ANTASIK THESIS REPORT: RATIONAL APPROXIMATION OF SURFACE GROUP REPRESENTATIONS

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1. Context

I will begin with some context for the work done in the thesis under review.

Let Γ denote a Fuchsian group (i.e. a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$) and G a semi-simple Lie group. Let

$$\mathrm{Hom}_0(\Gamma,\mathrm{G}) = \{\rho : \Gamma \to \mathrm{G} : \rho \text{ is an irreducible representation}\}$$

and $X(\Gamma,G)$ the quotient of $\mathrm{Hom}_0(\Gamma,\mathrm{G})$ under G-conjugation. We will refer to points in $X(\Gamma,G)$ as *characters*, and denote a character associated to a representaton ρ by χ_ρ. Then, depending on G, $X(\Gamma,G)$ is a real or complex algebraic set. A natural (somewhat vague) question is: If $X(\Gamma,G)$ contains a component of positive dimension, are there particularly interesting specializations on this component; for example, corresponding to faithful, discrete representations, and furthermore, with algebraic entries, or even rational entries.

This vague question has received some attention over the years, and one particular example of this is Takeuchi's result from 1971 who showed by very explicit computations that for Γ a Fuchsian surface group, then any representation ρ in the Teichmüller component of $X(\Gamma,\mathrm{SL}(2,\mathbb{R}))$ (ie those corresponding to faithful discrete representations) could be approximated by representations with rational entries. The results of thesis under review generalize this to higher Teichmüller spaces as we now describe.

2. Main result

The main results are contained in Chapters 3 and 4, the first two chapters consist of preliminaries, and necessary background on Hitchin components. Chapter 5 contains an explicit calculation in the case of $G = \mathrm{SL}(3,\mathbb{R})$.

The main result of the thesis is the following theorem. We first introduce some notation for ease of exposition. For $g \geq 2$, let $\Gamma = \pi_1(\Sigma_g)$, set $G = \mathrm{SL}(n,\mathbb{R})$ and let $\mathcal{H}_n \subset X(\Gamma,G)$ denote the Hitchin component; this is a connected component of $X(\Gamma,G)$ that consists only of characters of discrete and faithful representations.

Theorem 2.1. *For $n \geq 3$, let $\chi_\rho \in \mathcal{H}_n$, then ρ can be approximated by representations which are conjugate into* $\mathrm{SL}(n,\mathbb{Q})$

At this point I note that when $g > 3$ this result was also proved by Audibert and Zshornack. I will comment on the proof of Theorem 2.1, and the restricted version of Audibert and Zshornack below.

The proof of Theorem 2.1 given in Chapter 3 proceeds very much in the spirit of Takeuchi's approach for $\mathrm{SL}(2,\mathbb{R})$: namely given a Hitchin representation, explicitly build a rational approximation, which essentially boils down to solving a matrix equation of the form, $XY = ZYX$ where $Z \in \mathrm{SL}(n,\mathbb{Q})$ and X and Y are variables. I did not check carefully all the details on pp 18–22 where this is carried out, but what I did check looks good, and in broad terms is natural, and is certainly very plausible.

The proof of Audibert and Zshornack is of a different flavor, but does not work in the case of genus 2. This uses a well-known bending construction, together with an ingenious application of Goldman's twist flow to achieve the density on $\mathbb{Q}$-points. The failure in genus 2 arises from a part of the the proof which implictly uses a certain configuration of simple closed curves which can only exist when the genus is at least 3. The thesis under review also discusses the author's approach using this type of method. Although he was unable to carry this out in general, in Chapter 5 he does provide an open subset of $X(\pi_1(\Sigma_2), \mathrm{SL}(3,\mathbb{R}))$ for which representations can be rationally approximated. This used some significant computer aided calculation, and although the author did not prove the complete result here, I think this result and method of proof is interesting, especially in light of the fact that Audibert and Zshornack do not handle the genus 2 case.

3. Conclusion

This is a very good thesis proving some very good results. Overall, the thesis is also very well written and I don't have any comments to pass along concerning the writing. Therefore, I recommend the degree of doctor be awarded.

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