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Combinatorics of asymptotic representation theory of the symmetric groups

ABSTRACT

We investigate the structure of the Kerov characters polynomial, which is one of the main tools in the asymptotic representation theory of the symmetric groups. Description of this structure involves a connection to bipartite maps and investigation of their combinatorial structure. The structure of Jack deformation of Kerov character polynomials is also investigated in this dissertation. In particular, the proofs of conjectures of Lassalle describing this structure are presented. Moreover, several applications of these results, concerning behaviour of the random Young diagrams under Jack measure, are shown. The structure of this dissertation is the following.

Chapter 1 is a short introduction to the main subject of this dissertation.

In Chapter 2 we provide the whole background for the problems which appear in this thesis. In particular, in this chapter the reader can find the history of the problems as well as all necessary definitions.

In Chapter 3 we study asymptotics of characters of the symmetric groups on a fixed conjugacy class. It was proved by Kerov that such a character can be expressed as a polynomial in free cumulants of the Young diagram (certain functionals describing the shape of the Young diagram). We show that for each genus there exists a universal symmetric polynomial which gives the coefficients of the part of Kerov character polynomials with the prescribed homogeneous degree. The existence of such symmetric polynomials was conjectured by Lassalle [Las08a]. This result was published in the paper [DŚ12].

In Chapter 4 we consider a deformation of Kerov character polynomials, linked to Jack symmetric functions. It has been introduced recently by Lassalle, who formulated several conjectures on these objects, suggesting some underlying combinatorics. We give a partial result in this direction, showing that some quantities describing the structure of Kerov polynomials are polynomials with prescribed degree in the Jack parameter α. As a consequence we prove some of the conjectures of Lassalle. Our result has several interesting consequences in various

directions. Firstly, we give a new proof of the fact that the coefficients of Jack polynomials expanded in the monomial or power-sum basis depend polynomially in α. Secondly, a small part of Matching Jack conjecture from Goulden and Jackson is proved. Finaly, the last and main consequence is a proof of Law of Large Numbers and Central Limit Theorem for random Young diagrams under Jack measure, which is a one-parameter deformation of Plancherel measure. This result is a generalization of celebrated Vershik-Kerov's limit shape and Kerov's Central Limit Theorem and is proved using multivariate Stein's method. Some parts of this chapter can be found in the preprint [DF12].

In the last Chapter 5 we study combinatorial structure of Jack characters. We conjecture existence of a weight on maps (i.e., graphs drawn on surfaces), allowing to express Jack characters as weighted sums of some simple functions indexed by maps. We provide a candidate for this weight which gives a positive answer to our conjecture in some, but unfortunately not all, cases. This candidate weight measures non-orientability of a given map. We also show how our conjecture implies some of the conjectures of Lassalle. This chapter can be found as a preprint [DFŚ13].

References

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