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Prof. Adam Kwela
Institute of Mathematics
University of Gdańsk

Report on the PhD thesis of Tomasz Żuchowski

The PhD thesis by Tomasz Żuchowski, MSc, entitled "On sequences of measures and ideals in Boolean algebras" is 109 pages long and consists of five chapters (Introduction, Preliminaries, and three chapters containing the results), a bibliography, and an index.

The dissertation investigates the interplay between sequences of measures on Boolean algebras and the theory of ideals and filters on the natural numbers. The main focus is on the Nikodym and Grothendieck properties of Boolean algebras, which originate in measure theory and Banach space theory, and on their connections with the combinatorial structure of ideals. The thesis combines methods from set theory, topology, measure theory, and infinitary combinatorics to develop new characterizations of these properties, construct new examples of Boolean algebras with prescribed measure-theoretic behavior, and study related cardinal invariants.

Chapter 3 introduces and studies the fs-Nikodym property for filters on the natural numbers, motivated by topological spaces associated with free filters. The chapter establishes several equivalent characterizations of this property, revealing close connections with lower semicontinuous submeasures, summable ideals, and density ideals. It also investigates the class of ideals corresponding to filters without the fs-Nikodym property, analyzes its structure with respect to the Katětov and Tukey orders, compares it with other well-known classes of ideals, and provides numerous examples illustrating the richness of the resulting hierarchy.

Chapter 4 explores the relationship between the Nikodym and Grothendieck properties of Boolean algebras and investigates Boolean algebras generated by ideals. The chapter develops new techniques based on Nikodym concentration points. As a major application, it constructs large families of pairwise non-isomorphic Boolean algebras that possess the Nikodym property but fail the Grothendieck property, including examples of low descriptive complexity as well as non-Borel examples.

Chapter 5 investigates cardinal invariants related to Rosenthal families, motivated by the combinatorial version of Rosenthal's lemma on sequences of measures. It introduces two general schemes of cardinal invariants associated with classes of functions on the natural numbers and studies their relationships with classical cardinal characteristics of the continuum. The chapter establishes new upper and lower bounds, identifies several invariants with previously studied notions from large-scale topology, and proves consistency results showing that these invariants can consistently be strictly separated from the corresponding classical cardinal characteristics.

All proofs appear to be correct. Furthermore, the work contains new and interesting results – the considerations contained therein are motivated by and closely related to problems studied in the last few years by leading mathematicians from around the world. This is confirmed by the fact that most of the results of the dissertation have already been published by Tomasz Żuchowski and his supervisors in prestigious peer-reviewed journals. Personally, due to my scientific interests, I particularly appreciate the results of chapters 3 and 5. It is important to emphasize the thematic diversity of the thesis, which covers topics such as measure theory, Boolean algebras, ideals on ω, and cardinal invariants. This necessitates the use of a wide range of proof techniques, which is a major strength of the dissertation.

The weaknesses of the thesis are its excessive length and thematic diversity (however, the latter is also an advantage, as described in the previous paragraph). Perhaps the thesis would be more enjoyable to read in a somewhat shortened form. In particular, this would avoid introducing some concepts that the reader will have forgotten by the time they are needed in the proofs. If the Author had decided to include such extensive material in the thesis, some of the information from the Preliminaries could have been transferred to later chapters, for example, at the beginning of each chapter the Author could add a short subsection introducing the information necessary for that chapter (another option would have been to leave the Preliminaries in their current form and briefly review the necessary information at the beginning of each chapter, but this would have made the thesis even longer).

Below I list several minor errors and inaccuracies that I noticed in the thesis.

- First line of page 2: "where by Fr is the".
- Theorem 2.5.1 and Definition 2.5.6: The word "every" appears twice.
- Lemma 2.6.3: The word "be" appears twice. Moreover, this Lemma requires some additional assumptions: if $A_i = \{a\}$ and $\varphi_i(\{a\}) = i$ for all $i \in \omega$, then $\sup_{i\in\omega} \varphi_i(\{a\}\cap A_i) = \infty$, so $X \mapsto \sup_{i\in\omega} \varphi_i(X \cap A_i)$ is not a submeasure.
- Last paragraph of page 16: The supports of μ_n should be finite (this is used on the next page). It applies also to the third line of the proof of Lemma 2.6.6.
- Page 19: "(and so $\mathcal{I} \leq_K \mathcal{J}$)", not "(and so $\mathcal{I} \leq_{KB} \mathcal{J}$)".
- Lemma 2.6.13: In the second line of the proof it should be $f^{-1}[\{n\}]$, not $f^{-1}(\{n\})$.
- Fact 2.6.17: In the last line of the proof it should be [79, Prop. 1.5.3], not [79, Prop. 1.3.5].
- In the whole thesis there are many references to results from later chapters. An example is Proposition 3.5.1, which appears in the proof of Proposition 3.1.10. Another example is Corollary 4.4.5, which appears in the proof of Proposition 3.5.1. This is not very elegant.
- Beginning of the proof of Lemma 3.2.6: "disjointly supported finitely supported". Moreover, a measure on ω has to have support contained in ω.
- Beginning of subsection 3.3.1: It should be $\langle I_n : n \in \omega_+\rangle$, not $\langle I_n : n \in \omega\rangle$. Moreover, it would be more elegant to write that we define two sequences ($\langle I_n : n \in \omega_+\rangle$ and $\langle \mu_n^f : n \in \omega_+\rangle$), not only one.
- Page 35, line 14: $\mu_n(E_n) \geq \sqrt{n-1}$ instead of $\mu_n(E_n) = \sqrt{n-1}$.
- Page 35, line 17: $E_n \subseteq \bigcup_{j=1}^{i(n)^2} \varphi^{-1}[C_j^{i(n)}]$ instead of $E_n \subseteq \bigcup_{j=1}^{n^2} \varphi^{-1}[C_j^{i(n)}]$.
- Corollary 3.5.4: The proof uses Theorem 3.5.2, but does not mention it.
- Line 12 of subsection 3.6.2: $\min(\mathrm{supp}(\mu_n^f)) = \sum_{i<n} if(i)$, not $\min(\mathrm{supp}(\mu_n^f)) = \sum_{i<n} f(i)$.
- The proof of Theorem 4.1.20 is carefully written out, however the proof of Theorem 4.1.18 is very short, despite the fact that the reasoning is slightly more complicated. This inconsistency appears in several other places.

- Theorem 4.2.1: I believe that the function h frequently appearing in the proof is actually f.
- Proposition 5.2.5: In the proof we need to know that a local version of $\mathfrak{s}_2$ (defined with the use of partitions of the infinite set B instead of partitions of ω) has the same value as $\mathfrak{s}_2$. This is easy, but perhaps it would be good to point it out.
- Theorem 5.4.3: The proof is hard to follow, because $\prod_{i \in I_j} g(i)$ sometimes stands for a natural number, and sometimes denotes a family of finite sequences.
- Page 93: It should be $b_n^{-1}(h \restriction I_n) \in S_{f_\alpha}(n)$, not $b_n^{-1}(h \restriction I_n) \in S_{f_\alpha}$.
- Corollary 5.4.7: The proof also uses the equality $\max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\} = \mathrm{cof}(\mathcal{M})$, but does not mention it.
- Claim 5.4.11: What is f here? Is it still an injection (as in the previous Claim)? Moreover, in the proof, in the case $x_m > x_n$ we cannot write $f^{-1}(x_n)$, as it might happen that $x_n \notin f[\omega]$.
- Above Theorem 5.5.6: "(the definition" should be erased. Moreover, it should be written, what is the order on $\mathbf{D}$.

However, the above list is not long, taking into account the length of the entire dissertation. The presented results contribute meaningfully to the field and are highly non-trivial. For this reason, the above critical remarks do not affect my very good assessment of the entire thesis.

Summing up, I confirm that the PhD thesis "On sequences of measures and ideals in Boolean algebras" more than satisfies all the requirements for a doctoral dissertation and its Author Tomasz Żuchowski can be admitted to further steps of PhD degree conferment procedure. Moreover, I request awarding the dissertation a distinction.

Signed by /
Podpisano przez:

Adam Marek Kwela
Uniwersytet Gdański

Date / Data: 2026-07-15 12:56

Adam Kwela