{A}$ such that $K \subseteq [C]_\mathcal{A} \subseteq U$ and

$$|\widehat{\mu}|\big([C]_\mathcal{A}\big) \le |\widehat{\mu}|(K) + \varepsilon/2 = \mu^+(K) + \varepsilon/2.$$

Since we have $|\mu| = \mu^+ + \mu^-$ ([49, Section 29]), the above inequality yields that

$$\mu^-\big([C]_\mathcal{A}\big) = |\widehat{\mu}|\big([C]_\mathcal{A}\big) - \mu^+\big([C]_\mathcal{A}\big) \le |\widehat{\mu}|\big([C]_\mathcal{A}\big) - \mu^+(K) \le$$
$$\le \mu^+(K) + \varepsilon/2 - \mu^+(K) = \varepsilon/2,$$

and hence

$$-\mu^-\big([C]_\mathcal{A} \setminus B\big) \ge -\mu^-\big([C]_\mathcal{A}\big) \ge -\varepsilon/2. \tag{$**$}$$

Using $(*)$ and $(**)$, we have

$$|\widehat{\mu}(B)| = \big|\mu^+(B) - \mu^-(B)\big| = \mu^+(B) - \mu^-(B) \le \mu^+(K) + \varepsilon/2 - \mu^-(B) \le$$
$$\le \mu^+\big([C]_\mathcal{A}\big) + \varepsilon/2 - \mu^-\big([C]_\mathcal{A} \cap B\big) = \mu^+\big([C]_\mathcal{A}\big) + \varepsilon - \mu^-\big([C]_\mathcal{A} \cap B\big) - \varepsilon/2 \le$$
$$\le \mu^+\big([C]_\mathcal{A}\big) + \varepsilon - \mu^-\big([C]_\mathcal{A} \cap B\big) - \mu^-\big([C]_\mathcal{A} \setminus B\big) =$$
$$= \mu^+\big([C]_\mathcal{A}\big) + \varepsilon - \mu^-\big([C]_\mathcal{A}\big) = \widehat{\mu}\big([C]_\mathcal{A}\big) + \varepsilon = \mu(C) + \varepsilon \le |\mu(C)| + \varepsilon,$$

and the proof of the lemma is finished. □

Lemma 4.3.4. *Let $\mathcal{A}$ be a Boolean algebra and $\mu \in \mathrm{ba}(\mathcal{A})$. For every open subset U of $St(\mathcal{A})$, disjoint Borel sets $B, B' \subseteq U$, and $\varepsilon > 0$, there are disjoint $C, C' \in \mathcal{A}$ such that $[C]_{\mathcal{A}}, [C']_{\mathcal{A}} \subseteq U$, $|\widehat{\mu}(B)| < |\mu(C)| + \varepsilon$, and $|\widehat{\mu}(B')| < |\mu(C')| + \varepsilon$.*

Proof. Let U be an open subset of $St(\mathcal{A})$, B, B' disjoint Borel subsets of U, and $\varepsilon > 0$. By the regularity of $|\widehat{\mu}|$, there are compact sets $K \subseteq B$, $K' \subseteq B'$ such that $|\widehat{\mu}|(B \setminus K) < \varepsilon/2$ and $|\widehat{\mu}|(B' \setminus K') < \varepsilon/2$. Since $K \cap K' = \emptyset$, by the normality of $St(\mathcal{A})$ there are disjoint open sets $V, V' \subseteq U$ such that $K \subseteq V$ and $K' \subseteq V'$. Using Lemma 4.3.3 twice, we get elements $C, C' \in \mathcal{A}$ such that $[C]_{\mathcal{A}} \subseteq V$, $[C']_{\mathcal{A}} \subseteq V'$, as well as

$$|\widehat{\mu}(K)| \leq |\mu(C)| + \varepsilon/2 \quad \text{and} \quad |\widehat{\mu}(K')| \leq |\mu(C')| + \varepsilon/2. \tag{$*$}$$

Note that $C \wedge C' = 0_{\mathcal{A}}$ and $[C]_{\mathcal{A}}, [C']_{\mathcal{A}} \subseteq U$.

We have

$$\varepsilon/2 > |\widehat{\mu}|(B \setminus K) \geq |\widehat{\mu}(B \setminus K)| = |\widehat{\mu}(B) - \widehat{\mu}(K)| \geq |\widehat{\mu}(B)| - |\widehat{\mu}(K)|,$$

hence

$$|\widehat{\mu}(K)| > |\widehat{\mu}(B)| - \varepsilon/2.$$

Combining the latter inequality with $(*)$, we get

$$|\widehat{\mu}(B)| - \varepsilon/2 < |\mu(C)| + \varepsilon/2,$$

and hence $|\widehat{\mu}(B)| \leq |\mu(C)| + \varepsilon$. We similarly show that $|\widehat{\mu}(B')| < |\mu(C')| + \varepsilon$. □

Lemma 4.3.5. *Let $\mathcal{I}$ be an ideal in a Boolean algebra $\mathcal{A}$. Let $\mu \in \mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$ and $\nu \in \mathrm{ba}(\mathcal{I})$ be such measures that μ extends ν. Then,*

$$\Big\| \widehat{\mu} \restriction \bigcup_{A \in \mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle} \Big\| = \|\nu\|.$$

Proof. We have

$$\begin{aligned} \Big\| \widehat{\mu} \restriction \bigcup_{A \in \mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle} \Big\| &= |\widehat{\mu}|\Big(\bigcup_{A \in \mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle} \Big) \geq \\ &\geq \sup \big\{ \big|\widehat{\mu}\big([A]_{\mathcal{A}\langle\mathcal{I}\rangle} \big)\big| + \big|\widehat{\mu}\big([A']_{\mathcal{A}\langle\mathcal{I}\rangle} \big)\big| \colon\ A, A' \in \mathcal{I},\ A \wedge A' = 0_{\mathcal{A}} \big\} = \\ &= \sup \big\{ |\mu(A)| + |\mu(A')| \colon\ A, A' \in \mathcal{I},\ A \wedge A' = 0_{\mathcal{A}} \big\} = \\ &= \sup \big\{ |\nu(A)| + |\nu(A')| \colon\ A, A' \in \mathcal{I},\ A \wedge A' = 0_{\mathcal{A}} \big\} = \|\nu\|. \end{aligned}$$

For the reverse inequality, let $\varepsilon > 0$ and note that by Lemma 4.3.4 we have

$$\begin{aligned} \Big\| \widehat{\mu} \restriction \bigcup_{A \in \mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle} \Big\| &= |\widehat{\mu}|\Big(\bigcup_{A \in \mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle} \Big) = \\ &= \sup \Big\{ |\widehat{\mu}(B)| + |\widehat{\mu}(B')| \colon B, B' \in Bor\Big(\bigcup_{A \in \mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle} \Big),\ B \cap B' = \emptyset \Big\} \leq \\ &\leq \sup \big\{ |\mu(C)| + |\mu(C')| \colon C, C' \in \mathcal{I},\ C \wedge C' = 0_{\mathcal{A}} \big\} + 2\varepsilon = \\ &= \sup \big\{ |\nu(C)| + |\nu(C')| \colon C, C' \in \mathcal{I},\ C \wedge C' = 0_{\mathcal{A}} \big\} + 2\varepsilon = \|\nu\| + 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, we get

$$\Big\| \widehat{\mu} \restriction \bigcup_{A \in \mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle} \Big\| \leq \|\nu\|,$$

which finishes the proof of the equality. □

Using the above results, we get the following decomposition of the norms of extensions.

Proposition 4.3.6. *Let $\mathcal{I}$ be an ideal in a Boolean algebra $\mathcal{A}$. Let $\mu \in \mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$ and $\nu \in \mathrm{ba}(\mathcal{I})$ be such measures that μ extends ν. Then,*

$$\|\mu\| = \|\nu\| + \left|\widehat{\mu}\left(\{p_{\mathcal{I}^*}\}\right)\right| = \lim_{A\in\mathcal{I}} |\nu|(A) + \left|\widehat{\mu}\left(\{p_{\mathcal{I}^*}\}\right)\right|.$$

Proof. Combining equalities (+) with Lemma 4.3.5, we get that $\|\mu\| = \|\nu\| + \left|\widehat{\mu}\left(\{p_{\mathcal{I}^*}\}\right)\right|$. For every $A \in \mathcal{I}$ we have $|\nu|(A) = |\mu|(A)$ and so, again by (+), $\|\nu\| = \lim_{A\in\mathcal{I}} |\mu|(A) = \lim_{A\in\mathcal{I}} |\nu|(A)$. □

The next proposition essentially extends the observation by Drewnowski and Paúl mentioned at the beginning of this section.

Proposition 4.3.7. *Let $\mathcal{A}$ be an infinite Boolean algebra and let $\mathcal{I}$ be an ideal in $\mathcal{A}$. Let $\alpha \in \mathbb{R}$. For every $\mu \in \mathrm{ba}(\mathcal{I})$, the function $T_\alpha(\mu)\colon \mathcal{A}\langle\mathcal{I}\rangle \to \mathbb{R}$, given for every $A \in \mathcal{A}\langle\mathcal{I}\rangle$ by*

$$T_\alpha(\mu)(A) = \begin{cases} \mu(A), & \text{if } A \in \mathcal{I}, \\ -\mu(A^c) + \alpha, & \text{if } A \notin \mathcal{I}, \end{cases}$$

is a measure on $\mathcal{A}\langle\mathcal{I}\rangle$, i.e. $T_\alpha(\mu) \in \mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$, extending μ and satisfying:

(i) $\|(\mu,\alpha)\|_\infty \le \left\|T_\alpha(\mu)\right\| \le 2\|(\mu,\alpha)\|_1$,

(ii) $\widehat{T_\alpha(\mu)}\left(\{p_{\mathcal{I}^*}\}\right) = \alpha - \widehat{T_\alpha(\mu)}\left(\bigcup_{A\in\mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle}\right)$.

Consequently, if μ is non-negative, then we have

(ii') $\widehat{T_\alpha(\mu)}\left(\{p_{\mathcal{I}^*}\}\right) = \alpha - \lim_{A\in\mathcal{I}} \mu(A)$.

Proof. Fix $\mu \in \mathrm{ba}(\mathcal{I})$. Obviously, $T_\alpha(\mu)$ is a function extending μ onto $\mathcal{A}\langle\mathcal{I}\rangle$.

We prove that $T_\alpha(\mu)$ is finitely additive. It is enough to check that for every $A \in \mathcal{I}$ and $B \in \mathcal{I}^*$ such that $A \wedge B = 0_{\mathcal{A}}$ we have $T_\alpha(\mu)(A \vee B) = T_\alpha(\mu)(A) + T_\alpha(\mu)(B)$. So, let A and B be as required. We have $A \vee B \in \mathcal{I}^*$, so by definition

$$T_\alpha(\mu)(A \vee B) = -\mu((A \vee B)^c) + \alpha.$$

Since $A \le B^c \in \mathcal{I}$, it holds

$$T_\alpha(\mu)(B) = -\mu(B^c)+\alpha = -\mu(A\wedge B^c) - \mu(A^c\wedge B^c) + \alpha = -\mu(A\wedge B^c) - \mu((A\vee B)^c) + \alpha =$$

$$= -\mu(A) + T_\alpha(\mu)(A \vee B) = -T_\alpha(\mu)(A) + T_\alpha(\mu)(A \vee B),$$

hence $T_\alpha(\mu)(A \vee B) = T_\alpha(\mu)(A) + T_\alpha(\mu)(B)$.

We have

$$\left\|T_\alpha(\mu)\right\| = \sup\left\{\left|T_\alpha(\mu)(A)\right| + \left|T_\alpha(\mu)(B)\right|\colon\ A, B \in \mathcal{A}\langle\mathcal{I}\rangle\,,\ A \wedge B = 0_{\mathcal{A}}\right\} \le$$

$$\le \max\Big\{\sup\left\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \mathcal{I}\right\}, \sup\left\{|\mu(A)| + |-\mu(C) + \alpha|\colon\ A, C \in \mathcal{I}\right\}\Big\} \le$$

$$\le 2\sup\left\{|\mu(A)|\colon\ A \in \mathcal{I}\right\} + |\alpha| \le 2\|\mu\| + |\alpha| = \|(2\mu,\alpha)\|_1 \le 2\|(\mu,\alpha)\|_1.$$

Moreover, we also have

$$\big\|T_\alpha(\mu)\big\| \geq \sup\big\{\big|T_\alpha(\mu)(A)\big| + \big|T_\alpha(\mu)(B)\big|\colon\ A, B \in \mathcal{I},\ A \wedge B = 0_{\mathcal{A}}\big\} =$$

$$= \sup\big\{|\mu(A)| + |\mu(B)|\colon\ A, B \in \mathcal{I},\ A \wedge B = 0_{\mathcal{A}}\big\} = \|\mu\|.$$

Finally,

$$\big\|T_\alpha(\mu)\big\| \geq \big|T_\alpha(\mu)\big(0_{\mathcal{A}}\big)\big| + \big|T_\alpha(\mu)\big(1_{\mathcal{A}}\big)\big| = \big|\mu\big(0_{\mathcal{A}}\big)\big| + \big| - \mu\big(0_{\mathcal{A}}\big) + \alpha\big| = |\alpha|,$$

hence $\big\|T_\alpha(\mu)\big\| \geq \|(\mu, \alpha)\|_\infty$. Consequently, (i) holds and so also $T_\alpha(\mu) \in \mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$.

By $(\star)$, we have

$$\widehat{T(\mu)}\big(\{p_{\mathcal{I}^*}\}\big) = \widehat{T_\alpha(\mu)}\big(St(\mathcal{A}\langle\mathcal{I}\rangle)\big) - \widehat{T_\alpha(\mu)}\Big(\bigcup_{A\in\mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle}\Big) =$$

$$= T_\alpha(\mu)\big(1_{\mathcal{A}}\big) - \widehat{T_\alpha(\mu)}\Big(\bigcup_{A\in\mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle}\Big) = -\mu\big(0_{\mathcal{A}}\big) + \alpha - \widehat{T_\alpha(\mu)}\Big(\bigcup_{A\in\mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle}\Big) =$$

$$= \alpha - \widehat{T_\alpha(\mu)}\Big(\bigcup_{A\in\mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle}\Big),$$

which proves (ii).

Let $U = \bigcup_{A\in\mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle}$. If μ is non-negative, then the measure $\nu \in M(St(\mathcal{A}\langle\mathcal{I}\rangle))$ given by the formula

$$\nu = \Big(\widehat{T_\alpha(\mu)} \restriction U\Big) + 0 \cdot \delta_{p_{\mathcal{I}^*}}$$

is also non-negative. To see this, assume that there is a Borel set $B \subseteq U$ such that $\gamma = \nu(B) < 0$. By the regularity of ν, there is a compact set $K \subseteq B$ such that $|\nu|(B \setminus K) < |\gamma|/4$ as well as an open set $V \subseteq U$ such that $K \subseteq V$ and $|\nu|(V \setminus K) < |\gamma|/4$. By the compactness of K, we can assume that V is clopen, that is, $V = [A]_{\mathcal{A}}$ for some $A \in \mathcal{I}$. We have

$$|\nu(B) - \nu(V)| = \big|\nu(B) - \nu(K) + \nu(K) - \nu(V)\big| \leq |\nu(B) - \nu(K)| + |\nu(K) - \nu(V)| =$$

$$= |\nu(B \setminus K)| + |\nu(V \setminus K)| \leq |\nu|(B \setminus K) + |\nu|(V \setminus K) < |\gamma|/4 + |\gamma|/4 = |\gamma|/2.$$

As $\nu(B) = \gamma < 0$ and $A \in \mathcal{I}$, it follows from the above calculations that

$$0 < \mu(A) = T_\alpha(\mu)(A) = \widehat{T_\alpha(\mu)}\big([A]_{\mathcal{A}}\big) = \nu\big([A]_{\mathcal{A}}\big) = \nu(V) < \gamma/2 < 0,$$

which is clearly a contradiction.

Consequently, (ii') follows from (ii) and the τ-additivity of ν:

$$\widehat{T_\alpha(\mu)}\big(\{p_{\mathcal{I}^*}\}\big) = \alpha - \widehat{T_\alpha(\mu)}\Big(\bigcup_{A\in\mathcal{I}} [A]_{\mathcal{A}\langle\mathcal{I}\rangle}\Big) = \alpha - \lim_{A\in\mathcal{I}} \widehat{T_\alpha(\mu)}\big([A]_{\mathcal{A}}\big) =$$

$$= \alpha - \lim_{A\in\mathcal{I}} T_\alpha(\mu)(A) = \alpha - \lim_{A\in\mathcal{I}} \mu(A).$$

□

Note that by the above proof in condition (i) we even have the stronger inequality:

$$\|T_\alpha(\mu)\| \le \|(2\mu,\alpha)\|_1.$$

Proposition 4.3.7 yields the aforementioned isomorphism between the Banach spaces $\mathrm{ba}(\mathcal{I}) \oplus \mathbb{R}$ and $\mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$.

Theorem 4.3.8. *Let $\mathcal{A}$ be an infinite Boolean algebra and let $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then, the mapping*

$$T_{\cdot}\colon \mathrm{ba}(\mathcal{I}) \oplus \mathbb{R} \ni (\mu,\alpha) \longmapsto T_\alpha(\mu) \in \mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle),$$

defined in Proposition 4.3.7, is an isomorphism between the Banach spaces $\mathrm{ba}(\mathcal{I}) \oplus \mathbb{R}$ *and* $\mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$ *with* $\|T_{\cdot}\| \le 2$.

Proof. It is immediate that $T_{\cdot}$ is an injective linear mapping.

Let $\mu \in \mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$. Set $\nu = \mu \restriction \mathcal{I}$ and $\alpha = \mu\big(1_{\mathcal{A}}\big)$. As $T_\alpha(\nu)$ extends ν, we have $T_\alpha(\nu)(A) = \mu(A)$ for every $A \in \mathcal{I}$. For $A \in \mathcal{I}^*$ we have

$$T_\alpha(\nu)(A) = -\nu(A^c) + \alpha = -\mu(A^c) + \mu\big(1_{\mathcal{A}}\big) = \mu(A).$$

It follows that $T_\alpha(\nu) = \mu$ and so $T_{\cdot}$ is a bijection. By Proposition 4.3.7.(i), $T_{\cdot}$ is bounded with $\|T_{\cdot}\| \le 2$ and hence, by the Open Mapping Theorem (see e.g. [26, Theorems II.2.1-II.2.2]), $T_{\cdot}$ is an isomorphism. □

4.3.2 Properties of ideals and Boolean algebras generated by them

We now study the relation between the Nikodym property and the Grothendieck property of Boolean algebras of the form $\mathcal{A}\langle\mathcal{I}\rangle$ as well as provide a characterization of the former property in terms of non-negative measures. We first introduce the following auxiliary definitions, supplementing Definitions 2.5.2, 4.3.1, 4.3.2, and 4.1.1.

Definition 4.3.9. An ideal $\mathcal{I}$ in a Boolean algebra $\mathcal{A}$ has the *Nikodym property* if the ring $\mathcal{R}(\mathcal{I})$ has the Nikodym property.

Definition 4.3.10. Let $\mathcal{I}$ be an ideal in a Boolean algebra $\mathcal{A}$. A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{I}$ is *anti-Nikodym* if the sequence $\langle \mathring{\mu}_n\colon\ n \in \omega\rangle$ of the corresponding measures on $\mathcal{R}(\mathcal{I})$ is anti-Nikodym.

As we will need this result later, for the sake of completeness, we now reprove the aforementioned observation of Drewnowski and Paúl [25, Remark 2.2.(a)], in the case of ideals and Boolean algebras. We use the topological language developed in the previous subsection.

Theorem 4.3.11 (Drewnowski–Paúl [25])**.** *Let $\mathcal{A}$ be a Boolean algebra and let $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then, $\mathcal{I}$ has the Nikodym property if and only if the Boolean algebra $\mathcal{A}\langle\mathcal{I}\rangle$ has the Nikodym property.*

Proof. Suppose that $\mathcal{I}$ has the Nikodym property but there is an anti-Nikodym sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}\langle\mathcal{I}\rangle$. Of course,

$$\lim_{n\to\infty} \widehat{\mu}_n\big([A]_{\mathcal{A}\langle\mathcal{I}\rangle}\big) = 0$$

for every $A \in \mathcal{I}$, and so

$$\lim_{n\to\infty} \big(\widehat{\mu}_n \restriction \mathcal{R}(\mathcal{I})\big)(A) = 0$$

for every $A \in \mathcal{R}(\mathcal{I})$. On the other hand, Corollary 4.1.6 implies that there exists an antichain $\langle A_k\colon\ k \in \omega\rangle$ in $\mathcal{A}\langle\mathcal{I}\rangle$ and a strictly increasing sequence $\langle n_k\colon\ k \in \omega\rangle$ satisfying

$$\left|\widehat{\mu}_{n_k}\big(\,[A_k]_{\mathcal{A}\langle\mathcal{I}\rangle}\,\big)\right| = \left|\mu_{n_k}\big(A_k\big)\right| > k$$

for every $k \in \omega$. At most one of A_k's is an element of the dual filter $\mathcal{I}^*$, and so for almost all $k \in \omega$ we have $A_k \in \mathcal{I}$. It follows that

$$\sup_{k\in\omega} \left\|\widehat{\mu}_{n_k} \restriction \mathcal{R}(\mathcal{I})\right\| = \infty,$$

and so $\langle\widehat{\mu}_{n_k} \restriction \mathcal{R}(\mathcal{I})\colon\ k \in \omega\rangle$ is an anti-Nikodym sequence on $\mathcal{R}(\mathcal{I})$, which is a contradiction as $\mathcal{R}(\mathcal{I})$ has the Nikodym property.

Assume now that $\mathcal{A}\langle\mathcal{I}\rangle$ has the Nikodym property but there exists an anti-Nikodym sequence $\langle\mu_n\colon\ n \in \omega\rangle$ on $\mathcal{I}$. For each $n \in \omega$, let $\mu_n' = T_0\big(\mu_n\big)$ be the extension of μ_n onto $\mathcal{A}\langle\mathcal{I}\rangle$ given by Proposition 4.3.7 (for $\alpha = 0$). It is immediate that for each $A \in \mathcal{A}\langle\mathcal{I}\rangle$ we have

$$\lim_{n\to\infty} \mu_n'(A) = 0.$$

By Proposition 4.3.7.(i), it also holds

$$\infty = \sup_{n\in\omega} \left\|\mu_n\right\| \le \sup_{n\in\omega} \left\|\mu_n'\right\|.$$

Consequently, $\langle\mu_n'\colon\ n \in \omega\rangle$ is an anti-Nikodym sequence on $\mathcal{A}\langle\mathcal{I}\rangle$, which is again a contradiction. □

The following proposition, describing the relation between the Nikodym property and the Grothendieck property of Boolean algebras $\mathcal{A}\langle\mathcal{I}\rangle$, will be crucial for results in the next section.

Proposition 4.3.12. *Let $\mathcal{A}$ be a Boolean algebra and let $\mathcal{I}$ be an ideal in $\mathcal{A}$. Assume that*

- *for every $A \in \mathcal{I}$ the restricted algebra $\mathcal{A}_A$ has the Nikodym property,*
- *the subalgebra $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Nikodym property.*

Then, for any anti-Nikodym sequence $\langle\mu_n\colon\ n \in \omega\rangle$ on $\mathcal{A}\langle\mathcal{I}\rangle$, the point $p_{\mathcal{I}^*} \in St(\mathcal{A}\langle\mathcal{I}\rangle)$ is the only Nikodym concentration point of $\langle\mu_n\colon\ n \in \omega\rangle$.

Consequently, $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Grothendieck property either.

Proof. Let $\langle\mu_n\colon\ n \in \omega\rangle$ be an anti-Nikodym sequence of measures on $\mathcal{A}\langle\mathcal{I}\rangle$. By Lemma 4.1.3, $\langle\mu_n\colon\ n \in \omega\rangle$ has a Nikodym concentration point $t \in St(\mathcal{A}\langle\mathcal{I}\rangle)$.

Suppose that there is $A \in \mathcal{I} \subseteq \mathcal{A}\langle\mathcal{I}\rangle$ such that $t \in [A]_{\mathcal{A}\langle\mathcal{I}\rangle}$. By the definition of Nikodym concentration points and the fact that $\mathcal{A}_A = (\mathcal{A}\langle\mathcal{I}\rangle)_A$, we have

$$\sup_{n\in\omega} \left\|\mu_n \restriction (\mathcal{A}\langle\mathcal{I}\rangle)_A\right\| = \sup_{n\in\omega} \left\|\mu_n \restriction \mathcal{A}_A\right\| = \sup_{n\in\omega} \left\|\mu_n \restriction A\right\| = \infty.$$

Since we obviously have $\lim_{n\to\infty} \mu_n(B) = 0$ for every $B \in (\mathcal{A}\langle\mathcal{I}\rangle)_A \subseteq \mathcal{A}\langle\mathcal{I}\rangle$ as well, we get that the sequence

$$\big\langle\mu_n \restriction (\mathcal{A}\langle\mathcal{I}\rangle)_A\colon\ n \in \omega\big\rangle$$

is an anti-Nikodym sequence on $(\mathcal{A}\langle\mathcal{I}\rangle)_A$. But this is a contradiction, as $(\mathcal{A}\langle\mathcal{I}\rangle)_A = \mathcal{A}_A$ and $\mathcal{A}_A$ has the Nikodym property. Consequently, there is no $A \in \mathcal{I}$ such that $t \in [A]_{\mathcal{A}\langle\mathcal{I}\rangle}$. By ($\star$) (from the beginning of Section 4.3.1), it follows that $t = p_{\mathcal{I}^*}$. In particular, $p_{\mathcal{I}^*}$ is the only Nikodym concentration point of $\langle\mu_n\colon\ n\in\omega\rangle$.

The second conclusion follows from Corollary 4.1.21. $\square$

Recall that, by the Nikodym–Andô theorem, every σ-complete Boolean algebra has the Nikodym property.

Corollary 4.3.13. *Let $\mathcal{A}$ be a Boolean algebra with the Nikodym property (e.g. let $\mathcal{A}$ be σ-complete) and $\mathcal{I}$ an ideal in $\mathcal{A}$. If $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Nikodym property, then it does not have the Grothendieck property either.*

We will now characterize the Nikodym property of Boolean algebras $\mathcal{A}\langle\mathcal{I}\rangle$ in terms of non-negative measures. We start with the following preparatory proposition, rephrasing [99, Proposition 2.4] in the context of algebras $\mathcal{A}\langle\mathcal{I}\rangle$.

Proposition 4.3.14. *Let $\mathcal{I}$ be an ideal in a Boolean algebra $\mathcal{A}$. Then, the following conditions are equivalent:*

(A) the algebra $\mathcal{A}\langle\mathcal{I}\rangle$ has the Nikodym property;

(B) the restricted algebra $\mathcal{A}_A$ has the Nikodym property for every $A \in \mathcal{I}$ and there is no sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on $\mathcal{A}\langle\mathcal{I}\rangle$ satisfying the following three conditions:

- *(a)* $\sup_{n\in\omega}\|\mu_n\| = \infty$,
- *(b)* $\lim_{n\to\infty}\mu_n(1_{\mathcal{A}}) = 0$,
- *(c)* $\lim_{n\to\infty}\|\mu_n \restriction A\| = 0$ *for every* $A \in \mathcal{I}$*;*

(C) the restricted algebra $\mathcal{A}_A$ has the Nikodym property for every $A \in \mathcal{I}$ and there is no sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on $\mathcal{A}\langle\mathcal{I}\rangle$ satisfying the following three conditions:

- *(a')* $\sup_{n\in\omega}\|\mu_n\| = \infty$,
- *(b')* $\sup_{n\in\omega}|\mu_n(1_{\mathcal{A}})| < \infty$,
- *(c')* $\sup_{n\in\omega}\|\mu_n \restriction A\| < \infty$ *for every* $A \in \mathcal{I}$.

Proof. (A)$\Rightarrow$(B): Suppose that $\mathcal{A}\langle\mathcal{I}\rangle$ has the Nikodym property. Trivially, for every $A \in \mathcal{I}$ the Boolean algebra $\mathcal{A}_A = (\mathcal{A}\langle\mathcal{I}\rangle)_A$ has the Nikodym property, too. Assume however that there is a sequence $\langle\mu_n\colon\ n\in\omega\rangle$ of measures on $\mathcal{A}\langle\mathcal{I}\rangle$ satisfying conditions (a)–(c). By conditions (b) and (c), for every $A \in \mathcal{I}^*$, we have

$$\lim_{n\to\infty}|\mu_n(A)| = \lim_{n\to\infty}|\mu_n(1_{\mathcal{A}}) - \mu_n(1_{\mathcal{A}}\setminus A)| \le \lim_{n\to\infty}|\mu_n(1_{\mathcal{A}})| + \lim_{n\to\infty}|\mu_n(1_{\mathcal{A}}\setminus A)| \le$$

$$\le \lim_{n\to\infty}|\mu_n(1_{\mathcal{A}})| + \lim_{n\to\infty}\|\mu_n \restriction (1_{\mathcal{A}}\setminus A)\| = 0,$$

which implies that $\langle\mu_n\colon\ n\in\omega\rangle$ is an anti-Nikodym sequence on $\mathcal{A}\langle\mathcal{I}\rangle$. Hence, $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Nikodym property, which is a contradiction.

(B)$\Rightarrow$(C): Assume that there is a sequence $\langle\mu_n\colon\ n\in\omega\rangle$ in $\mathrm{ba}(\mathcal{A}\langle\mathcal{I}\rangle)$ satisfying conditions (a')–(c'). By going to a subsequence if necessary, by condition (a'), we may assume that

$$\lim_{n\to\infty}\|\mu_n\| = \infty, \tag{$*$}$$

and that each μ_n is non-zero. For each $n \in \omega$ set:

$$\nu_n = \mu_n / \sqrt{\|\mu_n\|}.$$

We then have $\lim_{n\to\infty} \|\nu_n\| = \infty$. Moreover, by $(*)$ and condition (b'), we have

$$\lim_{n\to\infty} \left|\nu_n(1_{\mathcal{A}})\right| = \lim_{n\to\infty} \frac{\left|\mu_n(1_{\mathcal{A}})\right|}{\sqrt{\|\mu_n\|}} \leq \lim_{n\to\infty} \frac{\sup_{m\in\omega} \left|\mu_m(1_{\mathcal{A}})\right|}{\sqrt{\|\mu_n\|}} = 0.$$

Similarly, by $(*)$ and condition (c'), for every $A \in \mathcal{I}$ we have

$$\lim_{n\to\infty} \left\|\nu_n \restriction A\right\| = \lim_{n\to\infty} \frac{\left\|\mu_n \restriction A\right\|}{\sqrt{\|\mu_n\|}} \leq \lim_{n\to\infty} \frac{\sup_{m\in\omega} \left\|\mu_m \restriction A\right\|}{\sqrt{\|\mu_n\|}} = 0.$$

It follows that the sequence $\langle \nu_n \colon\ n \in \omega \rangle$ satisfies conditions (a)–(c), a contradiction.

(C)⇒(A): Assume that each restricted algebra $\mathcal{A}_A$ has the Nikodym property and there is no sequence of measures on $\mathcal{A}\langle\mathcal{I}\rangle$ satisfying conditions (a')–(c'), but yet $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Nikodym property. Let $\langle \mu_n \colon\ n \in \omega \rangle$ be an anti-Nikodym sequence on $\mathcal{A}\langle\mathcal{I}\rangle$. Immediately by definition $\langle \mu_n \colon\ n \in \omega \rangle$ satisfies conditions (a') and (b'). For every $A \in \mathcal{I}$, since $\lim_{n\to\infty} \mu_n(B) = 0$ for each $B \leq A$ and $\mathcal{A}_A$ has the Nikodym property, we have

$$\sup_{n\in\omega} \left\|\mu_n \restriction A\right\| = \sup_{n\in\omega} \left\|\mu_n \restriction \mathcal{A}_A\right\| < \infty,$$

that is, $\langle \mu_n \colon\ n \in \omega \rangle$ satisfies also condition (c'), which is again a contradiction. □

Note that, by the Nikodym property of the restricted algebras, condition (c') in Proposition 4.3.14.(C) can be exchanged for the following one:

(c)* $\sup_{n\in\omega} \left|\mu_n(A)\right| < \infty$ *for every* $A \in \mathcal{I}$.

Theorem 4.3.15. *Let $\mathcal{I}$ be an ideal in a Boolean algebra $\mathcal{A}$. Then, the algebra $\mathcal{A}\langle\mathcal{I}\rangle$ has the Nikodym property if and only if the restricted algebra $\mathcal{A}_A$ has the Nikodym property for every $A \in \mathcal{I}$ and there is no (disjointly supported) sequence $\langle \mu_n \colon\ n \in \omega \rangle$ of non-negative measures on $St(\mathcal{A}\langle\mathcal{I}\rangle)$ such that:*

1. $p_{\mathcal{I}^*} \notin \operatorname{supp}(\mu_n)$ *for every* $n \in \omega$,
2. $\sup_{n\in\omega} \|\mu_n\| = \infty$,
3. $\lim_{n\to\infty} \mu_n\left([A]_{\mathcal{A}\langle\mathcal{I}\rangle}\right) = 0$ *for every* $A \in \mathcal{I}$.

Proof. Assume first that each restricted algebra $\mathcal{A}_A$ has the Nikodym property and there is no sequence of measures on $St(\mathcal{A}\langle\mathcal{I}\rangle)$ satisfying conditions (1)–(3), but yet $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Nikodym property. By Proposition 4.3.12, there is an anti-Nikodym sequence $\langle \nu_n \colon\ n \in \omega \rangle$ on $\mathcal{A}\langle\mathcal{I}\rangle$ with $p_{\mathcal{I}^*}$ being its only Nikodym concentration point. By Lemma 4.1.13, $p_{\mathcal{I}^*}$ is a strong Nikodym concentration point of $\langle \nu_n \colon\ n \in \omega \rangle$, and hence, by Theorem 4.1.18, there is a disjointly supported anti-Nikodym sequence $\langle \theta_n \colon\ n \in \omega \rangle$ of non-zero measures on $\mathcal{A}\langle\mathcal{I}\rangle$ such that $p_{\mathcal{I}^*} \notin \operatorname{supp}\left(\widehat{\theta}_n\right)$ for every $n \in \omega$ and

$$\lim_{n\to\infty} \|\theta_n\| = \infty. \qquad (*)$$

For each $n \in \omega$ let us define the non-negative measure μ_n on $St(\mathcal{A}\langle\mathcal{I}\rangle)$ by the formula:

$$\mu_n = \big|\widehat{\theta}_n\big| / \sqrt{\big\|\widehat{\theta}_n\big\|}.$$

Note that $p_{\mathcal{I}^*} \notin \operatorname{supp}\big(\mu_n\big)$ for every $n \in \omega$ and that it follows from $(*)$ that $\sup_{n\in\omega}\big\|\mu_n\big\| = \infty$.

Let $A \in \mathcal{I}$. For every $n \in \omega$ we have

$$\mu_n\big([A]_{\mathcal{A}\langle\mathcal{I}\rangle}\big) = \big|\theta_n\big|(A) / \sqrt{\big\|\widehat{\theta}_n\big\|} \le \Big(\sup_{k\in\omega}\big\|\theta_k \restriction A\big\|\Big) / \sqrt{\big\|\widehat{\theta}_n\big\|},$$

hence, by $(*)$ and the fact that the restricted algebra $\mathcal{A}_A$ has the Nikodym property, it also holds

$$\lim_{n\to\infty}\mu_n\big([A]_{\mathcal{A}\langle\mathcal{I}\rangle}\big) = 0.$$

Consequently, the sequence $\big\langle\mu_n\colon\ n \in \omega\big\rangle$ satisfies conditions (1)–(3), which is a contradiction.

Assume now that $\mathcal{A}\langle\mathcal{I}\rangle$ has the Nikodym property. Trivially, for every $A \in \mathcal{I}$ the Boolean algebra $\mathcal{A}_A$ has the Nikodym property, too. Assume however that there is a sequence $\big\langle\mu_n\colon\ n \in \omega\big\rangle$ of non-negative measures on $St(\mathcal{A}\langle\mathcal{I}\rangle)$ satisfying conditions (1)–(3). For each $n \in \omega$ let us define the measure ν_n on $\mathcal{A}\langle\mathcal{I}\rangle$ via its Radon extension $\widehat{\nu}_n$ on $St(\mathcal{A}\langle\mathcal{I}\rangle)$ as follows:

$$\widehat{\nu}_n = \mu_n - \mu_n\big(St(\mathcal{A}\langle\mathcal{I}\rangle)\big)\cdot\delta_{p_{\mathcal{I}^*}}.$$

The sequence $\big\langle\nu_n\colon\ n \in \omega\big\rangle$ satisfies conditions (a')–(c') of Proposition 4.3.14.(C). Indeed, by condition (1) for every $n \in \omega$ we have

$$\big\|\nu_n\big\| = \big\|\widehat{\nu}_n\big\| = \big\|\mu_n\big\| + \big|\mu_n\big(St(\mathcal{A}\langle\mathcal{I}\rangle)\big)\big| \ge \big\|\mu_n\big\|,$$

so by condition (2) we get

$$\sup_{n\in\omega}\big\|\nu_n\big\| \ge \sup_{n\in\omega}\big\|\mu_n\big\| = \infty,$$

hence condition (a') is satisfied. Condition (b') holds, too, as

$$\nu_n\big(1_{\mathcal{A}}\big) = \widehat{\nu}_n\big(St(\mathcal{A}\langle\mathcal{I}\rangle)\big) = \mu_n\big(St(\mathcal{A}\langle\mathcal{I}\rangle)\big) - \mu_n\big(St(\mathcal{A}\langle\mathcal{I}\rangle)\big)\cdot 1 = 0,$$

so $\sup_{n\in\omega}\big|\nu_n\big(1_{\mathcal{A}}\big)\big| < \infty$. Finally, by condition (3), for every $A \in \mathcal{I}$ and $B \le A$ we have

$$\lim_{n\to\infty}\nu_n(B) = \lim_{n\to\infty}\mu_n\big([B]_{\mathcal{A}\langle\mathcal{I}\rangle}\big) = 0,$$

hence, by the fact that $\mathcal{A}_A$ has the Nikodym property, we get that

$$\sup_{n\in\omega}\big\|\nu_n \restriction A\big\| = \sup_{n\in\omega}\big\|\nu_n \restriction \mathcal{A}_A\big\| < \infty,$$

that is, condition (c') is satisfied as well.

Consequently, by Proposition 4.3.14, $\mathcal{A}\langle\mathcal{I}\rangle$ does not have the Nikodym property, which is again a contradiction. $\square$

4.4 Definable Boolean algebras with the Nikodym property and without the Grothendieck property

In this section, exploiting the results from the previous one, we will study the Nikodym property for Boolean algebras of the form $\mathcal{P}(\omega)\langle\mathcal{F}\rangle$, where $\mathcal{F}$ is a filter on ω. As a result, we will obtain a large class of simple Boolean subalgebras of $\mathcal{P}(\omega)$ having the Nikodym property but not the Grothendieck property. Those subalgebras, when considered as subsets of the Cantor space 2^ω, will also have an interesting additional feature: they will belong to a low level of the Borel hierarchy of subsets of 2^ω.

Let $\mathcal{F}$ be a filter on ω. Following Marciszewski and Sobota [76, Section 3], by $\mathcal{A}_\mathcal{F}$ we shortly denote the Boolean algebra $\mathcal{P}(\omega)\langle\mathcal{F}\rangle$. Therefore,

$$\mathcal{A}_\mathcal{F} = \mathcal{F} \cup \mathcal{F}^* = \big\{A \in \mathcal{P}(\omega)\colon\ A \in \mathcal{F} \text{ or } A^c \in \mathcal{F}\big\}.$$

Trivially, $\mathcal{A}_\mathcal{F}$ contains the ideal Fin and $\mathcal{F}$ is an ultrafilter in $\mathcal{A}_\mathcal{F}$. Also, we have $\mathcal{A}_\mathcal{F} = \mathcal{P}(\omega)$ if and only if $\mathcal{F}$ is an ultrafilter on $\mathcal{P}(\omega)$.

Again following [76, Section 3], we put $S_\mathcal{F} = St\big(\mathcal{A}_\mathcal{F}\big)$. For every $A \in \mathcal{A}_\mathcal{F}$ we will shortly write $[A]_\mathcal{F} = [A]_{\mathcal{A}_\mathcal{F}}$. Note that $S_\mathcal{F}$ contains a countable discrete dense subspace consisting of isolated points (i.e. the principal ultrafilters) which we can naturally associate with ω, and therefore we can put $S^*_\mathcal{F} = S_\mathcal{F} \setminus \omega$. For $A \in \mathcal{A}_\mathcal{F}$ we also write $[A]^*_\mathcal{F} = [A]_\mathcal{F} \setminus \omega$.

Remark 4.4.1. Note that $\tilde{\mathcal{F}} = \big\{A^\bullet\colon\ A \in \mathcal{F}\big\}$ is a filter in $\mathcal{P}(\omega)/\operatorname{Fin}$ such that $\mathcal{A}_\mathcal{F}/\operatorname{Fin} = \tilde{\mathcal{F}} \cup \tilde{\mathcal{F}}^*$. It is not difficult to show that $S^*_\mathcal{F}$ is homeomorphic to the Stone space $St\big(\mathcal{A}_\mathcal{F}/Fin\big)$ (identifying the points $p_\mathcal{F}$ and $p_{\tilde{\mathcal{F}}}$), or, equivalently, that the Boolean algebra of clopen subsets of $S^*_\mathcal{F}$ is isomorphic to $\mathcal{A}_\mathcal{F}/\operatorname{Fin}$.

For every $A \in \mathcal{P}(\omega)$ we will also simply write A for the corresponding subset of $\omega \subseteq S_\mathcal{F}$. If $A \in \mathcal{A}_\mathcal{F}$, then $\overline{A}^{S_\mathcal{F}} = [A]_\mathcal{F}$, and conversely, if $A \in \mathcal{P}(\omega)$ is such that $\overline{A}^{S_\mathcal{F}}$ is clopen, then $A \in \mathcal{A}_\mathcal{F}$ (and hence again $\overline{A}^{S_\mathcal{F}} = [A]_\mathcal{F}$). Note that for every infinite set $A \in \mathcal{F}^*$ we have $\big(\mathcal{A}_\mathcal{F}\big)_A = \mathcal{P}(A)$ (resp. $\big(\mathcal{A}_\mathcal{F}\big)_A/[A]^{<\omega} = \mathcal{P}(A)/[A]^{<\omega}$) and so the clopen $[A]_\mathcal{F}$ (resp. $[A]^*_\mathcal{F}$) is homeomorphic to $\beta\omega$ (resp. to ω^*); consequently the Boolean algebras $\big(\mathcal{A}_\mathcal{F}\big)_A$ and $\big(\mathcal{A}_\mathcal{F}\big)_A/[A]^{<\omega}$ have the Nikodym property (by Seever's theorem [92]).

Recall that $p_\mathcal{F}$ denotes the point in $S^*_\mathcal{F}$ corresponding to the ultrafilter $\mathcal{F}$ in the algebra $\mathcal{A}_\mathcal{F}$. It is easy to see that the subspace $\omega \cup \big\{p_\mathcal{F}\big\}$ with the inherited topology is homeomorphic to the space $N_\mathcal{F}$ associated with the filter $\mathcal{F}$, defined in Chapter 3. Therefore, we will consider $N_\mathcal{F}$ as a dense subspace of $S_\mathcal{F}$.

Since $\mathcal{A}_\mathcal{F}$ is a subalgebra of $\mathcal{P}(\omega)$, $S_\mathcal{F} = St\big(\mathcal{A}_\mathcal{F}\big)$ is, by [68, Theorem 8.2], a continuous image of $\beta\omega \cong St(\mathcal{P}(\omega))$. A general relation between $\beta\omega$ and spaces $S_\mathcal{F}$ is described by the following proposition (which follows from a more general [68, Proposition 8.5]). Intuitively, $S_\mathcal{F}$ is made from $\beta\omega$ by gluing together all the ultrafilters in $\beta\omega$ extending $\mathcal{F}$.

Proposition 4.4.2. *Let $\mathcal{F}$ be a filter on ω. If $\mathfrak{F}$ denotes the subset of $\beta\omega$ consisting of all ultrafilters $x \in \beta\omega$ such that $\mathcal{F} \subseteq x$, then $\mathfrak{F}$ is closed in $\beta\omega$ and the mapping $\varphi\colon \beta\omega/\mathfrak{F} \to S_\mathcal{F}$ given for every $x \in \beta\omega$ by the formula $\varphi\big([x]_\mathfrak{F}\big) = x \cap \mathcal{A}_\mathcal{F}$, where $[x]_\mathfrak{F}$ denotes the equivalence class of x in the quotient space $\beta\omega/\mathfrak{F}$, is a homeomorphism.*

For more basic properties of space $S_\mathcal{F}$, see [76, Section 3]. We will use later the following topological property of the point $p_\mathcal{F}$ in $S^*_\mathcal{F}$ when $\mathcal{F}$ is a P-filter.

Lemma 4.4.3. *If $\mathcal{F}$ is a P-filter on ω, then $p_\mathcal{F}$ is a P-point in $S^*_\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a P-filter on ω, and let $\langle U_n : n \in \omega \rangle$ be a sequence of open neighbourhoods of $p_{\mathcal{F}}$ in $S^*_{\mathcal{F}}$. We can assume that each U_n is of the form $[A_n]^*_{\mathcal{F}}$ for some $A_n \in \mathcal{F}$. By the assumption about $\mathcal{F}$, there is $A \in \mathcal{F}$ such that $A \subseteq^* A_n$ for all $n \in \omega$. Then, $[A_n]^*_{\mathcal{F}}$ is an open neighbourhood of $p_{\mathcal{F}}$ and $[A]^*_{\mathcal{F}} \subseteq [A_n]^*_{\mathcal{F}}$ for every $n \in \omega$. $\square$

The main topological relations between the spaces $N_{\mathcal{F}}$ and $S_{\mathcal{F}}$ are described by the following results, which were already used in the proof of Proposition 3.5.1.

Lemma 4.4.4. *For every filter $\mathcal{F}$, the space $N_{\mathcal{F}}$ is C^*-embedded in $S_{\mathcal{F}}$.*

Proof. Clearly, $N_{\mathcal{F}}$ is dense in $S_{\mathcal{F}}$ since $\omega \subseteq N_{\mathcal{F}}$. Therefore, by [39, Theorem 6.4], it is enough to show that every two disjoint zero-sets in $N_{\mathcal{F}}$ have disjoint closures in $S_{\mathcal{F}}$. Let $A, B \subseteq N_{\mathcal{F}}$ be disjoint zero-sets. Without loss of generality we may assume that $p_{\mathcal{F}} \notin A$. Then, $A^c = A^c \cup \{p_{\mathcal{F}}\}$ is an open neighborhood of $p_{\mathcal{F}}$ in $N_{\mathcal{F}}$, and so $A^c \in \mathcal{F}$. Thus, $A, A^c \in \mathcal{A}_{\mathcal{F}}$. Using $B \subseteq A^c$, and the fact that $\overline{X}^{S_{\mathcal{F}}} = [X]_{\mathcal{F}}$ for any $X \in \mathcal{A}_{\mathcal{F}}$, we get:

$$\overline{A}^{S_{\mathcal{F}}} \cap \overline{B}^{S_{\mathcal{F}}} \subseteq \overline{A}^{S_{\mathcal{F}}} \cap \overline{A^c}^{S_{\mathcal{F}}} = [A]_{\mathcal{F}} \cap [A^c]_{\mathcal{F}} = [A \cap A^c]_{\mathcal{F}} = [\emptyset]_{\mathcal{F}} = \emptyset.$$

$\square$

Corollary 4.4.5. *$S_{\mathcal{F}} = \beta(N_{\mathcal{F}})$, i.e. $S_{\mathcal{F}}$ is the Čech–Stone compactification of $N_{\mathcal{F}}$.*

Proof. By Lemma 4.4.4, and the uniqueness of $\beta(N_{\mathcal{F}})$ as the compactification of $N_{\mathcal{F}}$ in which $N_{\mathcal{F}}$ is C^*-embedded ([39, Theorem 6.5]). $\square$

The following corollary is an immediate consequence of Theorem 4.3.15 for Boolean algebras $\mathcal{A}_{\mathcal{F}}$ and $\mathcal{A}_{\mathcal{F}}/\mathrm{Fin}$ together with Remark 4.4.1. We will use it frequently throughout this section.

Corollary 4.4.6. *Let $\mathcal{F}$ be a filter on ω and let $\mathcal{A} = \mathcal{A}_{\mathcal{F}}$ or $\mathcal{A} = \mathcal{A}_{\mathcal{F}}/\mathrm{Fin}$. Then, $\mathcal{A}$ has the Nikodym property if and only if there is no (disjointly supported) sequence $\langle \mu_n : n \in \omega \rangle$ of non-negative measures on $St(\mathcal{A})$ such that:*

1. *$p_{\mathcal{F}} \notin \operatorname{supp}(\mu_n)$ for every $n \in \omega$,*
2. *$\sup_{n\in\omega} \mu_n(St(\mathcal{A})) = \infty$,*
3. *if $\mathcal{A} = \mathcal{A}_{\mathcal{F}}$, then $\lim_{n\to\infty} \mu_n\big([\omega \setminus A]_{\mathcal{F}}\big) = 0$ for every $A \in \mathcal{F}$, or,*
   if $\mathcal{A} = \mathcal{A}_{\mathcal{F}}/\mathrm{Fin}$, then $\lim_{n\to\infty} \mu_n\big([\omega \setminus A]^*_{\mathcal{F}}\big) = 0$ for every $A \in \mathcal{F}$.

The corollary and its variant for spaces $N_{\mathcal{F}}$, presented in Theorem 3.2.1, are crucial for proving the following characterization of the Nikodym property of Boolean algebras $\mathcal{A}_{\mathcal{F}}$ as being a conjunction of the Nikodym property of the algebras $\mathcal{A}_{\mathcal{F}}/\mathrm{Fin}$ and of the spaces $N_{\mathcal{F}}$.

Theorem 4.4.7. *Let $\mathcal{F}$ be a filter on ω. Then, the algebra $\mathcal{A}_{\mathcal{F}}$ has the Nikodym property if and only if the algebra $\mathcal{A}_{\mathcal{F}}/\mathrm{Fin}$ has the Nikodym property and the space $N_{\mathcal{F}}$ has the finitely supported Nikodym property.*

Proof. Assume first that $\mathcal{A}_{\mathcal{F}}$ has the Nikodym property. Since the property is preserved by taking quotients (see Schachermayer [91, Proposition 2.11.(b)]), the algebra $\mathcal{A}_{\mathcal{F}}/\mathrm{Fin}$ has it as well. If the space $N_{\mathcal{F}}$ did not have the finitely supported Nikodym property, then $\mathcal{A}_\mathcal{F}$ would not have the Nikodym property by Corollary 4.2.2, as $S_\mathcal{F}$ contains a homeomorphic copy of $N_\mathcal{F}$. Hence, $N_\mathcal{F}$ has the required property, too.

Suppose now that $\mathcal{A}_\mathcal{F}$ does not have the Nikodym property. Let $\langle\mu_n\colon\ n\in\omega\rangle$ be a sequence of non-negative measures on $S_\mathcal{F}$ like in Corollary 4.4.6 (for $\mathcal{A}=\mathcal{A}_\mathcal{F}$). If $\sup_{n\in\omega}\mu_n\big(S^*_\mathcal{F}\big)=\infty$, then for each $n\in\omega$ we define the measure $\nu_n=\mu_n\restriction S^*_\mathcal{F}$, and it is not difficult to see that $\langle\nu_n\colon\ n\in\omega\rangle$ is a sequence of non-negative measures on $S^*_\mathcal{F}$ like in Corollary 4.4.6 (for $\mathcal{A}=\mathcal{A}_\mathcal{F}/\operatorname{Fin}$), which means that the algebra $\mathcal{A}_\mathcal{F}/\operatorname{Fin}$ does not have the Nikodym property either.

Thus, let us assume that $\sup_{n\in\omega}\mu_n\big(S^*_\mathcal{F}\big)<\infty$, which simply implies that $\sup_{n\in\omega}\mu_n(\omega)=\infty$. If for each $n\in\omega$ we set $\theta_n=\mu_n\restriction\omega$, then θ_n is a non-negative measure on $S_\mathcal{F}$ such that $\sup_{n\in\omega}\theta_n(\omega)=\infty$ and for every $A\in\mathcal{F}$ we have

$$\lim_{n\to\infty}\theta_n(\omega\setminus A)=\lim_{n\to\infty}\mu_n\big([\omega\setminus A]_\mathcal{F}\big)=0.$$

By the continuity of the measures, for every $n\in\omega$ there exists a finite set $F_n\subseteq\omega$ such that

$$\theta_n\big(\omega\setminus F_n\big)<1/n. \tag{$*$}$$

For each $n\in\omega$ let us set $\nu_n=\theta_n\restriction F_n$. We claim that the finitely supported sequence $\langle\nu_n\colon\ n\in\omega\rangle$ of non-negative measures on $N_\mathcal{F}$ satisfies conditions (1)–(3) from Theorem 3.2.1. First, for every $n\in\omega$ we have $\operatorname{supp}\big(\nu_n\big)\subseteq\omega$ and, by $(*)$,

$$\nu_n(\omega)=\theta\big(F_n\big)=\theta_n(\omega)-\theta_n\big(\omega\setminus F_n\big)>\theta_n(\omega)-1/n,$$

so $\sup_{n\in\omega}\nu_n(\omega)=\infty$.

Next, for every $A\in\mathcal{F}$ we have $\lim_{n\to\infty}\nu_n(\omega\setminus A)=0$, as $\lim_{n\to\infty}\theta_n(\omega\setminus A)=0$ and, again by $(*)$,

$$\lim_{n\to\infty}\big|\nu_n(\omega\setminus A)-\theta_n(\omega\setminus A)\big|\le\lim_{n\to\infty}\big\|\nu_n-\theta_n\big\|=\lim_{n\to\infty}\big\|\theta_n\restriction\big(\omega\setminus F_n\big)\big\|=\lim_{n\to\infty}\theta_n\big(\omega\setminus F_n\big)=0.$$

Therefore, by Theorem 3.2.1, the space $N_\mathcal{F}$ does not have the finitely supported Nikodym property. $\square$

Note that an ideal $\mathcal{I}$ on ω has the Nikodym property as an ideal in the Boolean algebra $\mathcal{P}(\omega)$ (i.e. in the sense of Definition 4.3.9) if and only if $\mathcal{I}$ has the Nikodym property as a ring of subsets of ω (i.e. in the sense of Definition 4.3.1)—this follows from the fact that the rings $\mathcal{R}(\mathcal{I})$ and $\mathcal{I}$ are isomorphic. Also, recall that by Theorem 4.3.11 an ideal $\mathcal{I}$ on ω has the Nikodym property if and only if the Boolean algebra $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property. Since the Boolean algebra $\mathcal{P}(\omega)$ has the Nikodym property, it follows that if an ideal $\mathcal{I}$ on ω is maximal (that is, $\mathcal{I}^*$ is an ultrafilter and so $\mathcal{A}_{\mathcal{I}^*}=\mathcal{P}(\omega)$), then $\mathcal{I}$ has the Nikodym property.

Corollary 4.4.8. *If $\mathcal{I}$ is a maximal ideal on ω, then $\mathcal{I}$ has the Nikodym property.*

Let us now turn our attention to P-ideals. We start with the following remark.

Example 4.4.9. In [25, Example 5.6] an ideal on ω which is not a P-ideal but has the Nikodym property was constructed. It immediately follows from the discussion above that every maximal ideal $\mathcal{I}$ on ω whose dual filter $\mathcal{I}^*$ is not a P-point in ω^* is also such an ideal. Since there are $2^\mathfrak{c}$ many non-isomorphic ultrafilters on ω which are not P-points in ω^* (see e.g. Jech [57, page 76]), we actually get $2^\mathfrak{c}$ many non-isomorphic such examples of ideals.

In the case of a P-ideal $\mathcal{I}$ on ω, Theorem 4.4.7 yields that the Nikodym property of the algebra $\mathcal{A}_{\mathcal{I}^*}$ is equivalent to the finitely supported Nikodym property of the space $N_{\mathcal{I}^*}$ as well as to the following two properties of $\mathcal{I}$, see Theorem 4.4.15 below.

Definition 4.4.10. Let $\mathcal{R}$ be a ring of subsets of a set X. A subset $\mathcal{Q} \subseteq \mathcal{R}$ is a *Nikodym set for* $\operatorname{ba}(\mathcal{R})$ if, for every sequence $\langle \mu_k \colon\ k \in \omega \rangle$ in $\operatorname{ba}(\mathcal{R})$, if $\lim_{k\in\omega} \mu_k(A) = 0$ for every $A \in \mathcal{Q}$, then $\sup_{k\in\omega} \|\mu_k\| < \infty$.

Definition 4.4.11. Let $\mathcal{R}$ be a ring of subsets of a set X. A collection $\langle \mathcal{R}_\sigma \colon\ \sigma \in \omega^{<\omega} \rangle$ of subsets of $\mathcal{R}$ with $\mathcal{R} = \mathcal{R}_{\langle\rangle}$ and such that for every $\sigma \in \omega^{<\omega}$ the sequence $\langle \mathcal{R}_{\sigma^\frown n} \colon\ n \in \omega \rangle$ is increasing and satisfies $\mathcal{R}_\sigma = \bigcup_{n\in\omega} \mathcal{R}_{\sigma^\frown n}$ is called a *web in* $\mathcal{R}$.

Definition 4.4.12. A ring $\mathcal{R}$ of subsets of a set X has the *strong Nikodym property* if for every increasing sequence $\langle \mathcal{R}_n \colon\ n \in \omega \rangle$ of subsets of $\mathcal{R}$ with $\mathcal{R} = \bigcup_{n\in\omega} \mathcal{R}_n$ there is $n_0 \in \omega$ such that $\mathcal{R}_{n_0}$ is a Nikodym set for $\operatorname{ba}(\mathcal{R})$.

A ring $\mathcal{R}$ of subsets of a set X has the *web Nikodym property* if for every web $\langle \mathcal{R}_\sigma \colon\ \sigma \in \omega^{<\omega} \rangle$ in $\mathcal{R}$ there is a sequence $\langle \sigma_m \in \omega^m \colon\ m \in \omega \rangle$ such that, for every $m \in \omega$, σ_{m+1} extends σ_m and $\mathcal{R}_{\sigma_m}$ is a Nikodym set for $\operatorname{ba}(\mathcal{R})$.

Both of the properties of course immediately imply the standard Nikodym property for rings. The strong Nikodym property and the web Nikodym property were first essentially studied by Valdivia [108] and López-Pellicer [75], respectively, who proved that σ-fields of sets have both the properties (see also [63, Chapter 21]). Later, Valdivia [109] showed that the field $\mathscr{J}_n$ of Jordan measurable subsets of the n-dimensional cube $[0,1]^n$ has the strong Nikodym property, which was further extended by López-Alfonso [74] to the web Nikodym property (consequently, both results strengthen Schachermayer's theorem [91], mentioned in Introduction). Ferrando [29] proved that the ideal $\mathcal{Z}$ has the latter property, too. For further results on the web Nikodym property for ideals, see also Ferrando *et al.* [46].

We translate Definition 4.4.12 in a natural way to the class of Boolean algebras and ideals.

Definition 4.4.13. Let $\mathcal{A}$ be a Boolean algebra. $\mathcal{A}$ has the *strong Nikodym property* (resp. the *web Nikodym property*) if the ring $\operatorname{Clopen}(St(\mathcal{A}))$ of all clopen subsets of $St(\mathcal{A})$ has the strong Nikodym property (resp. the web Nikodym property).

An ideal $\mathcal{I}$ in $\mathcal{A}$ has the *strong Nikodym property* (resp. the *web Nikodym property*) if the ring $\mathcal{R}(\mathcal{I})$ of clopen subsets of $St(\mathcal{A})$ has the strong Nikodym property (resp. the web Nikodym property).

Lemma 4.4.14. *Let $\mathcal{I}$ be an ideal on ω such that $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property. Let $\mathcal{Q} \subseteq \mathcal{R}(\mathcal{I})$ be non-empty and set $\mathcal{Q}^* = \{S_{\mathcal{I}^*} \setminus A \colon A \in \mathcal{Q}\}$. Then, the following statements are equivalent:*

(a) *$\mathcal{Q}$ is Nikodym for* $\operatorname{ba}(\mathcal{R}(\mathcal{I}))$,

(b) *$\mathcal{Q} \cup \{S_{\mathcal{I}^*}\}$ is Nikodym for* $\operatorname{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$,

(c) *$\mathcal{Q} \cup \mathcal{Q}^*$ is Nikodym for* $\operatorname{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$.

Proof. (a)$\Rightarrow$(b): Assume that $\mathcal{Q} \subseteq \mathcal{R}(\mathcal{I})$ is a Nikodym set for $\operatorname{ba}(\mathcal{R}(\mathcal{I}))$. Let $\langle \mu_n \colon\ n \in \omega \rangle$ be a sequence in $\operatorname{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$ such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{Q} \cup \{S_{\mathcal{I}^*}\}$. It follows that

$$\sup_{n\in\omega} \big\|\mu_n \restriction \mathcal{R}(\mathcal{I})\big\| < \infty,$$

hence for every $A \in \mathcal{R}(\mathcal{I})$ we have $\sup_{n\in\omega} \left|\mu_n(A)\right| < \infty$, and so

$$\sup_{n\in\omega} \left|\mu_n(A^c)\right| \le \sup_{n\in\omega} \left|\mu_n\big(S_{\mathcal{I}^*}\big)\right| + \sup_{n\in\omega} \left|\mu_n(A)\right| < \infty.$$

Thus, we have $\sup_{n\in\omega} \left|\mu_n(A)\right| < \infty$ for every $A \in \mathcal{A}_{\mathcal{I}^*}$. As $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property, we get that $\sup_{n\in\omega} \left\|\mu_n\right\| < \infty$. This proves that the set $\mathcal{Q} \cup \big\{S_{\mathcal{I}^*}\big\}$ is Nikodym for $\mathrm{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$.

(b)⇒(c): Let $\langle \mu_n\colon\ n \in \omega\rangle \subseteq \mathrm{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$ be such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{Q} \cup \mathcal{Q}^*$. Let $A \in \mathcal{Q}$. Then,

$$\lim_{n\to\infty} \mu_n\big(S_{\mathcal{I}^*}\big) = \lim_{n\to\infty} \big(\mu_n(A) + \mu_n\big(S_{\mathcal{I}^*} \setminus A\big)\big) = 0.$$

As $\mathcal{Q} \cup \big\{S_{\mathcal{I}^*}\big\}$ is Nikodym for $\mathrm{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$, we get that $\sup_{n\to\infty} \left\|\mu_n\right\| < \infty$. Consequently, $\mathcal{Q} \cup \mathcal{Q}^*$ is Nikodym for $\mathrm{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$, too.

(c)⇒(a) The implication follows by a similar argument as in the proof of Theorem 4.3.11, using the extension operator $T_0\colon\ \mathrm{ba}(\mathcal{R}(\mathcal{I})) \to \mathrm{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$. □

Theorem 4.4.15. *Let $\mathcal{I}$ be a P-ideal on ω. Then, the following conditions are equivalent:*

1. *$\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property,*
2. *$\mathcal{A}_{\mathcal{I}^*}$ has the strong Nikodym property,*
3. *$\mathcal{A}_{\mathcal{I}^*}$ has the web Nikodym property,*
4. *$\mathcal{I}$ has the Nikodym property,*
5. *$\mathcal{I}$ has the strong Nikodym property,*
6. *$\mathcal{I}$ has the web Nikodym property,*
7. *$N_{\mathcal{I}^*}$ has the finitely supported Nikodym property.*

Proof. Implications (3)⇒(2)⇒(1) and (6)⇒(5)⇒(4) are immediate.

Equivalence (1)⇔(4) follows from Theorem 4.3.11.

Using the nomenclature of [46], an ideal $\mathcal{J}$ on ω is a P-ideal if and only if the family $[\omega]^{<\omega}$ is *$\mathcal{J}$-singular*. Consequently, by [46, Theorem 3.3], if the ideal $\mathcal{I}$ has the Nikodym property, then $\mathcal{I}$ has the web Nikodym property, i.e. implication (4)⇒(6) holds.

We show implication (6)⇒(3). Let $\big\langle \mathcal{R}_\sigma\colon\ \sigma \in \omega^{<\omega}\big\rangle$ be a web in $Clopen\big(S_{\mathcal{I}^*}\big)$ (with $\mathcal{R}_{\langle\rangle} = Clopen\big(S_{\mathcal{I}^*}\big)$). Since the sequences $\big\langle \mathcal{R}_{\sigma^\frown n}\colon\ n \in \omega\big\rangle$ are increasing, without loss of generality we may assume that for each $\sigma \in \omega^{<\omega}$ we have $S_{\mathcal{I}^*} \in \mathcal{R}_\sigma$. For each $\sigma \in \omega^{<\omega}$ set $\mathcal{R}'_\sigma = \mathcal{R}_\sigma \cap \mathcal{R}(\mathcal{I})$. It follows that the collection $\big\langle \mathcal{R}'_\sigma\colon\ \sigma \in \omega^{<\omega}\big\rangle$ is a web in $\mathcal{R}(\mathcal{I})$ (with $\mathcal{R}'_{\langle\rangle} = \mathcal{R}(\mathcal{I})$). As $\mathcal{I}$ has the web Nikodym property, and so by definition the (isomorphic) ring $\mathcal{R}(\mathcal{I})$ has the web Nikodym property, too, there is a sequence $\big\langle \sigma_m \in \omega^m\colon\ m \in \omega\big\rangle$ such that, for every $m \in \omega$, σ_{m+1} extends σ_m and the set $\mathcal{R}'_{\sigma_m}$ is Nikodym for $\mathrm{ba}(\mathcal{R}(\mathcal{I}))$. Note that, by already established implications (6)⇒(4)⇒(1), the algebra $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property, so by implication (a)⇒(b) of Lemma 4.4.14 for each $m \in \omega$ the set $\mathcal{R}_{\sigma_m}$ is Nikodym for $\mathrm{ba}\big(Clopen\big(S_{\mathcal{I}^*}\big)\big)$, as $S_{\mathcal{I}^*} \in \mathcal{R}_{\sigma_m}$. Consequently, the ring $Clopen\big(S_{\mathcal{I}^*}\big)$ has the web Nikodym property and so the algebra $\mathcal{A}_{\mathcal{I}^*}$ does, too.

We now prove that implication (3)⇒(6) holds. Let $\big\langle \mathcal{R}_\sigma\colon\ \sigma \in \omega^{<\omega}\big\rangle$ be a web in $\mathcal{R}(\mathcal{I})$ (with $\mathcal{R}_{\langle\rangle} = \mathcal{R}(\mathcal{I})$); again, without loss of generality we may assume that $\emptyset \in \mathcal{R}_\sigma$ for every

$\sigma \in \omega^{<\omega}$. For every $\sigma \in \omega^{<\omega}$ set $\mathcal{R}_\sigma^* = \{S_{\mathcal{I}^*} \setminus A\colon A \in \mathcal{R}_\sigma\}$ and $\mathcal{S}_\sigma = \mathcal{R}_\sigma \cup \mathcal{R}_\sigma^*$. Then, $\langle \mathcal{S}_\sigma\colon \sigma \in \omega^{<\omega}\rangle$ is a web in $Clopen(S_{\mathcal{I}^*})$ (with $\mathcal{S}_{\langle\rangle} = Clopen(S_{\mathcal{I}^*})$). As $\mathcal{A}_{\mathcal{I}^*}$ has the web Nikodym property, there is a sequence $\langle \sigma_m \in \omega^m\colon m \in \omega\rangle$ such that, for every $m \in \omega$, σ_{m+1} extends σ_m and the set $\mathcal{S}_{\sigma_m}$ is Nikodym for $\mathrm{ba}(Clopen(S_{\mathcal{I}^*}))$. By implication (c)⇒(a) of Lemma 4.4.14, for each $m \in \omega$ the set $\mathcal{R}_{\sigma_m}$ is Nikodym for $\mathrm{ba}(\mathcal{R}(\mathcal{I}))$, which proves that $\mathcal{I}$ has the web Nikodym property as well.

Implication (1)⇒(7) follows from Theorem 4.4.7.

For implication (7)⇒(1), assume that $\mathcal{A}_{\mathcal{I}^*}$ does not have the Nikodym property. It follows by (7) and Theorem 4.4.7 that the algebra $\mathcal{A}_{\mathcal{I}^*}/\,\mathrm{Fin}$ does not have the Nikodym property, so let $\langle \mu_n\colon n \in \omega\rangle$ be an anti-Nikodym sequence on $\mathcal{A}_{\mathcal{I}^*}/\,\mathrm{Fin}$. As $\mathcal{I}$ is a P-ideal, Lemma 4.4.3 implies that $p_{\mathcal{I}^*}$ is a P-point in the space $S^*_{\mathcal{I}^*}$. Thus, by Sobota [97, Proposition 4.5], $p_{\mathcal{I}^*}$ cannot be a Nikodym concentration point of $\langle \mu_n\colon n \in \omega\rangle$. Consequently, $\langle \mu_n\colon n \in \omega\rangle$ has a concentration point $t \in S^*_{\mathcal{I}^*}$ different from $p_{\mathcal{I}^*}$. Let $A \in \mathcal{I}$ be such that $t \in [A]^*_{\mathcal{I}^*}$. Note that for the equivalence class $A^\bullet$ of A in the quotient algebra $\mathcal{A}_{\mathcal{I}^*}/\,\mathrm{Fin}$ we have $\left(\mathcal{A}_{\mathcal{I}^*}/\,\mathrm{Fin}\right)_{A^\bullet} = \mathcal{P}(A)/[A]^{<\omega}$. Consequently, the sequence $\langle \mu_n \restriction \mathcal{P}(A)/[A]^{<\omega}\colon n \in \omega\rangle$ is an anti-Nikodym sequence on $\mathcal{P}(A)/[A]^{<\omega}$, which is impossible as the latter algebra has the Nikodym property by the Seever theorem [92]. □

Theorem 4.4.15 will be frequently used in the sequel; for its particular applications to the class of the density ideals on ω, see Corollaries 4.4.19 and 4.4.21.

Example 4.4.16. A careful reader has surely noticed that the argument for implication (3)⇒(6) is superfluous in the proof of Theorem 4.4.15, as this implication follows from combinations of other ones, whose proofs are also given. We provided it however deliberately in order to show that the assumption that the considered ideal $\mathcal{I}$ is a P-ideal is actually not required for the validity of equivalences (3)⇔(6), (1)⇔(4), and (2)⇔(5) (which can be proved in a similar but easier way as (3)⇔(6)). Based on that, we obtain, e.g., that every maximal ideal $\mathcal{I}$ on ω has the web Nikodym property, since for such $\mathcal{I}$ we have $\mathcal{A}_{\mathcal{I}^*} = \mathcal{P}(\omega)$ and the σ-field $\mathcal{P}(\omega)$ has the latter property (by López-Pellicer [75]), strengthening Corollary 4.4.8. In particular, by an argument as in Example 4.4.9, we get that there are $2^{\mathfrak{c}}$ many non-isomorphic maximal ideals with the web Nikodym property (and which are not P-ideals).

Example 4.4.17. However, in general, Theorem 4.4.15 does not hold anymore if one drops the assumption that $\mathcal{I}$ is a P-ideal. Namely, as described in [76, Sections 4 and 6.3], there are free filters $\mathcal{F}$ on ω such that the spaces $S^*_{\mathcal{F}}$ contain non-trivial convergent sequences, but the spaces $N_{\mathcal{F}}$ do not have the bounded Josefson–Nissenzweig property (recall Definition 3.1.7). Consequently, for those filters $\mathcal{F}$, the algebras $\mathcal{A}_{\mathcal{F}}/\,\mathrm{Fin}$, $\mathcal{A}_{\mathcal{F}}$, and the ideals $\mathcal{F}^*$ do not have the Nikodym property, but, by Proposition 3.5.1, the spaces $N_{\mathcal{F}}$ do have the finitely supported Nikodym property. A similar example is also given in Remark 4.5.16.

By Proposition 3.3.1, given two filters $\mathcal{F}$ and $\mathcal{G}$ on ω such that $\mathcal{F} \leq_K \mathcal{G}$, if $N_{\mathcal{G}}$ does not have the finitely supported Nikodym property, then $N_{\mathcal{F}}$ does not have it either. Our last example shows that this does not hold for the general Nikodym property of ideals.

Example 4.4.18. Using again the constructions presented in [76, Sections 4 and 6.3] for each ultrafilter $\mathcal{F}$ on ω one can obtain a filter $\mathcal{G}$ such that $\mathcal{F} \leq_K \mathcal{G}$ and $S^*_{\mathcal{G}}$ contains a non-trivial convergent sequence (simply set $\mathcal{F}_n = \mathcal{F}$ for each $n \in \omega$ in the construction). As for each ultrafilter $\mathcal{F}$ we have $\mathcal{A}_{\mathcal{F}} = \mathcal{P}(\omega)$, we consequently get that there are ideals $\mathcal{I}$

$(=\mathcal{F}^*)$ and $\mathcal{J}$ $(=\mathcal{G}^*)$ on ω such that $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{I}$ has the Nikodym property, but $\mathcal{J}$ does not.

One of the first results connecting the Nikodym property with any of the above defined classes of ideals on ω was the result of Drewnowski, Florencio, and Paúl [22, Proposition 6] asserting that the asymptotic density zero ideal $\mathcal{Z}$ has the Nikodym property.

Combining Theorem 4.4.15 and Corollary 3.2.7, we get the following corollary.

Corollary 4.4.19. *If $\mathcal{I}$ is a P-ideal on ω such that $\mathcal{I} \not\leq_K \operatorname{tr}(\mathcal{N})$, then $\mathcal{I}$ has the web Nikodym property.*

We also obtain the following important result, being a variant of Drewnowski, Florencio, and Paúl [23, Example 4.3] implying that Erdős–Ulam ideals belonging to a special class (which includes $\mathcal{Z}$) have the Nikodym property (see also [25, Theorem 6.8]).

Theorem 4.4.20. *Let $\mathcal{I}$ be a density ideal on ω. Then, $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property if and only if $\mathcal{I}$ is isomorphic to an Erdős–Ulam ideal.*

Proof. If $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property, then by Theorem 4.4.7 $N_{\mathcal{I}^*}$ has the finitely supported Nikodym property, but, as $\mathcal{I}$ is a density ideal, this may only happen if $\mathcal{I}$ is isomorphic to an Erdős–Ulam ideal, by Theorem 3.3.3.

Assume then that $\mathcal{I}$ is isomorphic to an Erdős–Ulam ideal. Again, by Theorem 3.3.3, the space $N_{\mathcal{I}^*}$ has the finitely supported Nikodym property. As $\mathcal{I}$ is a P-ideal, Theorem 4.4.15 yields that the algebra $\mathcal{A}_{\mathcal{I}^*}$ has the Nikodym property. □

Theorems 4.4.15 and 4.4.20 immediately yield together the following result.

Corollary 4.4.21. *Let $\mathcal{I}$ be a density ideal on ω. Then, the following conditions are equivalent:*

1. *$\mathcal{I}$ has the Nikodym property,*
2. *$\mathcal{I}$ has the strong Nikodym property,*
3. *$\mathcal{I}$ has the web Nikodym property,*
4. *$\mathcal{I}$ is isomorphic to an Erdős–Ulam ideal.*

Remark 4.4.22. By Kwela *et al.* [73, Theorem 3], there are $\mathfrak{c}$ many Katětov incomparable density ideals, hence there are $\mathfrak{c}$ many Katětov incomparable density ideals which are not isomorphic to Erdős–Ulam ideals (see Farah [27, Lemma 1.13.10]). Consequently, by Corollary 4.4.21, there are $\mathfrak{c}$ many non-isomorphic density ideals without the Nikodym property.

On the other hand, by Kwela [72, Propositions 5 and 6] there are $\mathfrak{c}$ many pairwise non-isomorphic Erdős–Ulam ideals, and hence by Corollary 4.4.21 $\mathfrak{c}$ many pairwise non-isomorphic density ideals with the web Nikodym property.

Example 4.4.23. The preceding results give us several new examples of ideals on ω with the web Nikodym property. The first one is $\operatorname{tr}(\mathcal{N})$, which is a totally bounded non-pathological P-ideal (see Section 2.6) and so it has the web Nikodym property by Corollary 3.2.10 and Theorem 4.4.15. By Corollary 4.4.21, the logarithmic density zero ideal $\mathcal{Z}_{\log}$ has the web Nikodym property, too. Finally, a large class of ideals with the web Nikodym property is given in Section 4.5.3, see Remark 4.5.30.

The following example, appealing to the class $\mathcal{AN}$ of ideals, supplements Example 4.1.11 and expands Remark 4.4.22.

Example 4.4.24. (Supplement to Example 4.1.11). Let us consider the algebra $\mathcal{A}_{\mathcal{I}^*}$ for some $\mathcal{I} \in \mathcal{AN}$. By Theorem 4.4.7, it does not have the Nikodym property. On the other hand, for every $A \in \mathcal{I}$ the restricted algebra $\mathcal{P}(\omega)_A = \mathcal{P}(A)$ is σ-complete, so it has the Nikodym property. Therefore, by Proposition 4.3.12, every anti-Nikodym sequence on $\mathcal{A}_{\mathcal{I}^*}$ has a unique Nikodym concentration point, namely $p_{\mathcal{I}^*}$, which is then necessarily strong.

Recall that the class $\mathcal{AN}$ is large, that is, it contains $2^{\mathfrak{c}}$ many non-isomorphic non-maximal ideals (Corollary 3.6.3). By Lemma 4.4.25 below, it follows that there are $2^{\mathfrak{c}}$ many non-isomorphic Boolean subalgebras of $\mathcal{P}(\omega)$ without the Nikodym property and carrying only anti-Nikodym sequences with unique Nikodym concentration points.

4.4.1 Large families of algebras with the Nikodym property and without the Grothendieck property

In this subsection we construct two large families $\mathcal{H}_1$ and $\mathcal{H}_2$ of Boolean algebras with the Nikodym property and without the Grothendieck property. The family $\mathcal{H}_1$ consists of $\mathfrak{c}$ algebras of the form $\mathcal{A}_{\mathcal{F}}$ for a Borel (in fact, $\mathbb{F}_{\sigma\delta}$) filter $\mathcal{F}$ on ω, and the family $\mathcal{H}_2$ consists of $2^{\mathfrak{c}}$ algebras of the form $\mathcal{A}_{\mathcal{F}}$ for some non-analytic filter $\mathcal{F}$ on ω. These constructions supplement the results of Schachermayer [91], Graves and Wheeler [43], Valdivia [109], and Drewnowski, Florencio, and Paúl [22, Remark 4].

We start with the following lemma.

Lemma 4.4.25. *Let $\mathcal{F}$ and $\mathcal{G}$ be filters on ω which are not ultrafilters. Assume that $\varphi\colon S_{\mathcal{F}} \to S_{\mathcal{G}}$ is a homeomorphism. Then, the filters $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

Proof. By $\mathcal{U}_z$ let us denote the neighborhood filter of a point z in a topological space Z.

First, note that since φ is a homeomorphism and the points in ω are the only isolated points in both $S_{\mathcal{F}}$ and $S_{\mathcal{G}}$, we have $\varphi[\omega] = \omega$ and $\varphi\big[S_{\mathcal{F}}^*\big] = S_{\mathcal{G}}^*$. It follows that $\varphi \restriction \omega\colon \omega \to \omega$ is a bijection.

Second, for each points $x \in S_{\mathcal{F}}^*$ and $y \in S_{\mathcal{G}}^*$ we have $x = \big\{U \cap \omega\colon U \in \mathcal{U}_x\big\}$ and $y = \big\{V \cap \omega\colon V \in \mathcal{U}_y\big\}$. Moreover, as φ is a homeomorphism, for every $x \in S_{\mathcal{F}}^*$ it holds $\mathcal{U}_{\varphi(x)} = \big\{\varphi[U]\colon U \in \mathcal{U}_x\big\}$. Consequently, for every $x \in S_{\mathcal{F}}^*$ we have

$$\varphi(x) = \big\{V \cap \omega\colon V \in \mathcal{U}_{\varphi(x)}\big\} = \big\{\varphi[U] \cap \omega\colon U \in \mathcal{U}_x\big\} = \big\{\varphi[U \cap \omega]\colon U \in \mathcal{U}_x\big\} = \qquad (*)$$

$$= \{\varphi[A]\colon A \in x\}.$$

Third, observe that any points $x \in S_{\mathcal{F}}^* \setminus \big\{p_{\mathcal{F}}\big\}$ and $y \in S_{\mathcal{G}}^* \setminus \big\{p_{\mathcal{G}}\big\}$ extend to unique ultrafilters in $\mathcal{P}(\omega)$, whereas by the assumption both $p_{\mathcal{F}} = \mathcal{F}$ and $p_{\mathcal{G}} = \mathcal{G}$ are non-maximal filters in $\mathcal{P}(\omega)$. As $\varphi \restriction \omega$ is a bijection, $(*)$ implies that if $x \in S_{\mathcal{F}}^* \setminus \big\{p_{\mathcal{F}}\big\}$, then $\varphi(x)$ also extends to a unique ultrafilter in $\mathcal{P}(\omega)$, and so, consequently, $\varphi(x) \in S_{\mathcal{G}}^* \setminus \big\{p_{\mathcal{G}}\big\}$. It follows that $\varphi\big(p_{\mathcal{F}}\big) = p_{\mathcal{G}}$, and so $\varphi \restriction N_{\mathcal{F}}$ is a homeomorphism between $N_{\mathcal{F}}$ and $N_{\mathcal{G}}$. Thus, by Proposition 3.1.3, the filters $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.

□

Note that, for a filter $\mathcal{F}$ on ω, the Boolean algebra $\mathcal{A}_{\mathcal{F}} = \mathcal{F} \cup \mathcal{F}^*$ belongs to a Borel class Γ (resp. is analytic) provided that $\mathcal{F}$ belongs to the Borel class Γ (resp. is analytic).

Theorem 4.4.26. *There is a family $\{\mathcal{F}_\alpha\colon \alpha<\mathfrak{c}\}$ of $\mathbb{F}_{\sigma\delta}$ filters on ω such that the family $\mathcal{H}_1=\{\mathcal{A}_{\mathcal{F}_\alpha}\colon \alpha<\mathfrak{c}\}$ consists of pairwise non-isomorphic Boolean algebras, each of them having the Nikodym property but not the Grothendieck property.*

Proof. By the results from Kwela's [72, Section 4], there is a family $\{\mathcal{I}_\alpha\colon \alpha<\mathfrak{c}\}$ of pairwise non-isomorphic Erdős–Ulam ideals. Every Erdős–Ulam ideal, as a density ideal, is $\mathbb{F}_{\sigma\delta}$. Let $\mathcal{F}_\alpha=\mathcal{I}_\alpha^*$ for every $\alpha<\mathfrak{c}$, and set $\mathcal{H}_1=\{\mathcal{A}_{\mathcal{F}_\alpha}\colon \alpha<\mathfrak{c}\}$. By Lemma 4.4.25, the spaces $S_{\mathcal{F}_\alpha}$, $\alpha<\mathfrak{c}$, are pairwise non-homeomorphic, and so the Boolean algebras $\mathcal{A}_{\mathcal{F}_\alpha}$, $\alpha<\mathfrak{c}$, are pairwise non-isomorphic. Next, by Marciszewski and Sobota [76, Theorem C], for all $\alpha<\mathfrak{c}$ the space $N_{\mathcal{F}_\alpha}$ has the bounded Josefson–Nissenzweig property, and so, by [76, Corollary D], the algebra $\mathcal{A}_{\mathcal{F}_\alpha}$ does not have the Grothendieck property. Finally, by Theorem 4.4.20, the algebra $\mathcal{A}_{\mathcal{F}_\alpha}$ has the Nikodym property for every $\alpha<\mathfrak{c}$. □

Remark 4.4.27. Let us note that in the above proof we could use as well the family of ideals given by Theorem 4.5.26 instead of the one from [72, Section 4], since it also consists of analytic P-ideals (hence, $\mathbb{F}_{\sigma\delta}$ ideals) with the Nikodym property and for which [76, Theorem C and Corollary D] are applicable.

For a pair of topological spaces X, Y and points $x\in X$, $y\in Y$, by $(X\sqcup Y)/\{x,y\}$ we mean the topological disjoint union $X\sqcup Y$ of X and Y with the points x and y identified as a one point. Both of the spaces X and Y are naturally identified inside $(X\sqcup Y)/\{x,y\}$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be filters on ω. Fix a partition (Ω_1,Ω_2) of ω into two infinite sets and bijections $b_i\colon\omega\to\Omega_i$, $i=1,2$. Then, we define the filter $\mathcal{F}_1\oplus\mathcal{F}_2$ on ω as follows: for every $A\in\mathcal{P}(\omega)$, $A\in\mathcal{F}_1\oplus\mathcal{F}_2$ if and only if $b_i^{-1}[A\cap\Omega_i]\in\mathcal{F}_i$ for $i=1,2$. For each $i=1,2$ and the subspace $N'_{\mathcal{F}_i}=\Omega_i\cup\{p_{\mathcal{F}_1\oplus\mathcal{F}_2}\}$ of $N_{\mathcal{F}_1\oplus\mathcal{F}_2}$, the bijection b_i gives rise to the natural homeomorphism $h_i\colon N_{\mathcal{F}_i}\to N'_{\mathcal{F}_i}$ with $h_i(p_{\mathcal{F}_i})=p_{\mathcal{F}_1\oplus\mathcal{F}_2}$, identifying the spaces $N_{\mathcal{F}_i}$ and $N'_{\mathcal{F}_i}$. Consequently, the space $N_{\mathcal{F}_1\oplus\mathcal{F}_2}$ can be naturally identified with $(N_{\mathcal{F}_1}\sqcup N_{\mathcal{F}_2})/\{p_{\mathcal{F}_1},p_{\mathcal{F}_2}\}$, with the points $p_{\mathcal{F}_1}$ and $p_{\mathcal{F}_2}$ glued together as the point $p_{\mathcal{F}_1\oplus\mathcal{F}_2}$. The following lemma extends this identification to the spaces $S_{\mathcal{F}_1\oplus\mathcal{F}_2}$ and $(S_{\mathcal{F}_1}\sqcup S_{\mathcal{F}_2})/\{p_{\mathcal{F}_1},p_{\mathcal{F}_2}\}$.

Lemma 4.4.28. *For every filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on ω, the space $S_{\mathcal{F}_1\oplus\mathcal{F}_2}$ can be naturally identified with the quotient space $(S_{\mathcal{F}_1}\sqcup S_{\mathcal{F}_2})/\{p_{\mathcal{F}_1},p_{\mathcal{F}_2}\}$.*

Proof. We first show that

$$\overline{N'_{\mathcal{F}_1}}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}\cap\overline{N'_{\mathcal{F}_2}}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}=\{p_{\mathcal{F}_1\oplus\mathcal{F}_2}\}.\tag{$*$}$$

Of course, we have

$$\{p_{\mathcal{F}_1\oplus\mathcal{F}_2}\}=N'_{\mathcal{F}_1}\cap N'_{\mathcal{F}_2}\subseteq\overline{N'_{\mathcal{F}_1}}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}\cap\overline{N'_{\mathcal{F}_2}}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}.$$

So, assume there is

$$x\in\overline{N'_{\mathcal{F}_1}}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}\cap\overline{N'_{\mathcal{F}_2}}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}$$

such that $x\neq p_{\mathcal{F}_1\oplus\mathcal{F}_2}$. Then, there is a clopen subset U of $S_{\mathcal{F}_1\oplus\mathcal{F}_2}$ such that $x\in U$ and $p_{\mathcal{F}_1\oplus\mathcal{F}_2}\notin U$. Let $A_1=\Omega_1\cap U$ and $A_2=\Omega_2\cap U$. It follows that $U\cap\omega=A_1\cup A_2$, so $A_1\cup A_2\in\mathcal{A}_{\mathcal{F}_1\oplus\mathcal{F}_2}$ and

$$U=\overline{A_1\cup A_2}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}=[A_1\cup A_2]_{\mathcal{F}_1\oplus\mathcal{F}_2}.$$

Moreover, as for each $i=1,2$ the set A_i is clopen in $N'_{\mathcal{F}_i}$ and $p_{\mathcal{F}_1\oplus\mathcal{F}_2}\notin A_i$, we have $b_i^{-1}[A_i]\in\mathcal{F}_i^*$, and hence $A_i\in\mathcal{A}_{\mathcal{F}_1\oplus\mathcal{F}_2}$. Also, as $A_1\cap A_2\subseteq\Omega_1\cap\Omega_2=\emptyset$, we get that

$$[A_1]_{\mathcal{F}_1\oplus\mathcal{F}_2}\cap[A_2]_{\mathcal{F}_1\oplus\mathcal{F}_2}=\emptyset.$$

Since

$$x \in U = [A_1 \cup A_2]_{\mathcal{F}_1 \oplus \mathcal{F}_2} = [A_1]_{\mathcal{F}_1 \oplus \mathcal{F}_2} \cup [A_2]_{\mathcal{F}_1 \oplus \mathcal{F}_2},$$

we obtain that either $x \in [A_1]_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ and $x \notin [A_2]_{\mathcal{F}_1 \oplus \mathcal{F}_2}$, or $x \in [A_2]_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ and $x \notin [A_1]_{\mathcal{F}_1 \oplus \mathcal{F}_2}$. However, note then that in the first case we have $x \notin \overline{N'_{\mathcal{F}_2}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}}$, as $[A_1]_{\mathcal{F}_1 \oplus \mathcal{F}_2} \cap N'_{\mathcal{F}_2} = \emptyset$, and in the second case it holds $x \notin \overline{N'_{\mathcal{F}_1}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}}$, since $[A_2]_{\mathcal{F}_1 \oplus \mathcal{F}_2} \cap N'_{\mathcal{F}_1} = \emptyset$. In either case, we get a contradiction, which proves that $(*)$ indeed holds.

Second, we trivially have

$$S_{\mathcal{F}_1 \oplus \mathcal{F}_2} = \overline{N_{\mathcal{F}_1 \oplus \mathcal{F}_2}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}} = \overline{N'_{\mathcal{F}_1} \cup N'_{\mathcal{F}_2}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}} = \overline{N'_{\mathcal{F}_1}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}} \cup \overline{N'_{\mathcal{F}_2}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}}. \tag{$**$}$$

Finally, for every $i \in \{1, 2\}$ we have

$$\overline{N'_{\mathcal{F}_i}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}} = \beta\big(N'_{\mathcal{F}_i}\big). \tag{$***$}$$

Indeed, fix $i \neq j \in \{1, 2\}$ and note that every $f \in C_b\big(N'_{\mathcal{F}_i}\big)$ extends to the function $f' \in C_b\big(N_{\mathcal{F}_1 \oplus \mathcal{F}_2}\big)$ defined for every $n \in \Omega_j$ by $f'(n) = f\big(p_{\mathcal{F}_1 \oplus \mathcal{F}_2}\big)$. Since the space $N_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ is C^*-embedded in $S_{\mathcal{F}_1 \oplus \mathcal{F}_2}$, f' extends to some $f'' \in C\big(S_{\mathcal{F}_1 \oplus \mathcal{F}_2}\big)$. Consequently, $N'_{\mathcal{F}_i}$ is C^*-embedded in $S_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ and therefore, by [39, Section 6.9], we have $\overline{N'_{\mathcal{F}_i}}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}} = \beta\big(N'_{\mathcal{F}_i}\big)$.

Now, combining $(*)$, $(**)$, and $(***)$, we get the natural identification of the space $S_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ with the quotient space

$$\Big(\beta\big(N'_{\mathcal{F}_1}\big) \sqcup \beta\big(N'_{\mathcal{F}_2}\big)\Big) / \big\{p^1_{\mathcal{F}_1 \oplus \mathcal{F}_2}, p^2_{\mathcal{F}_1 \oplus \mathcal{F}_2}\big\},$$

where for $i = 1, 2$ the point $p^i_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ is the copy of the point $p_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ in the subspace $N'_{\mathcal{F}_i}$ of the disjoint union $\beta\big(N'_{\mathcal{F}_1}\big) \sqcup \beta\big(N'_{\mathcal{F}_2}\big)$. As for each $i = 1, 2$, using the homeomorphism h_i, we have the identification $\beta\big(N'_{\mathcal{F}_i}\big) = \beta\big(N_{\mathcal{F}_i}\big) = S_{\mathcal{F}_i}$ such that $p^i_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ is identified with $p_{\mathcal{F}_i}$, we get the required natural identification of the spaces $\big(S_{\mathcal{F}_1} \sqcup S_{\mathcal{F}_2}\big)/\big\{p_{\mathcal{F}_1}, p_{\mathcal{F}_2}\big\}$ and $S_{\mathcal{F}_1 \oplus \mathcal{F}_2}$. □

Theorem 4.4.29. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filters on ω. Then, the algebra $\mathcal{A}_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ has the Nikodym property if and only if both the algebras $\mathcal{A}_{\mathcal{F}_1}$ and $\mathcal{A}_{\mathcal{F}_2}$ have the Nikodym property.*

Proof. We will use the identification described in Lemma 4.4.28.

Assume first that both $\mathcal{A}_{\mathcal{F}_1}$ and $\mathcal{A}_{\mathcal{F}_2}$ have the Nikodym property, but $\mathcal{A}_{\mathcal{F}_1 \oplus \mathcal{F}_2}$ does not. Let $\langle \mu_n\colon\ n \in \omega \rangle$ be a sequence of non-negative measures on $St\big(\mathcal{A}_{\mathcal{F}_1 \oplus \mathcal{F}_2}\big)$ as in Corollary 4.4.6. By condition (2), there exists $i \in \{1, 2\}$ for which we have $\sup_{n \in \omega} \mu_n\big(S_{\mathcal{F}_i}\big) = \infty$. Let $j \in \{1, 2\} \setminus \{i\}$. For every $n \in \omega$, define the non-negative measure ν_n on $S_{\mathcal{F}_i}$ simply by $\nu_n = \mu_n \restriction S_{\mathcal{F}_i}$.

We claim that the sequence $\langle \nu_n\colon\ n \in \omega \rangle$ of measures on $S_{\mathcal{F}_i}$ too satisfies conditions (1)–(3) of Corollary 4.4.6. Indeed, (2) holds trivially. By condition (1) for $\langle \mu_n\colon\ n \in \omega \rangle$, for every $n \in \omega$ we have $p_{\mathcal{F}_1 \oplus \mathcal{F}_2} \notin \operatorname{supp}\big(\mu_n\big)$, so $p_{\mathcal{F}_i} \notin \operatorname{supp}\big(\nu_n\big)$, hence condition (1) is satisfied by $\langle \nu_n\colon\ n \in \omega \rangle$ as well. By condition (3) for $\langle \mu_n\colon\ n \in \omega \rangle$, for every $A \in \mathcal{F}_i$ we have

$$\lim_{n \to \infty} \nu_n\big([\omega \setminus A]_{\mathcal{F}_i}\big) = \lim_{n \to \infty} \nu_n\Big(\overline{\omega \setminus A}^{S_{\mathcal{F}_i}}\Big) = \lim_{n \to \infty} \mu_n\Big(\overline{\Omega_i \setminus b_i[A]}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}}\Big) =$$

$$= \lim_{n \to \infty} \mu_n\Big(\overline{\Omega_i \setminus b_i[A]}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}} \cup \overline{\Omega_j \setminus \Omega_j}^{S_{\mathcal{F}_1 \oplus \mathcal{F}_2}}\Big) =$$

$$= \lim_{n\to\infty} \mu_n\Big(\overline{\omega \setminus \big(b_i[A] \cup \Omega_j\big)}^{S_{\mathcal{F}_1\oplus\mathcal{F}_2}}\Big) = \lim_{n\to\infty} \mu_n\Big(\big[\omega \setminus \big(b_i[A] \cup \Omega_j\big)\big]_{\mathcal{F}_1\oplus\mathcal{F}_2}\Big) = 0;$$

it follows that condition (3) also holds for $\langle \nu_n \colon\ n \in \omega\rangle$. Consequently, $\mathcal{A}_{\mathcal{F}_i}$ does not have the Nikodym property, a contradiction.

Assume now that $\mathcal{A}_{\mathcal{F}_1\oplus\mathcal{F}_2}$ has the Nikodym property, but $\mathcal{A}_{\mathcal{F}_i}$ for some $i \in \{1,2\}$ does not, so that there exists a sequence $\langle \mu_n \colon\ n \in \omega\rangle$ of non-negative measures on $St(\mathcal{A}_{\mathcal{F}_i})$ as in Corollary 4.4.6. It is however immediate that $\langle \mu_n \colon\ n \in \omega\rangle$ is also a sequence on $St(\mathcal{A}_{\mathcal{F}_1\oplus\mathcal{F}_2})$ as in Corollary 4.4.6 and so $\mathcal{A}_{\mathcal{F}_1\oplus\mathcal{F}_2}$ does not have the Nikodym property either, which is a contradiction. □

We are ready to construct the second aforementioned family $\mathcal{H}_2$ of Boolean algebras.

Theorem 4.4.30. *There is a family $\{\mathcal{G}_\alpha \colon \alpha < 2^{\mathfrak{c}}\}$ of non-analytic filters on ω such that the family $\mathcal{H}_2 = \{\mathcal{A}_{\mathcal{G}_\alpha} \colon \alpha < 2^{\mathfrak{c}}\}$ consists of pairwise non-isomorphic Boolean algebras, each of them having the Nikodym property but not the Grothendieck property.*

Proof. Recall that every ultrafilter on ω is non-analytic (by [11, Theorem 4.1.1] and the fact that analytic sets have the Baire property). For each ultrafilter $\mathcal{U}$ on ω let $\mathcal{F}_\mathcal{U} = \mathcal{U} \oplus \mathcal{Z}^*$. Then, for every $\mathcal{U}$ the space $N_{\mathcal{F}_\mathcal{U}}$ has the bounded Josefson–Nissenzweig property by [76, Theorem C and Proposition 5.17], and so the algebra $\mathcal{A}_{\mathcal{F}_\mathcal{U}}$ does not have the Grothendieck property by [76, Corollary D]. On the other hand, by Theorem 4.4.29 each $\mathcal{A}_{\mathcal{F}_\mathcal{U}}$ has the Nikodym property, since both $\mathcal{A}_\mathcal{U} = \mathcal{P}(\omega)$ and $\mathcal{A}_{\mathcal{Z}^*}$ have the Nikodym property (by the Nikodym–Andô theorem and by Theorem 4.4.20, respectively).

Finally, since there are $2^{\mathfrak{c}}$ ultrafilters on ω and each family of pairwise isomorphic filters has the cardinality at most $\mathfrak{c}$, there is a family $\{\mathcal{G}_\alpha \colon \alpha < 2^{\mathfrak{c}}\}$ of pairwise non-isomorphic filters of the form $\mathcal{F}_\mathcal{U}$. Set $\mathcal{H}_2 = \{\mathcal{A}_{\mathcal{G}_\alpha} \colon \alpha < 2^{\mathfrak{c}}\}$. By Lemma 4.4.25, the spaces $S_{\mathcal{G}_\alpha}$, $\alpha < 2^{\mathfrak{c}}$, are pairwise non-homeomorphic, and so the algebras $\mathcal{A}_{\mathcal{G}_\alpha}$, $\alpha < 2^{\mathfrak{c}}$, are pairwise non-isomorphic. □

4.5 Boundedness, non-atomicity, and hypergraph ideals

In this section we study the relations between the Nikodym property of ideals on ω and miscellaneous additional notions of boundedness and non-atomicity of ideals. In connection with this, we also expand and investigate the construction of so-called hypergraph ideals.

4.5.1 Various boundedness properties of ideals on ω

We start with the following alike properties introduced by Drewnowski, Florencio, and Paúl ([23, Section 2]).

Definition 4.5.1 ([23])**.** An ideal $\mathcal{I}$ on ω has the *Local-to-Global Boudnedness Property for submeasures* (in short, *(LGBPs)*) if whenever φ is a submeasure on ω for which $\mathcal{I} \subseteq \mathrm{Fin}(\varphi)$, then $\sup_{A\in\mathcal{I}} \varphi(A) < \infty$.

Definition 4.5.2 ([23])**.** An ideal $\mathcal{I}$ on ω has the *Local-to-Global Boudnedness Property for measures* (in short, *(LGBPm)*) if whenever $\varphi \colon \mathcal{I} \to \mathbb{R}$ is a finitely additive set function which is bounded on $\mathcal{I}_A$ (i.e. $\sup_{B\subseteq A} |\varphi(B)| < \infty$) for every $A \in \mathcal{I}$, then φ is a measure on $\mathcal{I}$ (i.e. $\|\varphi\| < \infty$).

Intuitively, the Local-to-Global Boundedness Property for submeasures may be considered as the Nikodym property for a single submeasure. It was shown in [23, Proposition 2.1] that an ideal $\mathcal{I}$ on ω has the Nikodym property if and only if it has (LGBPm), and in [23, Proposition 2.2] that if $\mathcal{I}$ has (LGBPs), then it has the Nikodym property. However, it was conjectured in [23, Conjecture, page 147] that the Nikodym property does not imply (LGBPs). Theorem 4.5.4 shows that this conjecture indeed holds true (see also Remark 4.5.28). First, we have the following consequence of the Local-to-Global Boundedness Property for submeasures.

Proposition 4.5.3. *If an ideal $\mathcal{I}$ on ω has (LGBPs), then $\mathcal{I}$ cannot be extended to an $\mathbb{F}_\sigma$-ideal.*

Proof. Let $\mathcal{I}$ be an ideal on ω with (LGBPs). By Theorem 2.6.4, it is enough to prove that there is no lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ and $\mathcal{I} \subseteq \mathrm{Fin}(\varphi)$. Thus, let φ be an lsc submeasure on ω with $\varphi(\omega) = \infty$. By lsc, we have $\sup_{A\in\mathrm{Fin}} \varphi(A) = \infty$, and so $\sup_{A\in\mathcal{I}} \varphi(A) = \infty$. Therefore, by (LGBPs), we cannot have $\mathcal{I} \subseteq \mathrm{Fin}(\varphi)$. □

Theorem 4.5.4. *There exists an ideal $\mathcal{I}$ on ω with the Nikodym property but without the Local-to-Global Boundedness Property for submeasures.*

Proof. Let $\mathcal{I}$ be either of the two examples considered in Remark 3.2.11. $\mathcal{I}$ is then of the form $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ for some lsc submeasure φ on ω and the space $N_{\mathcal{I}*}$ has the finitely supported Nikodym property. Since $\mathcal{I}$ is an $\mathbb{F}_\sigma$ ideal, by Proposition 4.5.3 it does not have (LGBPs). As $\mathcal{I}$ is a P-ideal, Theorem 4.4.15 implies that $\mathcal{I}$ has the Nikodym property. □

Since $\mathrm{Exh}(\varphi) \subseteq \mathrm{Fin}(\varphi)$ for any lsc submeasure φ on ω, it follows that every analytic P-ideal $\mathcal{I}$ which is not totally bounded is contained in an $\mathbb{F}_\sigma$ ideal. We will show that these properties are in fact equivalent for analytic P-ideals. For this, we first prove a strengthening of Sakai [90, Lemma 2.6] (with a similar proof).

Lemma 4.5.5. *For every lsc submeasure φ on ω such that $\varphi(\omega) = \infty$ there exists an lsc submeasure ψ on ω satisfying $\psi(\omega) = \infty$ and $\mathrm{Fin}(\varphi) \subseteq \mathrm{Exh}(\psi)$.*

Proof. Let φ be an lsc submeasure on ω such that $\varphi(\omega) = \infty$. By induction on $n \in \omega$, we can define a sequence $\langle k_n\colon\ n \in \omega\rangle \subseteq \omega$ such that $k_0 = 0$ and $\varphi\big(k_{n+1} \setminus k_n\big) \geq n^2$ for each $n \in \omega$. For each $n \in \omega$ set $X_n = k_{n+1} \setminus k_n$ and let ψ_n be a submeasure on ω defined by $\psi_n(A) = \varphi(A)/n$ for every $A \subseteq \omega$. Then, using Lemma 2.6.3, we can define an lsc submeasure ψ on ω by setting

$$\psi(A) = \sup_{n\in\omega} \psi_n\big(A \cap X_n\big)$$

for each $A \subseteq \omega$. We have $\psi(\omega) = \infty$, since $\psi_n(X_n) \geq n$ for every $n \in \omega$.

We will show that $\mathrm{Fin}(\varphi) \subseteq \mathrm{Exh}(\psi)$. Let $A \in \mathrm{Fin}(\varphi)$. Then, for each $n \in \omega$ we have

$$\varphi\big(A \cap X_n\big) \leq \varphi(A) < \infty,$$

so $\psi_n\big(A \cap X_n\big) \leq \varphi(A)/n$. Thus, $\lim_{n\to\infty} \psi(A \setminus n) = 0$, that is, $A \in \mathrm{Exh}(\psi)$. □

Theorem 4.5.6. *An analytic P-ideal on ω is totally bounded if it is not contained in an $\mathbb{F}_\sigma$ ideal.*

Proof. Let $\mathcal{I}$ be an analytic P-ideal. We only have to prove that if $\mathcal{I}$ is contained in an $\mathbb{F}_\sigma$ ideal, then it is not totally bounded. Assume that $\mathcal{I} \subseteq \mathrm{Fin}(\varphi)$ for some lsc submeasure φ on ω such that $\varphi(\omega) = \infty$. By Lemma 4.5.5, there is an lsc submeasure ψ on ω satisfying $\psi(\omega) = \infty$ and $\mathrm{Fin}(\varphi) \subseteq \mathrm{Exh}(\psi)$. Therefore, $\mathcal{I} \subseteq \mathrm{Exh}(\psi)$, and so $\mathcal{I}$ is not totally bounded, by Proposition 3.2.9. □

Remark 4.5.7. By the proof of Lemma 4.5.5, if we have a non-pathological lsc φ such that $\varphi(\omega) = \infty$, then there is a non-pathological lsc ψ satisfying $\psi(\omega) = \infty$ and $\mathrm{Fin}(\varphi) \subseteq \mathrm{Exh}(\psi)$. Therefore, by Theorem 3.2.2, we have in this case $\mathcal{I} = \mathrm{Fin}(\varphi) \in \mathcal{AN}$.

The following corollary collects the most important conditions equivalent to the total boundedness. Note that most equivalences were already proved by Filipów *et al.* in [6] and [30].

Corollary 4.5.8. *For analytic P-ideal $\mathcal{I}$, the following are equivalent:*

1. *$\mathcal{I}$ is totally bounded,*
2. *$\mathcal{I}$ cannot be extended to an $\mathbb{F}_\sigma$-ideal,*
3. *$\mathcal{I}$ does not have the Bolzano–Weierstrass property,*
4. *The Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$ contains a countable splitting family,*
5. *$\mathrm{conv} \leq_K \mathcal{I}$.*

Proof. Equivalences (1)⇔(2) follow from Theorem 4.5.6. Equivalences (2)⇔(3)⇔(4) follow from [30, Theorems 4.2 and 5.1]. Finally, equivalence (2)⇔(5) is given by [6, Proposition 6.5]. □

Remark 4.5.9. As the ideal $\mathrm{tr}(\mathcal{N})$ is totally bounded, Corollary 4.5.8 immediately implies the (known) inequality $\mathrm{conv} \leq_K \mathrm{tr}(\mathcal{N})$.

In Corollary 4.5.15 we provide further "non-atomic" conditions for an ideal $\mathrm{Exh}(\varphi)$ to be totally bounded. Additional conditions equivalent to the Bolzano–Weierstrass property (and so to the lack of the total boundedness) of analytic P-ideals were also given e.g. in Filipów *et al.* in [31] and [35, Section 4] (in terms of colorings of ω and $[\omega]^2$), and in [32] (in terms of convergence of functions).

Moreover, Filipów *et al.* [30, Theorem 4.3] proved that under the assumption of Continuum Hypothesis an analytic P-ideal $\mathcal{I}$ has the Bolzano–Weierstrass property if and only if $\mathcal{I}$ can be extended to a maximal P-ideal. Combining this with Corollary 4.5.8, we get yet another (consistent) characterization of totally bounded ideals.

Corollary 4.5.10. *Assume the Continuum Hypothesis. Let $\mathcal{I}$ be an analytic P-ideal. Then, $\mathcal{I}$ is totally bounded if and only if $\mathcal{I}$ cannot be extended to a maximal P-ideal.*

Note that Corollary 4.5.10 requires some set-theoretic assumption, as there are models of ZFC without any maximal P-ideals (that is, equivalently, without P-points on ω), see e.g. Wimmers [111].

4.5.2 Non-atomic ideals and the Strong Nested Partition Property

Let us recall the following standard definition of strongly non-atomic submeasures and its counterpart for ideals.

Definition 4.5.11. A submeasure φ on ω is said to be *strongly non-atomic* if for every $\varepsilon > 0$ there exists a finite partition $A_1, \ldots, A_k$ of ω such that $\varphi(A_i) < \varepsilon$ for each $i = 1, \ldots, k$.

Given two partitions $\mathcal{P}$ and $\mathcal{P}'$ of ω, we write $\mathcal{P}' \preceq \mathcal{P}$ if for any $A \in \mathcal{P}$ and $B \in \mathcal{P}'$ we either have $A \cap B = \emptyset$ or $B \subseteq A$.

Definition 4.5.12. An ideal $\mathcal{I}$ on ω is *non-atomic* if there exists a $\preceq$-decreasing sequence $\langle \mathcal{P}_n \colon\ n \in \omega \rangle$ of finite partitions of ω such that, for every $\subseteq$-decreasing sequence $\langle A_n \in \mathcal{P}_n \colon\ n \in \omega \rangle$ and every set $E \subseteq \omega$ such that $E \setminus A_n$ is finite for every $n \in \omega$, we have $E \in \mathcal{I}$.

For basic information concerning non-atomic submeasures and ideals, see [5], [24], [30], [34]. In particular, note that for a strongly non-atomic submeasure φ on ω the ideal $\mathrm{Exh}(\varphi)$ is also non-atomic, but the converse may not hold—it holds provided that $\varphi = \psi^\bullet$ for some lsc submeasure ψ on ω (see Drewnowski and Łuczak [24, Section 3] for details).

The relations between the Nikodym property and non-atomic submeasures and ideals were first studied by Drewnowski, Florencio and Paúl [23], where it was proved, among others, that for a strongly non-atomic submeasure φ on ω the ideal $\mathcal{Z}(\varphi) = \{A \subseteq \omega \colon \varphi(A) = 0\}$ has the Nikodym property (so in particular the asymptotic density zero ideal $\mathcal{Z}$ has the Nikodym property, cf. [22, Proposition 6]).

The following non-atomicity property of ideals, shortly called (NPP), was introduced by Stuart [105] (for rings of sets). We also introduce its strong variant, because it is somewhat easier to handle than (NPP) and, as shown in Proposition 4.5.14, it unifies both (NPP) and non-atomicity of ideals.

Definition 4.5.13. An ideal $\mathcal{I}$ on ω has the *Nested Partition Property* (in short, *(NPP)*), if there exists a $\preceq$-decreasing sequence $\langle \mathcal{P}_n \colon\ n \in \omega \rangle$ of finite partitions of ω such that for every sequence $\langle E_n \colon\ n \in \omega \rangle$ of pairwise disjoint elements of $\mathcal{I}$ there is an infinite set $M \in \omega$ such that for every $\subseteq$-decreasing sequence $\langle A_n \in \mathcal{P}_n \colon\ n \in \omega \rangle$ we have

$$\bigcup_{n \in M} \left(E_n \cap A_n \right) \in \mathcal{I}.$$

If the set M can be always chosen to be ω, then we say that $\mathcal{I}$ has the *Strong Nested Partition Property* (in short, *(SNPP)*).

Of course, if an ideal has (SNPP), then it has (NPP).

Proposition 4.5.14. *If an ideal $\mathcal{I}$ on ω has (SNPP), then $\mathcal{I}$ is non-atomic.*

Proof. Let $\langle \mathcal{P}_n \colon\ n \in \omega \rangle$ be the $\preceq$-decreasing sequence of finite partitions of ω witnessing (SNPP) of $\mathcal{I}$. Assume that $\langle A_n \in \mathcal{P}_n \colon\ n \in \omega \rangle$ is a $\subseteq$-decreasing sequence and that $E \subseteq \omega$ is such that $E \setminus A_n$ is finite for every $n \in \omega$. Moreover, without loss of generality we may assume that $\mathcal{P}_0 = \{\omega\}$ and $A_0 = \omega$. Let $\kappa = \left| \bigcap_{n \in \omega} A_n \right|$ and enumerate $\bigcap_{n \in \omega} A_n = \left\{ x_m \colon m < \kappa \right\}$. For each $m \in \omega$ we define

$$E_m = \Big(\left\{ x_m \right\} \cup \left(A_m \setminus A_{m+1} \right) \Big) \cap E$$

if $m < \kappa$, and

$$E_m = (A_m \setminus A_{m+1}) \cap E$$

if $\kappa \leq m < \omega$. Then, the sets E_m's are pairwise disjoint, and we have $E = \bigcup_{m\in\omega} E_m$, since

$$\bigcup_{m<\kappa} \Big(\{x_m\} \cup (A_m \setminus A_{m+1})\Big) \cup \bigcup_{\kappa\leq m<\omega} (A_m \setminus A_{m+1}) = A_0 = \omega.$$

Furthermore, for each $m \in \omega$ the set E_m is finite, since $E_m \subseteq (E \setminus A_{m+1}) \cup \{x_m\}$, thus in particular $E_m \in \mathcal{I}$. For every $m \in \omega$ we also have $E_m \subseteq A_m$, so $E_m \cap A_m = E_m$. Consequently, $E = \bigcup_{m\in\omega} E_m \in \mathcal{I}$, as required. □

The next corollary, being a supplement of Corollary 4.5.8, shows that the converse to Proposition 4.5.14 also holds, provided that a considered ideal is of the form $\mathrm{Exh}(\varphi)$ for some lsc submeasure φ on ω.

Corollary 4.5.15. *For every lsc submeasure φ on ω and the ideal $\mathcal{I} = \mathrm{Exh}(\varphi)$, conditions (1)–(5) of Corollary 4.5.8 are equivalent to the following ones:*

6. *$\varphi^\bullet$ is strongly non-atomic,*
7. *$\mathcal{I}$ is non-atomic,*
8. *$\mathcal{I}$ has (SNPP).*

Proof. Equivalence (3)⇔(6) follows from [30, Theorem 3.6]. Implication (8)⇒(7) follows from Proposition 4.5.14. Implication (7)⇒(6) is given by [24, Corollary 3.2]. For implication (6)⇒(8) see [105, proof of Proposition 3]. □

Remark 4.5.16. Recall that an infinite set $\{a_0 < a_1 < a_2 < ...\} \subseteq \omega$ is *lacunary* if $\lim_{n\to\infty} a_{n+1} - a_n = \infty$. Let $\mathcal{L}$ denote the ideal generated by all lacunary subsets of ω. By [94, Proposition 2] (see also [66]) $\mathcal{L}$ is not contained in any summable ideal, so $\mathcal{L} \notin \mathcal{AN}$, but by [25, Example 4.2] it does not have the Nikodym property.

Drewnowski and Łuczak [24, Theorem 4.1] proved that if an lsc submeasure φ on ω is such that $\mathcal{L} \subseteq \mathrm{Exh}(\varphi)$, then the core $\varphi^\bullet$ is strongly non-atomic. Since $\mathrm{conv} \leq_K \mathcal{L}$ (by [66, Lemma 3.2] and [79, page 57]), the combination of Corollaries 4.5.8 and 4.5.15 provides a generalization of the latter fact: if an lsc submeasure φ on ω is such that $\mathrm{conv} \leq_K \mathrm{Exh}(\varphi)$, then $\varphi^\bullet$ is strongly non-atomic.

It was proved in [105] that every ideal on ω with (NPP) has the Nikodym property, the converse however does not hold—in the next section we provide an example of an ideal which has the Nikodym property but not (NPP), see Corollary 4.5.31. Below we show that an ideal with (SNPP) cannot be contained in an $\mathbb{F}_\sigma$ ideal.

Proposition 4.5.17. *If φ is an lsc submeasure on ω such that $\varphi(\omega) = \infty$, then for every $\preceq$-decreasing sequence $\langle \mathcal{P}_n\colon\ n \in \omega\rangle$ of finite partitions of ω there exists a $\subseteq$-decreasing sequence $\langle A_n \in \mathcal{P}_n\colon\ n \in \omega\rangle$ and a sequence $\langle F_n\colon\ n \in \omega\rangle$ of pairwise disjoint finite subsets of ω such that $F_n \subseteq A_n$ and $\varphi(F_n) \geq n + 1$ for every $n \in \omega$.*

Proof. Fix a lsc submeasure φ on ω such that $\varphi(\omega) = \infty$. Let $\langle \mathcal{P}_n\colon\ n \in \omega\rangle$ be a $\preceq$-decreasing sequence of finite partitions of ω. By the subadditivity of φ, there exists $A_0 \in \mathcal{P}_0$ such that $\varphi(A_0) = \infty$. As φ is lsc, we can choose a finite set $F_0 \subseteq A_0$ with $\varphi(F_0) \geq 1$. Now, since $\varphi(A_0 \setminus F_0) = \infty$, there exists $A_1 \in \mathcal{P}_1$ such that $A_1 \subseteq A_0$ and $\varphi(A_1 \setminus F_0) = \infty$.

Again, choose a finite set $F_1 \subseteq \big(A_1 \setminus F_0\big)$ with $\varphi\big(F_1\big) \geq 2$. Since $\varphi\big(A_1 \setminus \big(F_0 \cup F_1\big)\big) = \infty$, there exists $A_2 \in \mathcal{P}_2$ such that $A_2 \subseteq A_1$ and $\varphi\big(A_2 \setminus \big(F_0 \cup F_1\big)\big) = \infty$. And so on.

By repeating the above argument, we get a $\subseteq$-decreasing sequence $\langle A_n \in \mathcal{P}_n\colon\ n \in \omega\rangle$ and a sequence $\langle F_n\colon\ n \in \omega\rangle$ of pairwise disjoint finite subsets of ω such that $F_n \subseteq A_n$ and $\varphi\big(F_n\big) \geq n+1$ for every $n \in \omega$. $\square$

Proposition 4.5.18. *If $\mathcal{I}$ is a non-atomic ideal on ω, then it is not contained in any $\mathbb{F}_\sigma$ ideal.*

Proof. Assume that for an ideal $\mathcal{I}$ on ω we have $\mathcal{I} \subseteq \operatorname{Fin}(\varphi)$ for some lsc submeasure φ on ω such that $\varphi(\omega) = \infty$. To show that $\mathcal{I}$ is not non-atomic, let $\langle \mathcal{P}_n\colon\ n \in \omega\rangle$ be a $\preceq$-decreasing sequence of finite partitions of ω. Let $\langle A_n \in \mathcal{P}_n\colon\ n \in \omega\rangle$ and $\langle F_n\colon\ n \in \omega\rangle$ be sequences as in Proposition 4.5.17. For $E = \bigcup_{n\in\omega} F_n$, we have $\varphi(E) = \infty$, so $E \notin \mathcal{I}$, even though $E \setminus A_n$ is finite for every $n \in \omega$, which shows that $\mathcal{I}$ is not non-atomic. $\square$

Combining Propositions 4.5.14 and 4.5.18, we get the following aforementioned result.

Corollary 4.5.19. *If $\mathcal{I}$ is an ideal on ω with (SNPP), then it is not contained in any $\mathbb{F}_\sigma$ ideal.*

Let us note that using a similar method one can get the following variation of Proposition 4.5.17 (cf. [52, Lemma 3.17]); we use here the convention that, for a set $A \subseteq \omega$, we have $A^1 = A$ and $A^{-1} = \omega \setminus A$.

Proposition 4.5.20. *If φ is an lsc submeasure on ω such that $\varphi(\omega) = \infty$, then for every sequence $\langle A_n\colon\ n \in \omega\rangle$ of subsets of ω there exist sequences $\langle \varepsilon_n\colon\ n \in \omega\rangle \in \{-1,1\}^\omega$ and $\langle F_n\colon\ n \in \omega\rangle \subseteq [\omega]^{<\omega}$ with the following properties:*

(a) $F_n \cap F_m = \emptyset$ for every $n \neq m \in \omega$,

(b) $F_n \subseteq \bigcap_{k=0}^n A_n^{\varepsilon_n}$ for every $n \in \omega$,

(c) $\varphi\big(F_n\big) \geq n$ for every $n \in \omega$.

4.5.3 Hypergraph ideals

In this subsection we construct a family of $\mathfrak{c}$ many pairwise non-isomorphic non-pathological ideals on ω which have the Nikodym property but are not totally bounded or, even stronger, do not have (NPP). In order to do this, we exploit and expand the construction due to Alon, Drewnowski, and Łuczak [2] of an ideal which has the Nikodym property but is not non-atomic. This construction relies on the notion of a hypergraph ideal, presented below.

Recall that a *hypergraph* is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set and $E(H) \subseteq \mathcal{P}(V(H))$.

Definition 4.5.21. A submeasure φ on ω is *hypergraph* if there exists a sequence $\langle H_n\colon\ n \in \omega\rangle$ of hypergraphs, a sequence $\langle G_n\colon\ n \in \omega\rangle$ of finite non-empty disjoint subsets of ω, and for each $n \in \omega$ a bijection $b_n\colon G_n \to V\big(H_n\big)$ such that

$$\varphi(A) = \sup_{n\in\omega} \sup_{e\in E(H_n)} \frac{\big|b_n\big[A \cap G_n\big] \cap e\big|}{|e|}.$$

Recall that this formula defines an lsc submeasure by Lemma 2.6.3. In this case, we will say that φ is *induced* by the sequence $\langle H_n\colon\ n \in \omega\rangle$ of hypergraphs.

An ideal $\mathcal{I}$ on ω is *hypergraph* if there exists a hypergraph submeasure φ on ω such that $\mathcal{I} = \operatorname{Exh}(\varphi)$.

Remark 4.5.22. Note that every hypergraph ideal is an non-pathological analytic P-ideal, as a hypergraph submeasure is a supremum of a family of measures on ω. On the other hand, every density ideal induced by a sequence of uniform measures (i.e. measures with the same value on every singleton in the support) is hypergraph.

Let H be a hypergraph. Recall that the *chromatic number* $\chi(H)$ of H is the minimal number $n \in \omega$ for which there exists a partition $C_1, \ldots, C_n$ of $V(H)$ such that no $e \in E(H)$ is contained in some C_i. For a subset $W \subseteq V(H)$ we set $H \restriction W = (W, E(H) \cap \mathcal{P}(W))$. We will need the following lemma.

Lemma 4.5.23. *Let $\langle H_n\colon\ n \in \omega\rangle$ be a sequence of hypergraphs such that $\lim_{n\to\infty} \chi\big(H_n\big) = \infty$. Let $\langle G_n\colon\ n \in \omega\rangle$ be a sequence of finite non-empty disjoint subsets of ω and for each $n \in \omega$ let $b_n\colon G_n \to V\big(H_n\big)$ be a bijection. Let $A_1, \ldots, A_k$ be a finite partition of ω into infinite sets. Then, there are $1 \leq j_0 \leq k$ and $I \in [\omega]^\omega$ such that*

$$\lim_{\substack{n\to\infty\\ n\in I}} \chi\Big(H_n \restriction b_n\big[G_n \cap A_{j_0}\big]\Big) = \infty.$$

Proof. Assume for the sake of contradiction that for each $1 \leq j \leq k$ there is $K_j \in \omega$ such that

$$\chi\Big(H_n \restriction b_n\big[G_n \cap A_j\big]\Big) \leq K_j$$

for every $n \in \omega$. Then, for every $n \in \omega$ we have

$$\chi\big(H_n\big) \leq \sum_{j=1}^{k} \chi\Big(H_n \restriction b_n\big[G_n \cap A_j\big]\Big) \leq \sum_{j=1}^{k} K_j,$$

where the second inequality follows from the fact that we can construct a partition of $V(H_n)$ witnessing $\chi\big(H_n\big)$ from partitions witnessing $\chi\Big(H_n \restriction b_n\big[G_n \cap A_j\big]\Big)$ for $j = 1, \ldots, k$. Therefore, we get

$$\sup_{n\in\omega} \chi\big(H_n\big) \leq \sum_{j=1}^{k} K_j < \infty,$$

which is impossible. $\square$

The idea behind the proof of the next proposition closely follows the first part of the proof of [2, Theorem 2.3], showing that for no submeasure φ on ω induced by a sequence $\langle H_n\colon\ n \in \omega\rangle$ of hypergraphs such that $\lim_{n\to\infty} \chi\big(H_n\big) = \infty$ the ideal $\mathrm{Exh}(\varphi)$ is non-atomic.

Proposition 4.5.24. *If a hypergraph submeasure φ on ω is induced by a sequence $\langle H_n\colon\ n \in \omega\rangle$ of hypergraphs such that $\lim_{n\to\infty} \chi\big(H_n\big) = \infty$, then the ideal $\mathcal{I} = \mathrm{Exh}(\varphi)$ does not have the Nested Partition Property.*

Proof. Let us fix sequences $\langle G_n\colon\ n \in \omega\rangle$ and $\langle b_n\colon\ n \in \omega\rangle$ as in Definition 4.5.21. Let also $\langle \mathcal{P}_n\colon\ n \in \omega\rangle$ be a $\preceq$-decreasing sequence of finite partitions of ω. We will show that $\langle \mathcal{P}_n\colon\ n \in \omega\rangle$ does not satisfy the requirements of Definition 4.5.13, which will prove that $\mathcal{I}$ does not have (NPP).

We start as follows. By Lemma 4.5.23 there are $A_0 \in \mathcal{P}_0$ and $I_0 \in [\omega]^\omega$ such that

$$\lim_{\substack{n\to\infty\\ n\in I_0}} \chi\Big(H_n \restriction b_n\big[G_n \cap A_0\big]\Big) = \infty.$$

Let $n_0 \in I_0$ be such that there exists $e_0 \in E\Big(H_{n_0} \restriction b_{n_0}\big[G_{n_0} \cap A_0\big]\Big)$ (such n_0 exists, since otherwise the above limit would be 1).

Again, by Lemma 4.5.23, there are $A_1 \in \mathcal{P}_1$, $A_1 \subseteq A_0$, and $I_1 \in [I_0 \setminus n_0]^\omega$ such that

$$\lim_{\substack{n\to\infty\\ n\in I_1}} \chi\Big(H_n \restriction b_n\big[G_n \cap A_1\big]\Big) = \infty.$$

Let $n_1 \in I_1$ be such that there exists $e_1 \in E\Big(H_{n_1} \restriction b_{n_1}\big[G_{n_1} \cap A_1\big]\Big)$.

Repeat the above argument until you get a $\subseteq$-decreasing sequence $\big\langle A_k \in \mathcal{P}_k\colon\ k \in \omega\big\rangle$, a strictly increasing sequence $\big\langle n_k \in \omega\colon\ k \in \omega\big\rangle$, and a sequence $\big\langle e_k \in E\big(H_{n_k}\big)\colon\ k \in \omega\big\rangle$ such that $e_k \in E\Big(H_{n_k} \restriction b_{n_k}\big[G_{n_k} \cap A_k\big]\Big)$ for each $k \in \omega$.

For each $k \in \omega$ put $B_k = b_{n_k}^{-1}\big[e_k\big]$; as B_k is a finite set, it belongs to $\mathcal{I}$. Since $\big\langle n_k \in \omega\colon\ k \in \omega\big\rangle$ is strictly increasing, the sets in the sequence $\big\langle B_k\colon\ k \in \omega\big\rangle$ are pairwise disjoint. By the assumption about e_k, we also have $b_{n_k}^{-1}\big[e_k\big] \subseteq A_k$ for any $k \in \omega$. For each $M \in [\omega]^\omega$ let

$$B_M = \bigcup_{k\in M} \big(B_k \cap A_k\big) = \bigcup_{k\in M} B_k.$$

We have

$$\varphi^\bullet\big(B_M\big) = \limsup_{n\to\infty} \sup_{e\in E(H_n)} \frac{\Big|b_n\big[B_M \cap G_n\big] \cap e\Big|}{|e|} \geq \limsup_{\substack{k\to\infty\\ k\in M}} \frac{\Big|b_{n_k}\big[B_k \cap G_{n_k}\big] \cap e_k\Big|}{\big|e_k\big|} =$$

$$= \limsup_{\substack{k\to\infty\\ k\in M}} \frac{\big|e_k \cap e_k\big|}{\big|e_k\big|} = 1,$$

so $B_M \notin \mathcal{I}$. $\square$

Corollary 4.5.25. *If a hypergraph submeasure φ on ω is induced by a sequence $\big\langle H_n\colon\ n \in \omega\big\rangle$ of hypergraphs such that $\lim_{n\to\infty} \chi\big(H_n\big) = \infty$, then the ideal $\mathcal{I} = \mathrm{Exh}(\varphi)$ is not totally bounded.*

Proof. Let φ be a hypergraph submeasure satisfying the premises. By Proposition 4.5.24 it follows that the ideal $\mathcal{I}$ does not have (NPP) and so, by Corollary 4.5.15, it is not totally bounded. $\square$

Theorem 4.5.26. *There is a family of $\mathfrak{c}$ many pairwise non-isomorphic hypergraph ideals which have the Nikodym property but are not totally bounded.*

Proof. Let $\big\langle H_n\colon\ n \in \omega\big\rangle$ be the sequence of Kneser hypergraphs defined by Alon, Drewnowski, and Łuczak in [2, Section 4], and let φ_{ADL} be the hypergraph submeasure on ω induced by the sequence $\big\langle H_n\colon\ n \in \omega\big\rangle$, with sequences $\big\langle G_n\colon\ n \in \omega\big\rangle$ and $\big\langle b_n\colon\ n \in \omega\big\rangle$ as in Definition 4.5.21. Set $\mathcal{I}_{ADL} = \mathrm{Exh}\left(\varphi_{ADL}\right)$. As for every $k \in \omega$ the edges of H_k are of size 2^k, we can recursively define a strictly increasing sequence $\big\langle n_k\colon\ k \in \omega\big\rangle$ such that

$$|e| > k \cdot \sum_{i<k} \big|G_{n_i}\big| \tag{$*$}$$

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holds for every $e \in E(H_{n_k})$ and $k \in \omega$. For any $M \in [\omega]^\omega$ let us set $\mathcal{I}_M = \mathrm{Exh}(\varphi_M)$, where φ_M is the hypegraph submeasure on ω defined for every $A \in \mathcal{P}(\omega)$ as

$$\varphi_M(A) = \sup_{k \in M} \sup_{e \in E(H_{n_k})} \frac{\left| b_{n_k}\left[A \cap G_{n_k}\right] \cap e \right|}{|e|}.$$

By [2, Corollary 3.4] we have $\chi(H_{n_k}) \geq 2^{n_k}$ for every $k \in \omega$, and so, by Corollary 4.5.25, the ideal $\mathcal{I}_M$ is not totally bounded for any $M \in [\omega]^\omega$. Next, $\mathcal{I}_{ADL}$ has the Nikodym property by [2, Theorem 2.3], and so the space $N_{\mathcal{I}^*_{ADL}}$ has the finitely supported Nikodym property by Theorem 4.4.15. Thus, the space $N_{\mathcal{I}^*_M}$ has the finitely supported Nikodym property for every $M \in [\omega]^\omega$, by Remark 3.1.6 and the fact that $\mathcal{I}_{ADL} \subseteq \mathcal{I}_M$. Therefore, $\mathcal{I}_M$ also has the Nikodym property for every $M \in [\omega]^\omega$, by Theorem 4.4.15.

Let $\{M_\alpha \colon \alpha < \mathfrak{c}\}$ be a family of infinite pairwise almost disjoint subsets of ω, that is, the intersection $M_\alpha \cap M_\beta$ is finite for every $\alpha \neq \beta < \mathfrak{c}$. We will show that the ideals $\{\mathcal{I}_{M_\alpha} \colon \alpha < \mathfrak{c}\}$ are pairwise non-isomorphic. Fix $\alpha \neq \beta < \mathfrak{c}$ and a bijection $\phi \colon \omega \to \omega$. Let $\langle m_i \colon i \in \omega\rangle$ be an increasing enumeration of $M_\alpha \setminus M_\beta$. Let us fix an arbitrary edge $e_k \in E(H_{n_k})$ for any $k \in \omega$. By $(*)$, for every $k \geq 2$ there exists $A_k \subseteq G_{n_k}$ such that $b_{n_k}[A_k] \subseteq e_k$, $|A_k| \geq |e_k|/2$, and

$$\phi[A_k] \cap \bigcup_{i<k} G_{n_i} = \emptyset. \tag{$**$}$$

Let $A = \bigcup_{i\in\omega} A_{m_i}$. We have $A \notin \mathcal{I}_{M_\alpha}$, as for every $i \geq 2$ we have

$$\varphi^\bullet_{M_\alpha}(A) \geq \frac{\left| b_{n_{m_i}}\left[A \cap G_{n_{m_i}}\right] \cap e_{m_i}\right|}{\left|e_{m_i}\right|} = \frac{\left| b_{n_{m_i}}\left[A_{m_i}\right] \cap e_{m_i}\right|}{\left|e_{m_i}\right|} \geq 1/2.$$

We will show that $\phi[A] \in \mathcal{I}_{M_\beta}$, which will conclude the proof. By $(**)$, for every $k \geq 2$ we have

$$\phi[A] \cap G_{n_k} = \bigcup_{i\in\omega} \left(\phi\left[A_{m_i}\right] \cap G_{n_k}\right) \subseteq \bigcup_{\substack{i\in\omega \\ m_i \leq k}} \phi\left[A_{m_i}\right].$$

Thus, for any $k \in M_\beta \setminus M_\alpha$ such that $k \geq 2$ we get

$$\phi[A] \cap G_{n_k} \subseteq \bigcup_{i<k} \phi\left[A_i\right].$$

By $(*)$, for every $k \in \omega$ and $e \in E(H_{n_k})$, we have

$$\frac{\sum_{i<k} |A_i|}{|e|} \leq \frac{\sum_{i<k} |G_{n_i}|}{|e|} < 1/k,$$

and so, for almost all $k \in M_\beta$, it holds

$$\sup_{e \in E(H_{n_k})} \frac{\left| b_{n_k}\left[\phi[A] \cap G_{n_k}\right] \cap e\right|}{|e|} < 1/k.$$

Consequently, we get that $\phi[A] \in \mathcal{I}_{M_\beta}$, as required. $\square$

As a corollary, we get a positive answer to the author's earlier [112, Question 9.1].

Corollary 4.5.27. *There exists a non-pathological lsc submeasure φ such that the ideal* $\mathrm{Exh}(\varphi)$ *is not totally bounded and the space $N_{\mathrm{Exh}(\varphi)^*}$ has the finitely supported Nikodym property.*

Proof. Let $M \in [\omega]^\omega$ be arbitrary and let $\varphi = \varphi_M$ be the corresponding submeasure defined in the proof of Theorem 4.5.26 (we can here of course consider also the submeasure φ_{ADL}). By Remark 4.5.22, φ is a non-pathological lsc submeasure. Moreover, the ideal $\mathcal{I}_M = \mathrm{Exh}(\varphi)$ is not totally bounded but has the Nikodym property. Consequently, the space $N_{\mathrm{Exh}(\varphi)^*}$ has the finitely supported Nikodym property by Theorem 4.4.15. □

Remark 4.5.28. For every $M \in [\omega]^\omega$ the ideal $\mathcal{I}_M$ from the proof of Theorem 4.5.26 has the Nikodym property and does not have the Local-to-Global Boundedness Property for submeasures, as it is an analytic P-ideal which is not totally bounded, so by Theorem 4.5.6 it is contained in an $\mathbb{F}_\sigma$ ideal. This provides yet another class of examples confirming the conjecture of Drewnowski, Florencio, and Paúl [23, Conjecture, page 147], cf. Theorem 4.5.4.

Remark 4.5.29. Let us also note that, for each $M \in [\omega]^\omega$, even though the core $\varphi_M^\bullet$ of the submeasure φ_M from the proof of Theorem 4.5.26 is not strongly non-atomic (otherwise, the ideal $\mathcal{I}_M$ would be totally bounded, by Corollary 4.5.15), it still presents some degree of non-atomicity. Namely, $\varphi_M^\bullet$ is *convexly non-atomic*, that is, for every set $A \in \mathcal{P}(\omega)$ and number $\alpha \in \left[0, \varphi_M^\bullet(A)\right]$ there is a set $B \subseteq A$ such that $\varphi_M^\bullet(B) = \alpha$, see [25, Proposition 5.2 and Theorem 6.2] for details (note here that every strongly non-atomic submeasure is automatically convexly non-atomic, see [25, page 494]).

Remark 4.5.30. Recall that each hypergraph ideal is a P-ideal, therefore if a hypergraph ideal $\mathcal{I}$ has the Nikodym property, then by Theorem 4.4.15 $\mathcal{I}$ has the web Nikodym property. In particular, all the ideals considered in (the proof of) Theorem 4.5.26 have the web Nikodym property.

Stuart [105, page 153] asked if every hereditary ring of sets with the Nikodym property has the Nested Partition Property. Theorem 4.5.26 (and its proof) yields a negative answer to this question, since for every $M \in [\omega]^\omega$ the ideal $\mathcal{I}_M$ has the Nikodym property, but it does not have (NPP) by Proposition 4.5.24; the same concerns the ideal $\mathcal{I}_{ADL}$.

Corollary 4.5.31. *There exists an ideal on ω with the Nikodym property but without (NPP).*

4.6 Open questions

In the final section of this chapter we state several problems. The first two concern Nikodym concentration points, studied in Section 4.1.

In Example 4.1.8, Proposition 4.2.1, and Corollary 4.2.5, using the class $\mathcal{AN}$ of ideals on ω, we described a large class of countable spaces X such that if x is a (unique) non-isolated point of X and X homeomorphically embeds into the Stone space $St(\mathcal{A})$ of a Boolean algebra $\mathcal{A}$, then the image of x in $St(\mathcal{A})$ is a strong Nikodym concentration point of some anti-Nikodym sequence of measures on $\mathcal{A}$. This description can be naturally translated into the language of filters of neighborhoods of points in Stone spaces in the following way: given a point y in the Stone space $St(\mathcal{A})$ of a Boolean algebra $\mathcal{A}$, if there is a countable discrete set $Y \subseteq St(\mathcal{A})$ such that $y \in \overline{Y} \setminus Y$ and the filter $\{Y \cap [A]_{\mathcal{A}} : A \in y\}$ on Y is isomorphic to some filter $\mathcal{F}$ on ω such that $\mathcal{F}^* \in \mathcal{AN}$, then y is a strong Nikodym concentration point of some anti-Nikodym sequence of measures on $\mathcal{A}$. We are interested in other properties of points in Stone spaces implying that they are strong Nikodym concentration points and thus we ask the following general question.

Question 4.6.1. *What properties of points in the Stone spaces $St(\mathcal{A})$ of Boolean algebras $\mathcal{A}$ imply that they are strong Nikodym concentration points of some anti-Nikodym sequences on $\mathcal{A}$?*

Note that the above issue seems to be closely connected with yet another general problem asking what topological properties of points in compact spaces imply that they (do not) belong to the supports of non-atomic measures, see Borodulin-Nadzieja and Ranchynski [15].

In Example 4.1.12 we mentioned several consistent constructions of Boolean algebras without the Nikodym property which is witnessed only by anti-Nikodym sequence with no strong Nikodym concentration points. Since we are not aware of any ZFC example of such an algebra, we pose the following question.

Question 4.6.2. *Does there exist in* ZFC *a Boolean algebra $\mathcal{A}$ without the Nikodym property and such that every anti-Nikodym sequence of measures on $\mathcal{A}$ has no strong Nikodym concentration points in $St(\mathcal{A})$?*

There exist $\mathbb{F}_\sigma$ P-filters $\mathcal{F}$ on ω for which the Boolean algebras $\mathcal{A}_\mathcal{F}$ have the Nikodym property, for example one can take filters constructed in [36, Theorem 4.12] or in [76, Example 6.15], and appeal to Theorem 4.4.15. Similarly, the filter $\mathcal{Z}^*$, dual to the asymptotic density ideal $\mathcal{Z}$, is an $\mathbb{F}_{\sigma\delta}$ P-filter also such that the algebra $\mathcal{A}_{\mathcal{Z}^*}$ has the Nikodym property. We do not know however of any constructions of Borel filters $\mathcal{F}$ on ω which do not belong to the Borel class $\mathbb{F}_{\sigma\delta}$ and for which the Boolean algebras $\mathcal{A}_\mathcal{F}$ have the Nikodym property.

Question 4.6.3. *Does there exist a non-$\mathbb{F}_{\sigma\delta}$ Borel filter $\mathcal{F}$ on ω for which the Boolean algebra $\mathcal{A}_\mathcal{F}$ has the Nikodym property?*

In Section 4.5.3 we studied the class of hypergraph ideals. In particular, in Theorem 4.5.26 we showed that there exists $\mathfrak{c}$ many pairwise non-isomorphic hypergraph ideals which have the Nikodym property but are not totally bounded. We then ask if the non-isomorphicity may be strengthened to the Katětov incomparability.

Question 4.6.4. *Does there exist a family of $\mathfrak{c}$ many pairwise Katětov incomparable hypergraph ideals which have the Nikodym property and are not totally bounded?*

A weaker variant of Question 4.6.4 may read as follows.

Question 4.6.5. *Does there exist a family of $\mathfrak{c}$ many pairwise Katětov incomparable non-pathological ideals which have the Nikodym property and are not totally bounded?*

We finish the chapter with the following question connected with Corollary 3.4.4. Let $\mathbb{AN}$ denote the class of all ideals $\mathcal{I}$ on ω which do not have the Nikodym property (so, e.g., $\mathcal{AN} \subseteq \mathbb{AN}$).

Question 4.6.6. *What are the Tukey types of the ordered sets $(\mathbb{AN}, \leq_K)$ and $(\mathbb{AN}\setminus\mathcal{AN}, \leq_K)$?*

Chapter 5

Cardinal invariants related to Rosenthal families

5.1 Introduction

We say that a set $A \in [\omega]^\omega$ is *free* for given $f \in \omega^\omega$ if $f[A] \cap A = \emptyset$. It follows a well-established combinatorial terminology (e.g., [67]) in which, given a set mapping $f\colon X \to \mathcal{P}(X)$, a set $Y \subseteq X$ is called free if $y \notin f(y')$ for any $y, y' \in Y$. It was proved by Koszmider and Martinez-Celís in [69, Theorem 23] that $\mathfrak{r}$ is the smallest size of a family $\mathcal{A} \subseteq [\omega]^\omega$ such that for every $f\colon \omega \to \omega$ without fixed points there is $A \in \mathcal{A}$ which is free for f. Functions without fixed points correspond to Rosenthal matrices of zeros and ones (see Introduction and [69, Section 2.2]). We will study in this chapter two families of cardinal invariants related to the notion of freeness and to functions without fixed points.

Many of the classical cardinal invariants can be investigated using relational systems, introduced by Vojtáš in [110] (see also [12, Section 4]). By a *relational system* we mean a triple $\langle A, B, R\rangle$, where $R \subseteq A \times B$ is a binary relation on non-empty sets A and B. For $x \in A$ and $y \in B$, $x\,R\,y$ is read as y *R-dominates* x. A family $X \subseteq A$ is *R-unbounded* if there is no element of B that R-dominates every element of X, and $Y \subseteq B$ is *R-dominating* if every element of A is R-dominated by some element of Y. The cardinal $\mathfrak{d}(R)$ is the smallest size of an R-dominating family, and $\mathfrak{b}(R)$ is the smallest size of an R-unbounded family. For example, for the relation $\mathfrak{D} = \langle \omega^\omega, \omega^\omega, \leq^*\rangle$ we have $\mathfrak{d}(\mathfrak{D}) = \mathfrak{d}$ and $\mathfrak{b}(\mathfrak{D}) = \mathfrak{b}$. Another common example is the relation $\mathfrak{R} = \langle \mathcal{P}(\omega), [\omega]^\omega, \text{does not split}\rangle$, for which we have $\mathfrak{d}(\mathfrak{R}) = \mathfrak{r}$ and $\mathfrak{b}(\mathfrak{R}) = \mathfrak{s}$.

The previously stated characterization of $\mathfrak{r}$ by free sets yields the following natural relational system: If we denote by $f\,\mathcal{R}\,A$ that A is *free* for f, for $f \in \omega^\omega$ without fixed points and infinite $A \subseteq \omega$, then, the dominating number of this relational system is $\mathfrak{r}$. Observe that $\mathfrak{b}(\mathcal{R}) = \omega$. Indeed, for every $a \neq b \in \omega$ there is $f_{a,b} \in \omega^\omega$ without fixed points such that $f(a) = b$ and $f(b) = a$. The family $\{f_{a,b}\colon \{a,b\} \in [\omega]^2\}$ is $\mathcal{R}$-unbounded. Instead, we will look at the following modification of the relation $\mathcal{R}$.

Definition 5.1.1. Let $f \in \omega^\omega$ be without fixed points and $A \in [\omega]^\omega$. We say that A is **-free* for f if $f[A] \cap A$ is finite.

As with the relation $\mathcal{R}$, the notion of *-free yields a natural relation $\mathcal{R}'$: $f\mathcal{R}'A$ if and only if A is *-free for f. One can easily show that $\mathfrak{d}(\mathcal{R}') = \mathfrak{d}(\mathcal{R})$ and that $\mathfrak{b}(\mathcal{R}')$ is uncountable. In [69], the authors also consider a restriction of this relation, limiting the domain to finite-to-one functions in ω^ω without fixed points. They proved that the dominating number of this restriction, which they denote $\mathfrak{ros}(c_0)$, is equal to $\min\{\mathfrak{d}, \mathfrak{r}\}$ [69, Theorem 32], and asked about the case when the relation is restricted to injective functions without fixed points ([69, Question 3]).

To investigate the aforementioned question, we will consider the following schemes of notions.

Definition 5.1.2. Given a property φ about functions of ω^ω, let

$$\Box_\varphi = \{f \in \omega^\omega : f \text{ has no fixed points and } \varphi(f)\},$$

and let $\mathfrak{ros}_\varphi = \mathfrak{d}(\langle \Box_\varphi, [\omega]^\omega, {}^*\text{-free}\rangle)$. In other words,

$$\mathfrak{ros}_\varphi = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ and } \forall f \in \Box_\varphi\, \exists A \in \mathcal{A}\ (f[A] \cap A \text{ is finite})\}.$$

For example, if φ is the trivial property, i.e., a property that all function satisfy (which, from now on, will be denoted by $\varphi = \mathbf{1}$), or if φ is the property of being a finite to one function (from now on $\varphi = \text{Fin-1}$), then the results from [69] can be restated as follows:

$$\mathfrak{ros}_{\mathbf{1}} = \mathfrak{r} \text{ and } \mathfrak{ros}_{\text{Fin-1}} = \min\{\mathfrak{d}, \mathfrak{r}\}.$$

If φ is the property of being a one-to-one function ($\varphi = \text{1-1}$) then [69, Question 3] asks about the value of $\mathfrak{ros}_{\text{1-1}}$. We will also use e.g. $\varphi = \text{bijective}$ for the property of being a bijection.

We will also study the dual cardinal invariants of $\mathfrak{ros}_\varphi$, which will be called Δ_φ. These are defined in the following way.

Definition 5.1.3. Given a property φ of functions of ω^ω, let $\Delta_\varphi = \mathfrak{b}(\langle \Box_\varphi, [\omega]^\omega, {}^*\text{-free}\rangle)$. In other words,

$$\Delta_\varphi = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \Box_\varphi \text{ and } \forall A \in [\omega]^\omega\, \exists f \in \mathcal{F}\ (|f[A] \cap A| = \aleph_0)\}.$$

Clearly, we have $\mathfrak{ros}_{\text{1-1}} \leq \mathfrak{ros}_{\text{Fin-1}} \leq \mathfrak{ros}_{\mathbf{1}}$ and $\Delta_{\mathbf{1}} \leq \Delta_{\text{Fin-1}} \leq \Delta_{\text{1-1}}$.

Furthermore, we will show a close relationship between the relation of being *-free and some of the cardinal invariants considered by Banakh in [7], in which the author studies some set-theoretical problems related to *large-scale topology*. Large-scale topology (referred also as asymptology in [85]) studies properties of *coarse spaces*, which were introduced independently in [8] and [87]. The cardinal Δ is defined as the smallest weight of a finitary coarse structure on ω that contains no infinite discrete subspaces (for the definitions and more information about the subject, we refer the reader to Banakh and Protasov [9]). In [7], the author found the following two combinatorial characterizations of Δ, where S_ω denotes the permutation group of ω and $I_\omega \subseteq S_\omega$ is the set of involutions of ω (i.e. permutations that are their own inverse) that have at most one fixed point.

Theorem 5.1.4 (Banakh)**.**

$$\begin{aligned}\Delta = \min\{|\mathcal{F}| : \mathcal{F} \subseteq S_\omega \textit{ and } \forall A \in [\omega]^\omega\ \exists f \in \mathcal{F}\ (|\{x \in A : x \neq f(x) \in A\}| = \aleph_0)\} = \\ = \min\{|\mathcal{F}| : \mathcal{F} \subseteq I_\omega \textit{ and } \forall A \in [\omega]^\omega\ \exists f \in \mathcal{F}\ (|f[A] \cap A| = \aleph_0)\}.\end{aligned}$$

Banakh also considered the dual cardinal invariant, denoted by $\widehat{\Delta}$:

$$\widehat{\Delta} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ and } \forall h \in I_\omega\, \exists A \in \mathcal{A}\ (|h[A] \cap A| < \aleph_0)\}.$$

We will prove that the families of cardinal invariants $\mathfrak{ros}_\varphi$ and Δ_φ are closely connected with Δ and $\widehat{\Delta}$: by Corollary 5.3.4 we have $\Delta = \Delta_{\text{bijective}} = \Delta_{\text{1-1}}$ and $\widehat{\Delta} = \mathfrak{ros}_{\text{bijective}} = \mathfrak{ros}_{\text{1-1}}$.

5.2 $\Delta_{\mathbf{1}}$ and $\Delta_{\mathbf{Fin\text{-}1}}$.

In [69], Koszmider and Martinez-Celís proved that $\mathfrak{ros}_{\mathbf{1}} = \mathfrak{r}$ and $\mathfrak{ros}_{\text{Fin-1}} = \min\{\mathfrak{d}, \mathfrak{r}\}$. In this section, we will prove similar equalities for the dual invariants. One of the main tools that is used to prove that $\mathfrak{ros}_{\mathbf{1}} \geq \mathfrak{r}$ is the fact that, if $\kappa < \mathfrak{r}$ and $\mathcal{S} \subseteq [\omega]^\omega$ is a family of sets such that $|\mathcal{S}| = \kappa$, then there is a partition $\omega = \bigcup_{n\in\omega} P_n$ such that for each $S \in \mathcal{S}$ and each P_n, $S \cap P_n$ is infinite (the authors of [69] call such families *nowhere reaping*), and $\mathfrak{r}$ is the smallest cardinal with such a property. It is not clear that the dual of the property of non-existence of the nowhere reaping families is $\mathfrak{s}$. This issue was already addressed in [28]. In Section 8 of that work, the authors introduce a cardinal invariant, closely related to $\mathfrak{s}$, and also a family of invariants that appear to be equal to $\mathfrak{s}$. For each $n \in \omega$ let $\mathbf{P}_n$ be the family of partitions of ω into n infinite sets, and let $\mathbf{P}_\omega$ be the family of partitions of ω into $\aleph_0$ many infinite sets. Then, we define:

$$\begin{aligned}\mathfrak{s}_n = \min\big\{&|\mathcal{S}| : \mathcal{S} \subseteq \mathbf{P}_n \text{ and } \forall A \in [\omega]^\omega \\ &\exists P = \langle P_k\rangle_{k<n} \in \mathcal{S}\ \forall k < n\,(|A \cap P_k| = \aleph_0)\big\};\end{aligned}$$

$$\begin{aligned}\mathfrak{s}_{\text{mix}} = \min\big\{&|\mathcal{S}| : \mathcal{S} \subseteq \mathbf{P}_\omega \text{ and } \forall A \in [\omega]^\omega \\ &\exists P = \langle P_k\rangle_{k\in\omega} \in \mathcal{S}\ \forall k \in \omega\,(|A \cap P_k| = \aleph_0)\big\}.\end{aligned}$$

The name $\mathfrak{s}_{\text{mix}}$ was chosen to avoid confusion, as $\mathfrak{s}_\omega$ stands for another already studied cardinal invariant, and because of the connection with so-called *mixing reals* (see Farkas *et al.* [28]).

By the definition, we have $\mathfrak{s} = \mathfrak{s}_2$. We have the following easy observation concerning the comparison of different versions of the splitting number.

Proposition 5.2.1. $\mathfrak{s} \leq \mathfrak{s}_{\text{mix}}$.

Proof. Let $\mathcal{S}$ be a family of infinite partitions witnessing $\mathfrak{s}_{\text{mix}}$. We define a family $\mathcal{S}_2$ of the same size as $\mathcal{S}$ in the following way: for each $\langle P_k\rangle_{k\in\omega} \in \mathcal{S}$ we consider a partition $\langle P_0, \bigcup_{k>0} P_k\rangle$. Then, $\mathcal{S}_2$ is a witness for $\mathfrak{s}_2$. For if $A \in [\omega]^\omega$ then there is $\langle P_k\rangle_{k\in\omega} \in \mathcal{S}$ such that $|A \cap P_k| = \aleph_0$ for each $k \in \omega$, and so

$$|A \cap P_0| = \big|A \cap \bigcup_{k>0} P_k\big| = \aleph_0.$$

$\square$

Remark 5.2.2. We can prove analogously as in the previous proposition that $\mathfrak{s} \leq \mathfrak{s}_n$ for each $n \in \omega$, by gluing together all parts of the partition apart of the first one. It is not difficult to show (see e.g. [28, Section 8]) that it actually holds $\mathfrak{s} = \mathfrak{s}_n$ for each $n \in \omega$, but we will not need this fact.

By the remark about nowhere reaping families, the "dual invariant" of $\mathfrak{s}_{\text{mix}}$ is $\mathfrak{r}$, however, it is not known if $\mathfrak{s}_{\text{mix}} = \mathfrak{s}$ ([28, Question 8.9]). We prove that this invariant is an upper bound for $\Delta_{\mathbf{1}}$. The proof is analogous to the proof of the dual inequality (see [69, Lemma 17]).

Proposition 5.2.3. $\Delta_{\mathbf{1}} \leq \mathfrak{s}_{\text{mix}}$.

Proof. Let $\mathcal{S}_\omega$ be a family of partitions of ω into $\aleph_0$ many infinite sets witnessing the definition of $\mathfrak{s}_{\text{mix}}$. We will build a set of functions $\mathcal{F} \subseteq \square_{\mathbf{1}}$ such that for every $A \in [\omega]^\omega$ there is $f \in \mathcal{F}$ such that $f[A] \cap A$ is infinite. For every partition $P = (P_n)_{n\in\omega} \in \mathcal{S}_\omega$ we define the following function $f_P \colon \omega \to \omega$:

$$f_P(k) = \begin{cases} n, & \text{if } n \neq k \in P_n, \\ k+1, & \text{otherwise.} \end{cases}$$

It follows that f_P is a function without fixed points. Let $\mathcal{F} = \{f_P : P \in \mathcal{S}_\omega\}$. If $A \in [\omega]^\omega$ then there exists $P = (P_n)_{n\in\omega} \in \mathcal{S}_\omega$ such that $\left|A \cap P_n\right| = \aleph_0$ for all $n \in \omega$. Then, for every $n \in \omega$ we have:

$$f_P^{-1}\left[\{n\}\right] \supseteq \left(P_n \backslash \{n\}\right) \cap A \neq \emptyset,$$

which implies that $n \in f_P[A]$ for all $n \in \omega$, and so $f_P[A] = \omega$. In particular, $f_P[A] \cap A$ is infinite. □

For our next result, we will use the following well-known theorem, whose proof can be found in Katětov [62].

Theorem 5.2.4. *For every $f \in \square_{\mathbf{1}}$ there is a partition $\omega = A_0 \cup A_1 \cup A_2$ such that $f[A_i] \cap A_i = \emptyset$ for all $i \in \{0, 1, 2\}$.*

Proposition 5.2.5. $\mathfrak{s} \leq \Delta_{\mathbf{1}}$.

Proof. Let $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq \square_{\mathbf{1}}$ with $\kappa < \mathfrak{s}$. We will see that there is infinite $C \in [\omega]^\omega$ such that $f_\alpha[C] \cap C$ is finite for every $\alpha \in \kappa$. For each $\alpha < \kappa$ pick a partition $\omega = A_0^\alpha \cup A_1^\alpha \cup A_2^\alpha$ such that $f_\alpha\left[A_i^\alpha\right] \cap A_i^\alpha = \emptyset$ for $i \in \{0, 1, 2\}$ (which exists by the previous theorem).

Since $\kappa < \mathfrak{s} \leq \mathfrak{s}_3$ by Remark 5.2.2, there exists infinite $B \subseteq \omega$ such that, for every $\alpha < \kappa$, we have that $|B \cap A_i^\alpha| = \aleph_0$ for at most two $i \in \{0, 1, 2\}$. Thus, by taking finite modifications of some particular A_i^α's, for each $\alpha < \kappa$ we can define a partition $B = B_0^\alpha \cup B_1^\alpha$, where each B_i^α is a finite modification of some A_j^α. Again, since $\kappa < \mathfrak{s} = \mathfrak{s}_2$, there exists infinite $C \subseteq B$ such that, for every $\alpha < \kappa$, we have that $C \subseteq^* B_{i_C}^\alpha$ for some $i_C \in \{0, 1\}$, and so $C \subseteq^* A_{j_C}^\alpha$ for some $j_C \in \{0, 1, 2\}$. Therefore, for all $\alpha \in \kappa$ we have:

$$f_\alpha[C] \cap C \subseteq^* f_\alpha[A_{j_C}^\alpha] \cap A_{j_C}^\alpha = \emptyset.$$

□

As a consequence, we got the following bounds for $\Delta_{\mathbf{1}}$, which raises Question 5.6.1.

Corollary 5.2.6. $\mathfrak{s} \leq \Delta_{\mathbf{1}} \leq \mathfrak{s}_{\text{mix}}$.

We will now turn our focus on $\Delta_{\text{Fin-1}}$. For the next proposition, we will need a well-known characterization of $\mathfrak{b}$. Let $\mathcal{I}$ and $\mathcal{J}$ be interval partitions of ω. We say that the partition $\mathcal{I}$ *dominates* the partition $\mathcal{J}$ if for almost all $I \in \mathcal{I}$ there is $J \in \mathcal{J}$ such that $J \subseteq I$. The following widely known theorem appears in Blass [12, Theorem 2.10]:

Theorem 5.2.7. $\mathfrak{b}$ *is the smallest amount of interval partitions of ω that are not dominated by a single interval partition.* $\mathfrak{d}$ *is the smallest size of a dominating family of interval partitions.*

By taking the end points of every piece, we can consider interval partitions as infinite subsets of ω, and the domination relations translates to the following: A set $A \in [\omega]^\omega$ *dominates* a set $B \in [\omega]^\omega$ if and only if for almost every pair $n < m$ of elements of A there are $i, j \in B$ such that $n \le i < j \le m$. With this in mind, we will prove the following proposition.

Proposition 5.2.8. $\mathfrak{b} \le \Delta_{\text{Fin-1}}$.

Proof. Let $\mathcal{F} = \{f_\alpha \colon \alpha < \kappa\} \subseteq \square_{\text{Fin-1}}$ with $\kappa < \mathfrak{b}$. For every $\alpha < \kappa$, one can easily construct an increasing function $h_\alpha \in \omega^\omega$ such that

$$h_\alpha(n+1) > \max\Big(f_\alpha\big[\{0,\dots,h_\alpha(n)\}\big] \cup f_\alpha^{-1}\big[\{0,\dots,h_\alpha(n)\}\big]\Big).$$

As $\kappa < \mathfrak{b}$, there exists an infinite set $A \in [\omega]^\omega$ dominating the set $\big\{h_\alpha(i) \colon i \in \omega\big\}$ for each $\alpha < \kappa$, i.e. if $A = \{a_i : i \in \omega\}$ is an increasing enumeration, then, for every $\alpha < \kappa$, for sufficiently large $n \in \omega$, there exists $k \in \omega$ such that $[h_\alpha(k), h_\alpha(k+1)] \subseteq [a_n, a_{n+1}]$. We fix $\alpha < \kappa$ and we will show that $f_\alpha[A] \cap A$ is finite. Let $n \in \omega$ be sufficiently large, so that there is $k \in \omega$ satisfying $[h_\alpha(k), h_\alpha(k+1)] \subseteq [a_n, a_{n+1}]$. We have $a_{n+1} \ge h_\alpha(k+1)$, implying that a_{n+1} is bigger than any number of the form $f_\alpha(\ell)$ or an element of the set $f_\alpha^{-1}(\ell)$ for $\ell \le h_\alpha(k)$, in particular for $\ell = a_n$. Thus, we have shown that for almost all $n \in \omega$ we have $a_{n+1} > f_\alpha(a_i), \max f_\alpha^{-1}(a_i)$ for $i \le n$. This implies that $f_\alpha[A] \cap A$ is finite. $\square$

Therefore, as a conclusion we get that $\max\{\mathfrak{b}, \Delta_{\mathbf{1}}\} \le \Delta_{\text{Fin-1}}$. We will now prove the converse inequality, whose proof is inspired by the proof of the dual inequality in [69, Proposition 26].

Proposition 5.2.9. $\Delta_{\text{Fin-1}} \le \max\{\mathfrak{b}, \Delta_{\mathbf{1}}\}$.

Proof. Let $\mathcal{G} = \{g_\beta \colon \beta < \Delta_{\mathbf{1}}\} \subseteq \square_{\mathbf{1}}$ be a family of functions such that for every $A \in [\omega]^\omega$ there is $g \in \mathcal{G}$ satisfying $\big|g[A] \cap A\big| = \aleph_0$, and let $\{A_\alpha : \alpha < \mathfrak{b}\} \subseteq [\omega]^\omega$ be a witness for characterization of $\mathfrak{b}$ from Theorem 5.2.7. Then, for every $B \in [\omega]^\omega$ there is $\alpha < \mathfrak{b}$ such that for infinitely many pairs $n < m$ of elements of A_α there are $i, j \in B$ satisfying $n \le i < j \le m$, as in the other case the set $C = \{b_{2n} \colon n \in \omega\}$, where $B = \{b_n \colon n \in \omega\}$ is its increasing enumeration, would dominate all the sets A_α. For every $\alpha < \mathfrak{b}$ consider $A_\alpha = \{a_n^\alpha : n \in \omega\}$ with its increasing enumeration.

Let us define $f_{\alpha,\beta} \colon \omega \to \omega$ for each $\alpha < \mathfrak{b}$ and $\beta < \Delta_{\mathbf{1}}$ by setting:

$$f_{\alpha,\beta}(i) = \begin{cases} g_\beta(i) & \text{if there is an } n \in \omega \text{ such that } i, g_\beta(i) \in \big[a_n^\alpha, a_{n+1}^\alpha\big), \\ i+1 & \text{otherwise.} \end{cases}$$

Clearly, $f_{\alpha,\beta} \in \square_{\text{Fin-1}}$. We will show that the family $\mathcal{F} = \{f_{\alpha,\beta} : \alpha < \mathfrak{b}, \beta < \Delta_{\mathbf{1}}\}$ is a witness for $\Delta_{\text{Fin-1}}$. Let $A \in [\omega]^\omega$. Then, there is $\beta < \Delta_{\mathbf{1}}$ such that $\big|g_\beta[A] \cap A\big| = \aleph_0$. Thus, we can find a set $B = \{b_j : j \in \omega\} \in [\omega]^\omega$ such that for every $i \in \omega$ we have

$$g_\beta\big[[b_i, b_{i+1}) \cap A\big] \cap \big(A \cap [b_i, b_{i+1})\big) \neq \emptyset.$$

Then, there exists $\alpha < \mathfrak{b}$ such that for infinitely many pairs $n < m$ of elements of A_α there are $i, j \in B$ satisfying $n \le i < j \le m$. Clearly, this implies that

$$g_\beta\big[[a_n^\alpha, a_{n+1}^\alpha) \cap A\big] \cap \big(A \cap [a_n^\alpha, a_{n+1}^\alpha)\big) \neq \emptyset$$

holds for infinitely many $n \in \omega$. Finally, note that if

$$i \in g_\beta\big[[a^\alpha_n, a^\alpha_{n+1}) \cap A\big] \cap \big(A \cap [a^\alpha_n, a^\alpha_{n+1})\big),$$

then, taking some $j \in [a^\alpha_n, a^\alpha_{n+1}) \cap A$ such that $g(j) = i$, we have $j, g_\beta(j) \in [a^\alpha_n, a^\alpha_{n+1})$, and so $f_{\alpha,\beta}(j) = g_\beta(j)$. In conclusion, every such i is an element of $f_{\alpha,\beta}[A] \cap A$, thus this set is infinite. $\square$

As a corollary, we get the main result of this section.

Theorem 5.2.10. $\Delta_{\text{Fin-1}} = \max\{\mathfrak{b}, \Delta_{\mathbf{1}}\}$.

Together with Corollary 5.2.6, this implies that:

Corollary 5.2.11. $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \Delta_{\text{Fin-1}} \leq \max\{\mathfrak{b}, \mathfrak{s}_{\text{mix}}\}$.

We do not know the answer to Question 5.6.2, which asks about the exact value of $\Delta_{\text{Fin-1}}$.

5.3 $\Delta_{\text{1-1}}$, $\mathfrak{ros}_{\text{1-1}}$ and their relatives.

In this section, we will take a look at $\Delta_{\text{1-1}}$, $\Delta_{\text{bijective}}$, $\Delta_{\text{involution}}$ and its dual versions (where φ = bijective is the property of being a bijective function and φ = involution is the property of being an involution). Recall that an *involution of* ω is a permutation of ω which is equal to its own inverse. Since

$$\square_{\text{involution}} \subseteq \square_{\text{bijective}} \subseteq \square_{\text{1-1}} \subseteq \square_{\text{Fin-1}} \subseteq \square_{\mathbf{1}},$$

we have

$$\Delta_{\text{involution}} \geq \Delta_{\text{bijective}} \geq \Delta_{\text{1-1}} \geq \Delta_{\text{Fin-1}} \geq \Delta_{\mathbf{1}}$$

and

$$\mathfrak{ros}_{\text{involution}} \leq \mathfrak{ros}_{\text{bijective}} \leq \mathfrak{ros}_{\text{1-1}} \leq \mathfrak{ros}_{\text{Fin-1}} \leq \mathfrak{ros}_{\mathbf{1}}.$$

The main goal of this section is to prove that $\Delta_{\text{1-1}} = \Delta_{\text{involution}} = \Delta$ and that $\mathfrak{ros}_{\text{1-1}} = \mathfrak{ros}_{\text{involution}} = \widehat{\Delta}$, where Δ and $\widehat{\Delta}$ are the invariants defined by Banakh in [7], introduced at the end of Section 5.1.

We start with the following lemma, connecting one-to-one functions with involutions.

Lemma 5.3.1. *For every $f \in \square_{\text{1-1}}$ there are $\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3 \in \square_{\text{involution}}$ such that, for every $A \in [\omega]^\omega$, if $f[A] \cap A$ is infinite, then $\hat{f}_i[A] \cap A$ is infinite for some $i \leq 3$.*

Proof. Let $f \in \square_{\text{1-1}}$. We will need the following claim.

Claim 5.3.2. There is a partition $\omega = A_0 \cup A_1 \cup A_2 \cup A_3$ into infinite sets, such that for every $i \leq 3$ the sets A_i and $f[A_i]$ are disjoint and the complement of $A_i \cup f[A_i]$ is infinite.

Proof. Consider the graph G given by f, i.e. with ω as the set of vertices and with an edge in between n and m if and only if $f(n) = m$ or $f(m) = n$. Then, G is a disjoint union of its connected components, i.e., since f is one to one and without fixed points, it is a disjoint union of finite cycles $\{C_n : n \in N\}$ and infinite linear graphs $\{B_m : m \in M\}$ where $N, M \leq \omega$. We will deal with linear graphs and cycles in different ways:

For every $m \in M$, there is an enumeration of $B_m = \{b^m_j : j \in O\}$ such that O is either ω or $\mathbb{Z}$, and $f(b^m_j) = b^m_{j+1}$ for each $j \in O$. For each $i \leq 3$ and $m \in M$, let $A^m_i = \{b^m_j : j = i \mod 4\}$.

For the case of finite cycles, note first that for each cycle there is a partition into 3 sets such that no two adjacent vertices are in the same part, and the same is true for any disjoint union of cycles. Let $N = P_0 \cup P_1 \cup P_2 \cup P_3$ be any partition into 4 sets (such that each part is infinite if N is infinite), and for each $i \leq 3$ we define $K_i = \bigcup_{j\in P_i} C_j$. Since K_i is a disjoint union of cycles, there exists a partition $K_i = \bigcup_{k\leq 3, k\neq i} Z^i_k$ into 3 sets such that no adjacent vertices are in the same Z^i_k, and each Z^i_k contains some elements of every cycle in K_i. For each $k \leq 3$, let $A'_k = \bigcup_{i\leq 3, i\neq k} Z^i_k$.

Finally, for each $i \leq 3$ let

$$A_i = A'_i \cup \bigcup_{m\in M} A^m_i,$$

and notice that each A_i is infinite (if N is infinite then so is each A'_i, if not, then there exists $m_0 \in M$, and $A^{m_0}_i$ is infinite for each $i \leq 4$). To check that this partition meets the requirements of the Claim, let $i \leq 3$. We have $A_i \cap f[A_i] = \emptyset$, since $f[A^m_i] \subseteq A^m_{(i+1) \mod 4}$ for any $m \in M$ and $f[Z^k_i] \subseteq K_k \setminus Z^k_i$ for any $k \leq 3, k \neq i$. Moreover, if N is infinite then $A_i \cup f[A_i]$ is disjoint from the infinite set K_i, and in the other case there exists $m_0 \in M$ and $A_i \cup f[A_i]$ it is disjoint from the infinite set $\{a^{m_0}_k : k \equiv (i + 2) \mod 4\}$. $\dashv$

To finish the proof of the lemma, for each $i \leq 3$, define $f'_i(n) = m$ and $f'_i(m) = n$ for every $n \in A_i$ and $m \in f[A_i]$ such that $f(n) = m$. Then, by the Claim, each f'_i is an involution without fixed points and the complement of its domain is infinite, so there exists an involution $\hat{f}_i \in \square_{\text{involution}}$ extending it. In conclusion, if $f[A] \cap A$ is infinite for some $A \subseteq \omega$, then there is $i \leq 3$ such that $f[A \cap A_i] \cap A$ is infinite, and for every $x \in A \cap A_i$ we have $\hat{f}_i(x) = f'_i(x) = f(x)$, so $\hat{f}_i[A] \cap A$ is infinite, too. $\square$

This lemma will be used to prove the first main result of this section.

Theorem 5.3.3. $\Delta_{\text{1-1}} = \Delta_{\text{involution}}$ *and* $\mathfrak{ros}_{\text{1-1}} = \mathfrak{ros}_{\text{involution}}$.

Proof. It is enough to prove that $\Delta_{\text{1-1}} \geq \Delta_{\text{involution}}$ and $\mathfrak{ros}_{\text{1-1}} \leq \mathfrak{ros}_{\text{involution}}$.

($\Delta_{\text{1-1}} \geq \Delta_{\text{involution}}$): Let $\{f_\alpha : \alpha \in \Delta_{\text{1-1}}\}$ be a witness for $\Delta_{\text{1-1}}$. Using the previous lemma, for each $\alpha \in \Delta_{\text{1-1}}$ it is possible to find $\hat{f}^\alpha_0, \hat{f}^\alpha_1, \hat{f}^\alpha_2, \hat{f}^\alpha_3$ such that, for all $A \in [\omega]^\omega$, if $f_\alpha[A] \cap A$ is infinite, then there is $i \in \{0, 1, 2, 3\}$ such that $\hat{f}^\alpha_i[A] \cap A$ is infinite. It follows that $\left\{\hat{f}^\alpha_i : \alpha \in \Delta_{\text{1-1}}, i \in \{0, 1, 2, 3\}\right\}$ is a witness for $\Delta_{\text{involution}}$ of cardinality $\Delta_{\text{1-1}}$.

($\mathfrak{ros}_{\text{1-1}} \leq \mathfrak{ros}_{\text{involution}}$): Let $\kappa < \mathfrak{ros}_{\text{1-1}}$ and $\{A_\alpha : \alpha \in \kappa\} \subseteq [\omega]^\omega$, we will show that this family is not a witness for $\mathfrak{ros}_{\text{involution}}$. Since $\kappa < \mathfrak{ros}_{\text{1-1}}$, there is a function $f \in \square_{\text{1-1}}$ such that, for all $\alpha \in \kappa$, $f[A_\alpha] \cap A_\alpha$ is infinite. Then, using the previous lemma, there are $\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3 \in \square_{\text{involution}}$ such that for every $\alpha \in \kappa$ there is $i_\alpha \in \{0, 1, 2, 3\}$ such that $\hat{f}_{i_\alpha}[A_\alpha] \cap A_\alpha$ is infinite. For every $\alpha \in \kappa$, one can easily construct an interval partition $\mathcal{I}_\alpha$, such that, for every $I \in \mathcal{I}_\alpha$, the set $\hat{f}_{i_\alpha}[I \cap A_\alpha] \cap (I \cap A_\alpha)$ has a pair of elements of the form $n, \hat{f}_{i_\alpha}(n)$. Modify each $\mathcal{I}_\alpha$ to $\hat{\mathcal{I}}_\alpha$ by gluing consecutive terms of the partition together. Since $\kappa < \mathfrak{ros}_{\text{1-1}} \leq \mathfrak{ros}_{\text{Fin-1}} \leq \mathfrak{d}$, by Theorem 5.2.7, there exists an interval partition $\mathcal{J} = \{J_n : n \in \omega\}$ such that, for all $\alpha \in \kappa$, infinitely many intervals from $\hat{\mathcal{I}}_\alpha$ do not contain an interval from $\mathcal{J}$; that is, infinitely many intervals from $\hat{\mathcal{I}}_\alpha$ are contained in consecutive intervals from $\mathcal{J}$. Therefore, the set

$$D_\alpha = \left\{n \in \omega : J_n \text{ contains an interval from } \mathcal{I}_\alpha\right\}$$

is infinite. If an interval partition $\mathcal{J}'$ dominates $\mathcal{J}$, then the family of partitions $\{\mathcal{I}_\alpha \colon \alpha < \kappa\}$ cannot dominate the partition $\mathcal{J}'$, and so, by taking a bigger partition, we can assume that $|J_n|$ is odd for each $n \in \omega$. Since $\kappa < \mathfrak{ros}_{1\text{-}1} \leq \mathfrak{ros}_{\mathrm{Fin}\text{-}1} \leq \mathfrak{r}$, there is a partition of $\omega = P_0 \cup P_1 \cup P_2 \cup P_3$ such that, for all $\alpha \in \kappa$ and all $i \in \{0,1,2,3\}$, the set $P_i \cap D_\alpha$ is infinite (see the remark about the nowhere reaping families at the beginning of Section 5.2). Let

$$D = \big\{n \in \omega : n, \hat{f}_i(n) \in J_k \text{ for } k \in P_i, i \in \{0,1,2,3\}\big\}.$$

Note that $\omega \setminus D$ is infinite, since each J_k has an odd number of elements and D picks an even amount of elements in J_k. Observe that, for all $i \in \{0,1,2,3\}$ and every $k \in P_i$, $n \in D$ if and only if $f_i(n) \in D$, and therefore the function

$$\hat{f}' = \bigcup_{i \in \{0,1,2,3\}} \bigcup_{k \in P_i} \hat{f}_i \restriction (J_k \cap D)$$

is an involution on D. Let $\hat{f} \in \square_{\text{involution}}$ be any function that extends $\hat{f}'$. We will see that, for every $\alpha \in \kappa$, $\hat{f}[A_\alpha] \cap A_\alpha$ is infinite. Let $\alpha \in \kappa$, so $P_{i_\alpha} \cap D_\alpha$ is infinite. For every $k \in P_{i_\alpha} \cap D_\alpha$, the set J_k contains an interval from $\mathcal{I}_\alpha$, thus $\hat{f}_{i_\alpha}[J_k \cap A] \cap (J_k \cap A)$ has at least 2 elements of the form $n_0, \hat{f}_{i_\alpha}(n_0)$. Finally, observe that $n_0 \in D$, so $\hat{f}_{i_\alpha}(n_0) = \hat{f}(n_0)$ and therefore

$$\hat{f}[J_k \cap A] \cap (J_k \cap A) \neq \emptyset.$$

This implies that $\hat{f}[A] \cap A$ is infinite, which is what we wanted to prove. $\square$

Clearly, the previous equality holds for any property φ in between being one to one and being an involution. In particular, $\Delta_{\text{bijective}} = \Delta_{1\text{-}1}$ and $\mathfrak{ros}_{\text{bijective}} = \mathfrak{ros}_{1\text{-}1}$.

We will now relate these cardinal invariants to Δ and $\widehat{\Delta}$. By their definition in Banakh [7], these are slight modifications of $\Delta_{\text{involution}}$ and $\mathfrak{ros}_{\text{involution}}$, where a single fixed point was allowed. Clearly an involution with a single fixed point can be modified in a finite way to be a bijection: For instance, if f is an involution such that $n \in \omega$ is its fixed point, and if $m \neq n$, then setting $\hat{f}(n) = m$, $\hat{f}(m) = f(n)$ and leaving the rest of $\hat{f}$ the same as f yields a bijection without fixed points such that, for every $A \in [\omega]^\omega$, $f[A] \cap A$ is infinite if and only if $\hat{f}[A] \cap A$ is infinite. Therefore, $\Delta_{\text{involution}} \leq \Delta \leq \Delta_{\text{bijection}}$ and $\mathfrak{ros}_{\text{involution}} \geq \widehat{\Delta} \geq \mathfrak{ros}_{\text{bijection}}$, giving the following as a consequence.

Corollary 5.3.4. *$\Delta = \Delta_{1\text{-}1}$ and $\widehat{\Delta} = \mathfrak{ros}_{1\text{-}1}$.*

Proof. It follows from Theorem 5.3.3 and the remark above. $\square$

We can now apply the results from [7] to $\Delta_{1\text{-}1}$ to get the following.

Theorem 5.3.5. *$\max\{\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{N})\} \leq \Delta_{1\text{-}1} \leq \mathrm{non}(\mathcal{M})$ and $\min\{\mathfrak{d}, \mathfrak{r}, \mathrm{non}(\mathcal{N})\} \geq \mathfrak{ros}_{1\text{-}1} \geq \mathrm{cov}(\mathcal{M})$.*

Proof. This is a direct consequence of Corollary 5.3.4 and [7, Theorems 3.2 and 7.1]. Alternatively one can use Proposition 5.2.5, Theorem 5.2.10, the fact that $\mathfrak{ros}_{\mathrm{Fin}\text{-}1} = \min\{\mathfrak{d}, \mathfrak{r}\}$ ([69, Theorem 27]), Proposition 5.3.6 and Theorem 5.4.3. $\square$

Later in this chapter, we will show the consistency of the strict ineqalities
$\max\{\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{N})\} < \Delta_{1\text{-}1}$ and $\min\{\mathfrak{d}, \mathfrak{r}, \mathrm{non}(\mathcal{N})\} > \mathfrak{ros}_{1\text{-}1}$.

To conclude this section, we will look at the case where the functions are strictly increasing (φ =increasing). The set $\square_{\text{increasing}}$ consists of strictly increasing functions

$f \in \omega^\omega$ such that $f(0) > 0$. Since all these functions are injective, then it follows that $\Delta_{\text{1-1}} \le \Delta_{\text{increasing}}$ and $\mathfrak{ros}_{\text{1-1}} \ge \mathfrak{ros}_{\text{increasing}}$. The following is proved in a standard way of comparing other cardinal invariants to $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$.

Proposition 5.3.6. *$\Delta_{\text{increasing}} \le \mathrm{non}(\mathcal{M})$ and $\mathfrak{ros}_{\text{increasing}} \ge \mathrm{cov}(\mathcal{M})$.*

Proof. It is easy to see that the space $\square_{\text{increasing}} \subseteq \omega^\omega$ with the subspace topology is Polish (since it is a closed subspace) and has no isolated points, and so the cardinal invariants of the ideal of meager subsets of $\square_{\text{increasing}}$ are the same as the ones of the ideal $\mathcal{M}$.

Let $\kappa < \Delta_{\text{increasing}}$ and let $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq \square_{\text{increasing}}$. Then, there exists an infinite set $A \subseteq \omega$ such that $f_\alpha[A] \cap A$ is finite for every $\alpha < \kappa$. For every $n \in \omega$, the set

$$E_n = \left\{ f \in \square_{\text{increasing}} \colon |f[A] \cap A| \le n \right\}$$

is closed nowhere dense in $\square_{\text{increasing}}$. It follows that $\mathcal{F}$ is meager, since we have $\mathcal{F} \subseteq \bigcup_{n\in\omega} E_n$.

Now, let $\mathcal{A} \subseteq [\omega]^\omega$ be a witness for $\mathfrak{ros}_{\text{increasing}}$. As in the previous paragraph, the set

$$E_A = \left\{ f \in \square_{\text{increasing}} \colon f[A] \cap A \text{ is finite} \right\}$$

is meager in $\square_{\text{increasing}}$ for every $A \in \mathcal{A}$. By the definition of $\mathfrak{ros}_{\text{increasing}}$, $\left\langle E_A \colon A \in \mathcal{A} \right\rangle$ forms a cover of $\square_{\text{increasing}}$ of size $\mathfrak{ros}_{\text{increasing}}$ by meager subsets. $\square$

5.4 Eventually different and infinitely equal reals.

Recall that, given $f, g \in \omega^\omega$, f is *infinitely equal to* g, denoted by $f =^\infty g$, if $f \cap g$ is infinite. This naturally yields a relational system $\mathcal{E} = \langle \omega^\omega, \omega^\omega, =^\infty \rangle$ and, by a well-known theorem of Bartoszyński [10] and Miller [82], $\mathfrak{d}(\mathcal{E}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{b}(\mathcal{E}) = \mathrm{cov}(\mathcal{M})$ (see also [11, Section 2.4]). We will make use of the following bounded variation of this relation:

Definition 5.4.1. Given a strictly increasing $g \in \omega^\omega$ such that $g(0) > 0$, the following is the *g-bounded version of the infinitely equal number*.

$$\mathfrak{e}_g = \min\left\{ |\mathcal{F}| \colon \mathcal{F} \subseteq \prod_{n\in\omega} g(n) \text{ and } \forall h \in \prod_{n\in\omega} g(n)\, \exists f \in \mathcal{F}\, (f =^\infty h) \right\}.$$

and its dual, the *g-bounded version of the eventually different number*:

$$\widehat{\mathfrak{e}_g} = \min\left\{ |\mathcal{F}| \colon \mathcal{F} \subseteq \prod_{n\in\omega} g(n) \text{ and } \forall h \in \prod_{n\in\omega} g(n)\, \exists f \in \mathcal{F}\, (f \cap h \text{ is finite}) \right\}.$$

From now on, whenever we consider $\mathfrak{e}_g$ or $\widehat{\mathfrak{e}_g}$, we will assume that $g \in \square_{\text{increasing}}$, i.e. g is strictly increasing and $g(0) > 0$. These invariants have been widely studied in the literature, even in a much more general setting (see Miller [81, Section 2] and Cardona, Mejía [19]). Clearly $\mathfrak{e}_g \le \mathrm{non}(\mathcal{M})$ and $\widehat{\mathfrak{e}_g} \ge \mathrm{cov}(\mathcal{M})$. Consistently, these invariants can have many different values (see Goldstern, Shelah [41] and Kamo, Osuga [61]). Let us observe that $\mathfrak{e}_g \le \mathfrak{e}_f$ whenever $g \le^* f$, as if $\mathcal{F}$ is a witness for $\mathfrak{e}_f$ then $\mathcal{G} = \{\min(h, g) : h \in \mathcal{F}\}$ is a witness for $\mathfrak{e}_g$. By an analogous reason, we have that $\widehat{\mathfrak{e}_g} \ge \widehat{\mathfrak{e}_f}$ whenever $g \le^* f$. Another well-known fact about these invariants is their following relation to the ideal $\mathcal{N}$ of sets of measure zero (see e.g. [61, Lemma 2] and [19, Theorem 3.21]).

Proposition 5.4.2. $\operatorname{cov}(\mathcal{N}) \leq \mathfrak{e}_g$ *and* $\operatorname{non}(\mathcal{N}) \geq \widehat{\mathfrak{e}_g}$ *for every* $g \in \square_{\text{increasing}}$ *such that* $\sum_{n\in\omega} 1/g(n) < \infty$.

Proof. Let λ_g be the standard product probability measure on $\prod_{n\in\omega} g(n)$, where $g \in \square_{\text{increasing}}$ satisfies $\sum_{n\in\omega} 1/g(n) < \infty$, and let $f \in \omega^\omega$. For every $k \in \omega$, we have

$$\lambda_g\Big(\big\{h \in \prod_{n\in\omega} g(n) \colon h(k) = f(k)\big\}\Big) = 1/g(k).$$

Therefore, by the Borel–Cantelli lemma, we get that:

$$\lambda_g\Big(\big\{h \in \prod_{n\in\omega} g(n) : (h =^\infty f)\big\}\Big) = 0.$$

Thus, a family witnessing $\mathfrak{e}_g$ is also a witness for $\operatorname{cov}(\mathcal{N})$.

The second inequality follows dually: if $\mathcal{F} \subseteq \prod_{n\in\omega} g(n)$ is not null, then it cannot be covered by any set of the form $\big\{f \in \prod_{n\in\omega} g(n) : (f =^\infty h)\big\}$ for $h \in \omega^\omega$, and so $\mathcal{F}$ is a witness for $\widehat{\mathfrak{e}_g}$. $\square$

Our next goal will be to prove $\Delta_{\text{1-1}} \geq \mathfrak{e}_g$ and $\mathfrak{ros}_{\text{1-1}} \leq \widehat{\mathfrak{e}_g}$ for every increasing $g \in \omega^\omega$, which, by Proposition 5.4.2 and Theorem 5.3.3, improve the bounds $\Delta_{\text{1-1}} \geq \operatorname{cov}(\mathcal{N})$ and $\mathfrak{ros}_{\text{1-1}} \leq \operatorname{non}(\mathcal{N})$ obtained by Banakh [7, Lemma 3.7 and Lemma 7.4].

Theorem 5.4.3. $\Delta_{\text{1-1}} \geq \mathfrak{e}_g$ *and* $\mathfrak{ros}_{\text{1-1}} \leq \widehat{\mathfrak{e}_g}$ *for every* $g \in \square_{\text{increasing}}$.

Proof. Let $g \in \square_{\text{increasing}}$. Before we begin the proof, we will build a couple of interval partitions, functions and some claims that will help us with the proof. First, pick an interval partition $\{I_n \colon n \in \omega\}$ of ω such that for every $n \in \omega$ we have:

$$|I_n| > 2 \cdot \sum_{j<n} \prod_{i\in I_j} g(i).$$

Let us define a function F by setting $F(n) = \prod_{i\in I_n} g(i)$. By the condition for the value of $|I_n|$ we have

$$|I_n| > 2 \cdot \sum_{j<n} F(j).$$

Let $\mathcal{P} = \{J_n \colon n \in \omega\}$ be the interval partition of ω such that $|J_n| = F(n)$ for every $n \in \omega$, and for each $n \in \omega$ fix a bijection $b_n : J_n \to \prod_{i\in I_n} g(i)$.

For every $f \in \square_{\text{1-1}}$ let us define a function $S_f : \omega \to [\omega]^{<\omega}$ in the following way:

$$S_f(n) = \Bigg(f\Big[\bigcup_{i<n} J_i\Big] \cup f^{-1}\Big[\bigcup_{i<n} J_i\Big]\Bigg) \cap J_n.$$

Note that, by the injectivity of f we have

$$|S_f(n)| \leq 2 \cdot \sum_{i<n} |J_i| = 2 \cdot \sum_{j<n} F(j) < |I_n|.$$

Thus, for each $n \in \omega$, $b_n[S_f(n)]$ is a subset of $\prod_{i\in I_n} g(i)$ of size less than $|I_n|$. We will need the following two observations.

Claim 5.4.4. Let $h \in \prod_{n\in\omega} J_n$ and denote $A = h[\omega]$. Then, for all $f \in \square_{1\text{-}1}$ satisfying $h(n) \notin S_f(n)$ for almost all $n \in \omega$, we have that $f[A] \cap A$ is finite.

Proof of the claim. Assume that $n \in \omega$ is such that $h(n) \in f[A]$. Then, there is $m \neq n$ such that $f(h(m)) = h(n)$, since f does not have a fixed point.

- if $m < n$, then $h(n) \in f[\bigcup_{i<n} J_i]$ and so $h(n) \in S_f(n)$,
- if $n < m$, then $h(m) \in f^{-1}[\bigcup_{i<m} J_i]$ and so $h(m) \in S_f(m)$.

Since $h(i) \in S_f(i)$ only for finitely many $i \in \omega$, the conclusion follows. $\dashv$

Claim 5.4.5. For every $f \in \square_{1\text{-}1}$ there exists $\ell_f \in \prod_{n\in\omega} g(n)$ such that, for all $n \in \omega$ and all $h \in b_n[S_f(n)]$, we have that $(\ell_f \restriction I_n) \cap h \neq \emptyset$.

Proof of the claim. For each $n \in \omega$, since there are more points in I_n than in $b_n[S_f(n)]$, there exists $\ell_f^n \in \prod_{i\in I_n} g(i)$ such that $\ell_f^n \cap h \neq \emptyset$ for all $h \in b_n[S_f(n)]$. Clearly, $\ell_f = \bigcup_{n\in\omega} \ell_f^n$ is a function we are looking for. $\dashv$

We are ready to prove our results. First, we will prove that $\Delta_{1\text{-}1} \geq \mathfrak{e}_g$. Let $\{f_\alpha \colon \alpha < \Delta_{1\text{-}1}\} \subseteq \square_{1\text{-}1}$ be a witness for $\Delta_{1\text{-}1}$. We will see that $\{\ell_{f_\alpha} : \alpha < \Delta_{1\text{-}1}\}$ witnesses $\mathfrak{e}_g$. Let $h \in \prod_{n\in\omega} g(n)$ and denote

$$A = \left\{b_n^{-1}(h \restriction I_n) : n \in \omega\right\}.$$

Clearly, $A \in [\omega]^\omega$, so there is $\alpha < \Delta_{1\text{-}1}$ such that $f_\alpha[A] \cap A$ is infinite. Therefore, by the first claim, there are infinitely many $n \in \omega$ such that $b_n^{-1}(h \restriction I_n) \in S_{f_\alpha}$. For every such n, by the second claim, we have that

$$(\ell_{f_\alpha} \restriction I_n) \cap (h \restriction I_n) \neq \emptyset,$$

and therefore $\ell_{f_\alpha} \cap h$ is infinite.

Finally, we will prove that $\mathfrak{ros}_{1\text{-}1} \leq \hat{\mathfrak{e}_g}$. Let $\kappa < \mathfrak{ros}_{1\text{-}1}$, and let $\{h_\alpha : \alpha \in \kappa\} \subseteq \prod_{n\in\omega} g(n)$. We will find a function $\ell \in \prod_{n\in\omega} g(n)$ such that $\ell \cap h_\alpha$ is infinite for all $\alpha \in \kappa$, so that $\{h_\alpha : \alpha \in \kappa\}$ is not a witness for $\hat{\mathfrak{e}_g}$. Let us define

$$A_\alpha = \left\{b_n^{-1}(h_\alpha \restriction I_n) \colon n \in \omega\right\} \in [\omega]^\omega$$

for each $\alpha \in \kappa$. Then, since $\kappa < \mathfrak{ros}_{1\text{-}1}$, there is $f \in \square_{1\text{-}1}$ such that $f[A_\alpha] \cap A_\alpha$ is infinite for all $\alpha \in \kappa$. By the first claim, for every $\alpha \in \kappa$ there are infinitely many $n \in \omega$ such that $b_n^{-1}(h_\alpha \restriction I_n) \in S_f$. Using the second claim, we conclude that $\ell_f \cap h_\alpha$ is infinite for every $\alpha \in \kappa$. $\square$

For our final result of this section, we will need to recall the following widely-known result, which we prove for the sake of completeness of the thesis.

Theorem 5.4.6. $\max\{\mathfrak{d}, \sup_{g\in\omega^\omega} \mathfrak{e}_g\} = \mathrm{cof}(\mathcal{M})$ *and* $\min\{\mathfrak{b}, \min_{g\in\omega^\omega} \widehat{\mathfrak{e}_g}\} = \mathrm{add}(\mathcal{M})$.

Proof. By Miller's theorem [81, Theorem 1.2] (see also [12, Theorem 5.6]),

$$\max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\} = \mathrm{cof}(\mathcal{M}) \text{ and } \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} = \mathrm{add}(\mathcal{M}).$$

Thus, it is enough to prove that $\max\{\mathfrak{d}, \sup(\mathfrak{e}_g)\} \geq \mathrm{non}(\mathcal{M})$ and $\min\{\mathfrak{b}, \min(\widehat{\mathfrak{e}_g})\} \leq \mathrm{cov}(\mathcal{M})$. We are going to use the characterizations of $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$ as $\mathfrak{d}(\mathcal{E})$ and $\mathfrak{b}(\mathcal{E})$ mentioned at the beginning of this section.

$(\max\{\mathfrak{d}, \sup(\mathfrak{e}_g)\} \geq \mathfrak{d}(\mathcal{E}))$. Let $\{f_\alpha \colon \alpha < \mathfrak{d}\}$ be a dominating family in ω^ω consisting of strictly increasing functions. For every $\alpha < \mathfrak{d}$, pick a witness $\mathcal{F}_\alpha \subseteq \prod_{n\in\omega} f_\alpha(n)$ for $\mathfrak{e}_{f_\alpha}$. Clearly, $\bigcup_{\alpha<\mathfrak{d}} \mathcal{F}_\alpha$ is a witness for $\mathfrak{d}(\mathcal{E})$ of the required size.

$(\min\{\mathfrak{b}, \min(\widehat{\mathfrak{e}_g})\} \leq \mathfrak{b}(\mathcal{E}))$. Let $\kappa < \min\{\mathfrak{b}, \mathfrak{ros}_{1\text{-}1}\}$ and let $\{f_\alpha \colon \alpha < \kappa\} \in \omega^\omega$. Since $\kappa < \mathfrak{b}$, there exists a strictly increasing $g \in \omega^\omega$ such that $f_\alpha \leq^* g$ for every $\alpha < \kappa$, and so for each $\alpha < \kappa$ there is a finite modification f'_α of f_α satisfying $f'_\alpha \in \prod_{n\in\omega} g(n)$. As $\kappa < \widehat{\mathfrak{e}_g}$, there is $h \in \prod_{n\in\omega} g(n)$ such that $f'_\alpha =^\infty h$ for every $\alpha < \kappa$, thus $f_\alpha =^\infty h$ for every $\alpha < \kappa$ and therefore $\{f_\alpha \colon \alpha < \kappa\}$ is not an $\mathcal{E}$-unbounded family. □

As a corollary, we get the following result connecting $\Delta_{1\text{-}1}$ with $\mathrm{cof}(\mathcal{M})$, and $\mathfrak{ros}_{1\text{-}1}$ with $\mathrm{add}(\mathcal{M})$, respectively.

Corollary 5.4.7. $\max\{\mathfrak{d}, \Delta_{1\text{-}1}\} = \mathrm{cof}(\mathcal{M})$ *and* $\min\{\mathfrak{b}, \mathfrak{ros}_{1\text{-}1}\} = \mathrm{add}(\mathcal{M})$.

Proof. It follows from Theorems 5.3.5, 5.4.3 and 5.4.6. □

Given an interval partition $\mathcal{J} = \{J_n \colon n \in \omega\}$, and a sequence $\bar{\mu} = \{\mu_n : n \in \omega\}$ such that, for each $n \in \omega$, μ_n is a measure on J_n, its value on singletons is at most 1 and $\mu_n(J_n) \geq n$ (thus in particular $|J_n| \geq n$), the ideal $\mathcal{ED}_{\bar{\mu}}$ is defined in the following way:

$$\mathcal{ED}_{\bar{\mu}} = \big\{A \subseteq \omega \colon \exists k \in \omega\, \forall n \in \omega\, \big(\mu_n(J_n \cap A) \leq k\big)\big\}.$$

One can easily see that all the ideals of the form $\mathcal{ED}_{\bar{\mu}}$ are tall and $\mathbb{F}_\sigma$. In the literature, the ideal $\mathcal{ED}_{\mathrm{Fin}}$ is defined as an ideal on $\{\langle m, n\rangle : m \leq n\}$ generated by the graphs of functions of ω^ω. Clearly ideal $\mathcal{ED}_{\mathrm{Fin}}$ is of the form $\mathcal{ED}_{\bar{\mu}}$, where J_n is a set of size n and μ_n is the counting measure on J_n for every $n \in \omega$. In [55, Proposition 3.6], the authors proved the following characterization of $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$.

Theorem 5.4.8 (Hrušák, Meza, Minami)**.** $\mathrm{cov}(\mathcal{M}) = \min\{\mathfrak{d}, \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})\}$ *and* $\mathrm{non}(\mathcal{M}) = \max\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})\}$.

Since $\mathfrak{ros}_{1\text{-}1} \leq \mathfrak{d}$ and $\Delta_{1\text{-}1} \geq \mathfrak{b}$, the question whether $\mathfrak{ros}_{1\text{-}1} = \mathrm{cov}(\mathcal{M})$ and $\Delta_{1\text{-}1} = \mathrm{non}(\mathcal{M})$ reduces to find out if $\mathfrak{ros}_{1\text{-}1} \leq \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$ and $\Delta_{1\text{-}1} \geq \mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})$. We were not able to prove it for $\mathcal{ED}_{\mathrm{Fin}}$, but we were able to prove those inequalities for some other ideal of the form $\mathcal{ED}_{\bar{\mu}}$ (see also Question 5.6.3).

Theorem 5.4.9. *There is an ideal of the form* $\mathcal{ED}_{\bar{\mu}}$ *such that* $\Delta_{1\text{-}1} \geq \mathrm{cov}^*(\mathcal{ED}_{\bar{\mu}})$ *and* $\mathfrak{ros}_{1\text{-}1} \leq \mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})$.

Proof. Fix an interval partition $\{J_n : n \in \omega\}$ of ω such that $|J_0| = 1$ and for every $n \in \omega$:

$$|J_{n+1}| = 2(n+1) \cdot \sum_{k\leq n} |J_k|.$$

For every $n \in \omega$, define a measure μ_n on J_n by setting $\mu_0(J_0) = 0$ and

$$\mu_{n+1}(\{i\}) = \frac{1}{\sum_{k\leq n} |J_k|} \text{ for any } n \in \omega \text{ and } i \in J_{n+1}.$$

By the assumption about values of $|J_n|$, we have $\mu_{n+1}(J_{n+1}) = 2(n+1)$ for every $n \in \omega$. We will show that, if $\bar{\mu} = \langle \mu_n : n \in \omega\rangle$, then $\mathcal{ED}_{\bar{\mu}}$ is the ideal we are looking for. First, we will need the following claims:

Claim 5.4.10. For every $f \in \square_{1\text{-}1}$, we have $B_f \in \mathcal{ED}_{\bar{\mu}}$, where

$$B_f = \bigcup_{n\in\omega} \Big(J_n \cap \bigcup_{k<n} \left(f[J_k] \cup f^{-1}[J_k] \right) \Big).$$

Proof of the Claim. This follows from the following calculations for any $n > 0$:

$$\mu_n(B_f \cap J_n) = \mu_n \Big(J_n \cap \bigcup_{k<n} \left(f[J_k] \cup f^{-1}[J_k] \right) \Big) =$$

$$= \frac{|J_n \cap \bigcup_{k<n} \left(f[J_k] \cup f^{-1}[J_k] \right)|}{\sum_{k<n} |J_k|} \leq \frac{2 \cdot \sum_{k<n} |J_k|}{\sum_{k<n} |J_k|} \leq 2.$$

$\dashv$

Claim 5.4.11. If $X \subseteq \omega$ is such that $|X \cap J_n| \leq 1$ for every $n \in \omega$ and $X \cap B_f$ is finite, then $f[X] \cap X$ is finite.

Proof. This proof is similar to the first claim from the proof of Theorem 5.4.3. We may assume that $|X \cap J_n| = 1$ for every $n \in \omega$. Let $\{x_k : k \in \omega\}$ be an enumeration of X such that $x_k \in J_k$. Let $N \in \omega$ be such that $X \cap B_f \subseteq N$. If $m, n \in \omega$ are such that $x_m, x_n > N$, then:

- if $x_m < x_n$ then $f(x_m) \in f[\bigcup_{i<n} J_i]$. Since $x_n \in J_n \setminus B_f$, we have $x_n \neq f(x_m)$,
- if $x_m > x_n$ then $f^{-1}(x_n) \in f^{-1}[\bigcup_{i<m} J_i]$. Since $x_m \in J_m \setminus B_f$, we have $x_n \neq f(x_m)$.

Therefore, $x_n \neq f(x_m)$ for any $x_m, x_n > N$, and so $f[X] \cap X$ is finite. $\dashv$

We will now continue with the proof of the theorem.

($\Delta_{1\text{-}1} \geq \mathrm{cov}^*(\mathcal{ED}_{\bar{\mu}})$). Let $\{f_\alpha \colon \alpha < \Delta_{1\text{-}1}\} \subseteq \square_{1\text{-}1}$ be a witness for $\Delta_{1\text{-}1}$. For the sake of contradiction, assume that $\{B_{f_\alpha} : \alpha \in \kappa\} \subseteq \mathcal{ED}_{\bar{\mu}}$ is not a witness for $\mathrm{cov}^*(\mathcal{ED}_{\bar{\mu}})$. Then, there exists $X \in [\omega]^\omega$ such that $X \cap B_{f_\alpha}$ is finite for each $\alpha < \kappa$. By shrinking X, we may assume that $|X \cap J_n| \leq 1$. Then, by the second Claim, $f_\alpha[X] \cap X$ is finite for every $\alpha \in \kappa$, which contradicts the definition of $\Delta_{1\text{-}1}$.

($\mathfrak{ros}_{1\text{-}1} \leq \mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})$). Let $\{X_\alpha : \alpha \in \mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})\}$ be a witness for $\mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})$. By shrinking each X_α, we may assume that, for each $\alpha \in \mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})$ and each $n \in \omega$, $|X_\alpha \cap J_n| \leq 1$. We will show that $\{X_\alpha : \alpha \in \mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})\}$ is a witness for $\mathfrak{ros}_{1\text{-}1}$: Let $f \in \square_{1\text{-}1}$. Then $B_f \in \mathcal{ED}_{\bar{\mu}}$, so there is α such that $B_f \cap X_\alpha$ is finite. Then, by the second claim, $f[X_\alpha] \cap X_\alpha$ is finite. $\square$

We would like to point out that for every ideal of the form $\mathcal{ED}_{\bar{\mu}}$, we have $\mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}}) \leq \mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})$ and $\mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}}) \geq \mathrm{cov}^*(\mathcal{ED}_{\bar{\mu}})$, which follow from Proposition 2.6.16 and the reduction $\mathcal{ED}_{\mathrm{Fin}} \leq_{KB} \mathcal{ED}_{\bar{\mu}}$ (a more general version of this statement can be found in [55, Theorem 3.3]). This raises Question 5.6.3.

5.5 Consistency results

The final section of this chapter will be dedicated to prove consistency results about all the cardinal invariants that we used in this chapter. We will start by looking at an equivalent variant of having the Laver property.

Definition 5.5.1. Let V be the ground model. A forcing $\mathbb{P} \in V$ has the *Laver property* if, for every $\mathbb{P}$-name $\dot{g}$ for a function from ω to ω such that there is $f \in \omega^\omega \cap V$ satisfying

$$\mathbb{P} \Vdash \text{"}\dot{g} \leq f\text{"},$$

there is a function $S \in [[\omega]^{<\omega}]^\omega \cap V$ with domain ω such that, for all $n \in \omega$, we have $|S(n)| \leq n+1$, all elements of $S(n)$ are bounded by $f(n)$ and

$$\mathbb{P} \Vdash \text{"}\dot{g}(n) \in S(n)\text{"}.$$

It is well-known that the Laver property is preserved under countable support iterations of proper forcings (see Bartoszyński, Judah [11, Theorem 6.3.34]). Our next result is connected with behaviour of $\mathfrak{ros}_{1\text{-}1}$ in the extensions of forcings that have the Laver property. By Theorem 5.4.3, it is enough to look at $\widehat{\mathfrak{e}_g}$. The following implicitly appears in Kada's Ph.D. thesis [59] (see also [60]).

Proposition 5.5.2. *Assume that the ground model V satisfies CH and that $\mathbb{P}$ is a forcing that has the Laver property. Let $g \in \square_{\text{increasing}}$. Then,*

$$\mathbb{P} \Vdash \text{"}V \cap \prod_{n\in\omega} g(n) \text{ witnesses both } \mathfrak{e}_g \text{ and } \widehat{\mathfrak{e}_g}\text{"}.$$

In particular, $\mathbb{P} \Vdash$ "$\mathfrak{e}_g = \widehat{\mathfrak{e}_g} = \aleph_1$".

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name for a function in $V \cap \prod_{n\in\omega} g(n)$. We will show that there are $\hat{f}, \bar{f} \in \prod_{n\in\omega} g(n)$ such that $\mathbb{P} \Vdash$ "$\hat{f} =^\infty \dot{f}$ and $\bar{f} \cap \dot{f}$ is finite".

First, let $\{I_n : n \in \omega\}$ be an interval partition of ω such that $|I_n| \geq n+1$ for all $n \in \omega$. Let $\dot{F}$ be a $\mathbb{P}$-name for a function from ω to $\omega^{<\omega}$ satisfying

$$\mathbb{P} \Vdash \text{"}\dot{F}(n) = \dot{f} \restriction I_n\text{"}$$

for each $n \in \omega$. By a countable enumeration of $\omega^{<\omega}$, we can use the Laver property of $\mathbb{P}$ for $\dot{F}$, since

$$\mathbb{P} \Vdash \text{"}\dot{F}(n) \in \prod_{m\in I_n} g(m)\text{"}$$

for all $n \in \omega$, where $\prod_{m\in I_n} g(m)$ is a finite set in V, and so $\dot{F}$ is bounded by a ground model function. Therefore, there exists a function $S \in V$ with domain ω such that, for every $n \in \omega$, $S(n)$ is a family of functions in $\prod_{m\in I_n} g(m)$ satisfying $\mathbb{P} \Vdash$ "$\dot{F}(n) \in S(n)$" and $|S(n)| \leq n+1$. For each $n \in \omega$, since $|S(n)| \leq |I_n|$, there exists a function $\hat{f}_n \in \prod_{m\in I_n} g(m)$ such that $\hat{f}_n \cap h \neq \emptyset$ for every $h \in S(n)$, and so

$$\mathbb{P} \Vdash \text{"}\hat{f}_n \cap \dot{F}(n) \neq \emptyset\text{"}.$$

Thus, if we define $\hat{f} = \bigcup_{n\in\omega} \hat{f}_n$, then $\hat{f} \in \prod_{n\in\omega} g(n)$ and $\mathbb{P} \Vdash$ "$\hat{f} =^\infty \dot{f}$", which finishes the first part of the proof.

To find the function $\bar{f} \in V \cap \prod_{n\in\omega} g(n)$, we use the Laver property of $\mathbb{P}$ for $\dot{h}$—a $\mathbb{P}$-name for the translated $\dot{f}$, satisfying

$$\mathbb{P} \Vdash \text{“}\dot{h}(n) = \dot{f}(n+1)\text{”}$$

for each $n \in \omega$ ($\dot{h}$ is bounded by the ground model function $n \mapsto g(n+1)$). So, there is a function $S \in [[\omega]^{<\omega}]^\omega \cap V$ such that, for all $n \in \omega$, $|S(n)| \le n+1$ and $\mathbb{P} \Vdash \text{“}\dot{h}(n) \in S(n)\text{”}$. Since

$$|S(n)| \le n+1 < g(n+1)$$

for every $n \in \omega$, there exists a function $\bar{f} \in V \cap \prod_{n\in\omega} g(n)$ such that, for each $n \in \omega$, $\bar{f}(n+1) \notin S(n)$. Therefore $\mathbb{P} \Vdash \text{“}\bar{f} \cap \dot{f}$ is finite”, which finishes the proof. □

For our next results, we will need the following widely known class of forcings.

Definition 5.5.3. Let $\mathcal{X}$ be either a filter on ω, or $[\omega]^\omega$. The *Mathias forcing of* $\mathcal{X}$ is the following:

$$\mathbf{M}(\mathcal{X}) = \left\{ \langle s, A\rangle : s \in [\omega]^{<\omega}, A \in \mathcal{X}, \max(s) < \min(A) \right\},$$

and the order is defined by $\langle s, A\rangle \le \langle s', A'\rangle$ if and only if $s' \subseteq s$, $A \subseteq A'$ and $s \setminus s' \subseteq A'$. $\mathbf{M}([\omega]^\omega)$ is known simply as the *Mathias forcing* and is denoted by $\mathbf{M}$.

Notice that, for all filters $\mathcal{F}$, $\mathbf{M}(\mathcal{F})$ is a σ-centered forcing notion, thus ccc. $\mathbf{M}$ is not ccc, but it is proper and has the Laver property (see e.g. Halbeisen [48, Corollaries 24.7–24.8]).

We are ready to show our first consistency result.

Theorem 5.5.4. *It is consistent with ZFC that* $\mathfrak{ros}_{1\text{-}1} < \min\{\mathfrak{d}, \mathfrak{r}, \mathrm{non}(\mathcal{N})\}$.

Proof. We force with $\mathbf{M}_{\omega_2}$, the countable support iteration of length ω_2 of the Mathias forcing, over a model of CH. By the properness of $\mathbf{M}_{\omega_2}$ (which is preserved by such an iteration by Shelah's [11, Theorem 6.1.4]) and CH in the ground model, all cardinals are preserved. Since the Mathias forcing is proper and has the Laver property, $\mathbf{M}_{\omega_2}$ also has the Laver property, thus $\mathbf{M}_{\omega_2} \Vdash \text{“}\mathfrak{ros}_{1\text{-}1} \le \widehat{e}_g = \aleph_1\text{”}$ by Proposition 5.5.2. On the other hand, one can easily show that the generic real for Mathias forcing is reaping and codes a dominating function, and so

$$\mathbf{M}_{\omega_2} \Vdash \text{“}\aleph_2 = \mathfrak{b} \le \min\{\mathfrak{r}, \mathfrak{d}\} = 2^{\aleph_0}\text{”}$$

and

$$\mathbf{M}_{\omega_2} \Vdash \text{“}\aleph_2 = \mathfrak{s} \le \mathrm{non}(\mathcal{N}) = 2^{\aleph_0}\text{”}.$$

Alternatively, one can check the values in the Mathias model in [11, Model 7.6.11]. □

Let us note the values of some other considered invariants in the Mathias model. By Proposition 5.5.2, we have that, in the extension, $\mathfrak{e}_g = \aleph_1$, and by [55, Lemma 4.4], it holds that $\mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}}) = \aleph_1$. The value of $\mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})$ and Δ_1 is $\aleph_2$ in the Mathias model, since they are both bounded below by $\mathfrak{s} = \aleph_2$ (we have $\mathfrak{s} \le \mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})$ by combining Proposition 0.1, Theorem 2.1 and Theorem 3.3 from [55]).

Our next consistency result is the following.

Theorem 5.5.5. *It is consistent with ZFC that* $\Delta_{1\text{-}1} > \max\{\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{N})\}$.

Proof. We start with a model V of CH, and consider $\mathbf{P}_{\omega_2}$, the finite support iteration of length ω_2 of the forcing $\mathbf{M}(\mathcal{F})$ with $\mathcal{F} = \mathcal{I}^*$, where $\mathcal{I}$ is the $\mathbb{F}_\sigma$ ideal obtained in Theorem 5.4.9. Since $\mathbf{M}(\mathcal{F})$ is ccc, $\mathbf{P}_{\omega_2}$ is also ccc (see e.g. [11, Theorem 1.5.12]), and so it preserves all cardinals. By Theorem 5.4.9 and the basic property of the $\mathbf{M}(\mathcal{F})$-generic set, we have

$$\mathbf{P}_{\omega_2} \Vdash \text{“}\aleph_2 \leq \mathrm{cov}^*(\mathcal{I}) \leq \Delta_{\text{1-1}}\text{”}$$

(cf. [52, Theorem 4.5.(ii)]). We only have to argue that

$$\mathbf{P}_{\omega_2} \Vdash \text{“}\aleph_1 = \mathfrak{b} = \mathfrak{s} = \mathrm{cov}(\mathcal{N})\text{”} :$$

- $\mathbf{M}(\mathcal{F})$ for an $\mathbb{F}_\sigma$ filter $\mathcal{F}$ preserves well-ordered unbounded families (see either Brendle [17, Theorem 3.1(Case 1)] or Guzmán-González *et al.* [45, Proposition 5]), and so $\mathbf{P}_{\omega_2}$ does not add *dominating reals*, i.e. $V \cap \omega^\omega$ is unbounded in the extension (see Bartoszyński, Judah [11, Lemma 6.5.7]), thus $\mathbf{P}_{\omega_2} \Vdash \text{“}\mathfrak{b} = \aleph_1\text{”}$,
- $\mathbf{M}(\mathcal{F})$ is a ccc Souslin forcing, as defined by Ihoda and Shelah [56] (since $\mathcal{F}$ is $\mathbb{F}_\sigma$), and so the finite support iteration $\mathbf{P}_{\omega_2}$ does not add reaping reals, i.e. $V \cap [\omega]^\omega$ is a splitting family in the extension (see [56, Claim 3.6]), thus $\mathbf{P}_{\omega_2} \Vdash \text{“}\mathfrak{s} = \aleph_1\text{”}$,
- $\mathbf{M}(\mathcal{F})$ is a σ-centered, and so the finite support iteration $\mathbf{P}_{\omega_2}$ does not add random reals, i.e. $\mathcal{N} \cap V$ covers 2^ω (see Bartoszyński, Judah [11, Theorems 6.5.29 and 6.5.30]), so $\mathbf{P}_{\omega_2} \Vdash \text{“}\mathrm{cov}(\mathcal{N}) = \aleph_1\text{”}$.

By all of this, it follows that $\mathbf{P}_{\omega_2} \Vdash \text{“}\Delta_{\text{1-1}} > \max\{\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{N})\}\text{”}$, so our proof is complete. $\square$

We would like to point out some remarks about the previous model. Since we are forcing with finite support iteration, Cohen reals are always added, so $\mathrm{cov}(\mathcal{M}) = \aleph_2$ in the extension, and therefore

$$\mathbf{P}_{\omega_2} \Vdash \text{“}\mathfrak{ros}_{\text{1-1}} = \widehat{\mathfrak{e}_g} = \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}}) = \aleph_2\text{”}.$$

The fact that $\mathbf{P}_{\omega_2} \Vdash \text{“}\mathfrak{e}_g = \aleph_1\text{”}$ follows from a general preservation theorem (see Cardona, Mejía [19, Theorem 4.16 and Example 4.17.(2)]).

We finish this chapter by showing the consistency of $\Delta_{\text{1-1}} > \mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})$ and the consistency of $\mathfrak{ros}_{\text{1-1}} < \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$. To achieve this, we will take a look at a slight variation of Hechler's forcing (the definition

$$\mathbf{D} = \Big\{T \subseteq \omega^{<\omega} \colon T \text{ is a tree with stem } s, \text{ i.e. } s \subseteq t \text{ or } t \subseteq s \text{ for all } t \in T, \\ \text{and for every } t \in T, \text{ if } s \subseteq t, \text{ then } \{n \in \omega : t^\frown\langle n\rangle \notin T\} \text{ is finite}\Big\}.$$

One can easily show that $\mathbf{D}$ is a σ-centered forcing notion and that the generic real is a dominating real. Again by ccc, all cardinals in are preserved by the finite support iterations of $\mathbf{D}$. We will need the following theorem about the iterations of $\mathbf{D}$, which is a direct consequence of Theorem 12 and Lemma 16 in [18].

Theorem 5.5.6 (Brendle, Löwe). *Let $\mathbf{P}_\lambda$ be the finite support iteration of length λ of the forcing $\mathbf{D}$. Then, for every $\mathbf{P}_\lambda$-name of an infinite partial function $\dot{f}$ such that*

$$\mathbf{P}_\lambda \Vdash \text{“}\forall n \in \mathrm{dom}(f)\ (\dot{f}(n) \leq n)\text{”},$$

there is a sequence $\{f_n : n \in \omega\}$ of partial functions such that $f_n(m) \leq m$ for all $m, n \in \omega$, and for every $y \in \omega^\omega \cap V$, if $f_n \cap y$ is infinite for all $n \in \omega$, then $\mathbf{P}_\lambda \Vdash \text{“}y \cap \dot{f}$ is infinite”.

We will also need the following result, which is a direct consequence of Lemma 3.9 in [55], and gives an equivalent characterization of $\mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$.

Proposition 5.5.7 (Hrušák, Meza, Minami)**.** *If $\kappa < \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$, and $\mathcal{F}$ is a family of size κ of infinite partial functions such that every $f \in \mathcal{F}$ is bounded by $g(n) = n$ on its domain, then there exists a function $y \in \omega^\omega$ such that $f \cap y$ is infinite for all $f \in \mathcal{F}$.*

We are ready to show the last result of this chapter.

Theorem 5.5.8. *Both $\Delta_{1\text{-}1} > \mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})$ and $\mathfrak{ros}_{1\text{-}1} < \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$ are consistent with ZFC.*

Proof. (Consistency of $\Delta_{1\text{-}1} > \mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})$): We start with a model V of CH and we force with $\mathbf{P}_{\omega_2}$, the finite support iteration of length ω_2 of the forcing $\mathbf{D}$. Since $\mathbf{D}$ adds dominating real, a standard reflection argument shows that $\mathbf{P}_{\omega_2} \Vdash$ "$\aleph_2 \le \mathfrak{b} \le \Delta_{1\text{-}1}$", and so it is enough to show that

$$\mathbf{P}_{\omega_2} \Vdash \text{“}\mathcal{ED}_{\mathrm{Fin}} \cap V \text{ witnesses } \mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}})\text{”}.$$

Let $\dot{X}$ be a $\mathbf{P}_{\omega_2}$-name for an infinite subset of $\{\langle n, m\rangle \in \omega \times \omega \colon n \le m\}$. Without loss of generality, we can assume that

$$\mathbf{P}_{\omega_2} \Vdash \text{“}\dot{X} \text{ is an infinite partial function”}.$$

By Theorem 5.5.6, there is a sequence $\{f_n \colon n \in \omega\}$ of partial functions such that, for every $y \in \omega^\omega \cap V$, if $f_n \cap y$ is infinite for all $n \in \omega$, then $\mathbf{P}_{\omega_2} \Vdash$ "$y \cap \dot{X}$ is infinite". Since $\mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$ is uncountable (by Theorem 5.4.8), by Proposition 5.5.7 there is a function $y \in \mathcal{ED}_{\mathrm{Fin}}$ such that $f_n \cap y$ is infinite for all $n \in \omega$. Then, we have $\mathbf{P}_{\omega_2} \Vdash$ "$y \cap \dot{X}$ is infinite", and so we proved that $\mathbf{P}_{\omega_2} \Vdash$ "$\mathrm{cov}^*(\mathcal{ED}_{\mathrm{Fin}} = \aleph_1)$".

(Consistency $\mathfrak{ros}_{1\text{-}1} < \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$): We start with a model V of Martin's Axiom and $2^{\aleph_0} = \aleph_2$, and we force with $\mathbf{P}_{\omega_1}$, the finite support iteration of length ω_1 of the forcing $\mathbf{D}$. If $\{\dot{d}_\alpha \colon \alpha \in \omega_1\}$ is the collection of dominating reals added on each stage of the iteration, then

$$\mathbf{P}_{\omega_1} \Vdash \text{“}\{\dot{d}_\alpha \colon \alpha \in \omega_1\} \text{ is a witness for } \mathfrak{d}\text{”}$$

and therefore $\mathbf{P}_{\omega_1} \Vdash$ "$\mathfrak{ros}_{1\text{-}1} \le \mathfrak{d} = \aleph_1$", so we only need to show that

$$\mathbf{P}_{\omega_1} \Vdash \text{“}\mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}}) \ge \aleph_2\text{”}.$$

Let $\{\dot{X}_\alpha \colon \alpha \in \omega_1\}$ be a sequence of $\mathbf{P}_{\omega_1}$-names for infinite subsets of $\{\langle n, m\rangle \in \omega \times \omega \colon n \le m\}$. Without loss of generality, we may assume that

$$\mathbf{P}_{\omega_1} \Vdash \text{“}\dot{X}_\alpha \text{ is an infinite partial function”}$$

for each $\alpha \in \omega_1$. Then, we can use Theorem 5.5.6 to find a family $\{f_n^\alpha \colon \alpha \in \omega_1, n \in \omega\}$ of partial functions such that $f_n^\alpha(m) \le m$, and whenever $y \in \omega^\omega \cap V$ and $\alpha \in \omega_1$ are such that $f_n^\alpha \cap y$ is infinite for all $n \in \omega$, then $\mathbf{P}_{\omega_1} \Vdash$ "$y \cap \dot{X}_\alpha$ is infinite". Since in V we have $\omega_1 < \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$ (as Martin's Axiom holds in V, which imply $2^{\aleph_0} = \mathrm{cov}(\mathcal{M}) \le \mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$, see e.g. Blass [12, Corollary 7.4]), by Proposition 5.5.7 there exists a function $y \in \mathcal{ED}_{\mathrm{Fin}}$ such that $f_n^\alpha \cap y$ is infinite for each $\alpha \in \omega_1$ and $n \in \omega$. Therefore, we have

$$\mathbf{P}_{\omega_1} \Vdash \text{“}\forall \alpha < \omega_1\ (y \cap \dot{X}_\alpha \text{ is infinite})\text{”},$$

i.e. $\mathbf{P}_{\omega_1} \Vdash$ "$\{\dot{X}_\alpha \colon \alpha \in \omega_1\}$ is not a witness for $\mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}})$", and so we proved that $\mathbf{P}_{\omega_1} \Vdash$ "$\mathrm{non}^*(\mathcal{ED}_{\mathrm{Fin}}) \ge \aleph_2$". $\square$

An alternative proof of Theorem 5.5.8 can be found in Meza-Alcántara's Ph.D. thesis [79, Theorem 1.6.12].

5.6 Open questions

After the results of this chapter were published, Brech, Brendle and Telles proved in [16, Corollary 14] that $\Delta_{\text{1-1}} = \mathrm{non}(\mathcal{M})$ and that $\mathfrak{ros}_{\text{1-1}} = \mathrm{cov}(\mathcal{M})$, which answers the main open problem stemming from of our work, about the values of $\Delta_{\text{1-1}}$ and $\mathfrak{ros}_{\text{1-1}}$.

The other open questions consider the values of $\Delta_{\mathbf{1}}$ and of $\Delta_{\text{Fin-1}}$.

Question 5.6.1. *Is it true that* $\mathfrak{s} = \Delta_{\mathbf{1}} = \mathfrak{s}_{\text{mix}}$*?*

Question 5.6.2. *Is it true that* $\max\{\mathfrak{b}, \mathfrak{s}\} = \Delta_{\text{Fin-1}} = \max\{\mathfrak{b}, \mathfrak{s}_{\text{mix}}\}$*?*

The last question asks about the possibility that Theorem 5.4.9 already proves the result from [16].

Question 5.6.3. *Is it consistent that* $\mathrm{non}^*(\mathcal{ED}_{\text{Fin}}) < \mathrm{non}^*(\mathcal{ED}_{\bar{\mu}})$ *or* $\mathrm{cov}^*(\mathcal{ED}_{\text{Fin}}) > \mathrm{cov}^*(\mathcal{ED}_{\bar{\mu}})$ *for the sequence* $\bar{\mu}$ *obtained in Theorem 5.4.9?*

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