

Report on the Doctoral Thesis **Asymptotics of functionals of Gaussian and Lévy processes with a view towards risk and queueing models**

by **Krzysztof Kępczyński**

The dissertation by Krzysztof Kępczyński titled "**Asymptotics of functionals of Gaussian and Lévy processes with a view towards risk and queueing models**" is structured into five chapters, starting with an Introduction. In the introductory chapter, the author outlines the motivation for the research, presents the essential notation, and provides a brief overview of the four main chapters that follow.

Chapter 2 is titled "**Sojourn Time for Correlated Brownian Motion with Drift**" and is dedicated to the analysis of the sojourn time for a two-dimensional correlated Brownian motion with a drift.

Let $\{\mathbf{W}(t) \equiv (W_1(t), W_2(t)) : t \geq 0\}$ be a two-dimensional correlated Brownian motion with a constant correlation $\rho \in (-1, 1)$, such that $\text{Cov}(W_1(t), W_2(t)) = \rho t$. For a deterministic drift vector $\mathbf{c} = (c_1, c_2) > (0, 0)$ and a threshold vector $\mathbf{a} = (a_1, a_2) > (0, 0)$, the sojourn-time functional:

$$S_{\rho,T}(\mathbf{c}, \mathbf{a}, u) = \int_{[0,T]} \mathbb{I}\{W_1(s) - c_1 u s > a_1 u, W_2(s) - c_2 u s > a_2 u\} ds$$

is considered. This chapter analyzes the exact asymptotic behavior of

$$s_{\rho,T}(\mathbf{c}, \mathbf{a}, u; H(u)) = \mathbb{P}\{S_{\rho,T}(\mathbf{c}, \mathbf{a}, u) > H(u)\}$$

as $u \rightarrow \infty$, with $H(u) = zu^{-2}$ for $z > 0$.

Such models are relevant in various fields, including risk theory for modeling surplus processes, and in financial mathematics for pricing exotic derivatives like Parisian options.

The analysis distinguishes between two main cases based on an essential index set $I \subseteq \{1, 2\}$, which determines which coordinates of the process asymptotically dominate: the dimension-reduction case and the full-dimensional case.

The proofs of the main theorems rely on a series of lemmas that analyze the behavior of the probability of interest.

- It is shown that the dominant contribution to the sojourn time probability comes from a small interval around the time point $t_0^* = \min(t_0, T)$, where t_0 minimizes $q(t) = \min_{\mathbf{x} > (\mathbf{a} + \mathbf{c}t)} \mathbf{x} \Sigma_t^{-1} \mathbf{x}^\top$ over $[0, \infty)$. Lemma 2.3.2 provides exponential bounds showing that the probability of the event occurring outside a small neighborhood of t_0^* is negligible.

- The interval around t_0^* is discretized into smaller intervals of length Δu^{-2} . The probability of the sojourn time exceeding zu^{-2} within each of these small intervals, denoted $\mathcal{P}_k(u)$, is analyzed. Lemmas 2.3.3 and 2.3.4 provide the precise asymptotic behavior of $\mathcal{P}_k(u)$ as $u \rightarrow \infty$ for the cases $t_0 \leq T$ and $t_0 > T$, respectively. This involves conditioning on the value of the Brownian motion at the start of the interval and using properties of Brownian motion increments.

The combination of these steps allows the derivation of the precise asymptotic formulas presented in Theorems 2.2.2 and 2.2.3, and the results generalize earlier univariate findings.

Chapter 3, titled "**Stationary Gaussian queues over a random time interval**" consists of an asymptotic analysis of a stationary Gaussian queue with a fractional Brownian motion input.

The model considers a fluid queue whose content process, $\{Q(t) : t \geq 0\}$, is the stationary solution to a Skorokhod problem with a fractional Brownian motion input (Hurst index $H \in (0, 1)$) and a constant output rate c . The central problem is to find the asymptotics of the probability:

$$p(\mathcal{T}_u; u) := \mathbb{P} \left\{ \sup_{t \in [0, \mathcal{T}_u]} Q(t) > u \right\}, \quad \text{as } u \rightarrow \infty,$$

where the time $\mathcal{T}_u = u^\gamma \sum_{k=1}^{u^\beta} T_k$ ($\beta \geq 0, \gamma \in \mathbb{R}$) is a random variable independent of the queueing process. The asymptotic behavior of this probability depends critically on the tail distribution of the i.i.d. random variables T_k . The results are classified into three cases:

- **The random time is integrable**, $\mathbb{E}\{T\} < \infty$: The randomness of the time horizon does not change the exponential decay rate of the probability. The logarithmic asymptotics are consistent with a fixed time interval, with the mean $\mathbb{E}\{T\}$ affecting the leading constant. For $\beta > 0$, the problem simplifies to that of a deterministic time horizon scaled by the mean, $u^{\gamma+\beta} \mathbb{E}\{T\}$.
- **Regularly varying tail**, $\mathbb{P}\{T > x\} \sim x^{-\alpha} L(x)$, $\alpha \in (0, 1)$: The probability of exceedance is dominated by the event $\{T > 1/p(1; u)\}$. In this regime, the index of regular variation α directly impacts the logarithmic asymptotics:

$$\log(p(\mathcal{T}_u; u)) \sim \alpha \log(p(1; u)) \sim -\frac{\alpha}{2} A^2 u^{2(1-H)}.$$

- **Slowly varying tail** $\mathbb{P}\{T > x\}$: The final case considers a slowly varying tail for T , which is even heavier than the regularly varying case. Under an additional technical condition (D4), the asymptotic behavior is determined. In this heaviest-tailed scenario, the exceedance probability can decay polynomially rather than exponentially. The asymptotics are driven by the tail probability of the random variable T itself. For instance, if $\mathbb{P}\{T > x\} \sim 1/\log(x)$, the resulting probability $p(\mathcal{T}_u; u)$ is regularly varying in u .

The proofs of Theorems 3.2.1, 3.2.5, and 3.2.7 are distinct for each case and rely on conditioning on the random time and applying principles from extreme value theory for Gaussian processes, as well as Lebesgue's dominated convergence theorem and the law of large numbers.

Chapter 4 has a title "**Tail Asymptotics for Functionals of Stationary Lévy Queues**" and analyzes a stationary fluid queue, $\{Q(t), t \geq 0\}$, which has the representation:

$$Q(t) = \sup_{s \leq t} (X(t) - X(s) - c(t - s)).$$

Here, $\{X(t), t \in \mathbb{R}\}$ is a Lévy process, and $c > 0$ is a constant. The study is conducted under the light-tailed assumption, where the net input process $X(t) - ct$ satisfies the Cramér condition (A1). Specifically, it is assumed there exists an adjustment coefficient $\omega > 0$ such that $\mathbb{E}\{e^{\omega(X(1)-c)}\} = 1$ and $\mathbb{E}\{e^{\omega(X(1)-c)} | X(1) - c|\} \in (0, \infty)$. Also, it is assumed that $\{X(t) - ct : t \geq 0\}$ has non-monotone paths and either 0 is regular for $\mathbb{R}_+ \setminus \{0\}$ or the Lévy measure of $\{X(t) - ct : t \geq 0\}$ is non-lattice (A2).

The objective is to analyze the tail probability of a functional Θ applied to the queueing process over a set $E \subset [0, \infty)$:

$$p^\Theta(E; u) := \mathbb{P}\{\Theta(\{Q(t) : t \in E\}) > u\}.$$

The functional Θ is assumed to be scale- and shift-invariant (i.e., $\Theta(af + b) = a\Theta(f) + b$) and bounded by the supremum (i.e., $\Theta(f) \leq \sup(f)$), covering key examples like the supremum, infimum, and their linear combinations. The analysis considers two distinct scenarios for the time interval E .

[Fixed-interval case - Theorem 4.2.1] For a fixed time interval $E = [0, T]$, the probability that a functional of the queueing process exceeds a high threshold u is asymptotically proportional to the probability of the initial queue state exceeding u . As $u \rightarrow \infty$,

$$\mathbb{P}\{\Theta(\{Q(t) : t \in [0, T]\}) > u\} \sim \mathbb{E}\{e^{\omega\Theta(\{X(t)-ct:t \in [0,T]\})}\} \cdot \mathbb{P}\{Q(0) > u\}.$$

[Growing-interval case - Theorem 4.2.3] For a time interval $E = [0, n(u)]$ that grows with the threshold u , the probability of the supremum exceeding u accumulates linearly with the length of the interval. For the supremum functional, as $u \rightarrow \infty$,

$$\mathbb{P}\left\{\sup_{t \in [0, n(u)]} Q(t) > u\right\} \sim \mathcal{H}_{\omega X_c}^{\sup} \cdot n(u) \cdot \mathbb{P}\{Q(0) > u\},$$

where $\mathcal{H}_{\omega X_c}^{\sup}$ is the limiting Pickands constant.

The proofs rely on two main strategies:

- For the **fixed-interval case**, the key is showing that the queue is unlikely to empty. This allows the application of Breiman's lemma to the product of two independent random variables representing the initial state and the subsequent process evolution.
- For the **growing-interval case**, a discretization argument is used. The long interval is broken into smaller blocks, and it is shown that the correlation between exceedance events in distant blocks is negligible, leading to the linear scaling with the interval length $n(u)$.

The title of **Chapter 5** is "**Extension of Breiman's Lemma**". The chapter's central contribution is a powerful extension of Breiman's lemma, Theorem 5.2.1:

Let $\{\mathbf{X}_{u,\tau_u}\}$ and $\{\mathbf{M}_{u,\tau_u}\}$ be families of random vectors and matrices, respectively, satisfying a set of technical assumptions (A1-A3) concerning their joint convergence and moment conditions. Then, as $u \rightarrow \infty$,

$$b_{u,\tau_u} \mathbb{P}\{\mathbf{M}_{u,\tau_u} \mathbf{X}_{u,\tau_u} \in a_{u,\tau_u} \cdot\} \rightarrow \nu_G(\cdot)$$

vaguely, uniformly with respect to τ_u . The limiting measure ν_G is defined by averaging the measure associated with $\mathbf{X}$ over the distribution of $\mathbf{M}$:

$$\nu_G(\cdot) = \mathbb{E}\{\nu(\mathbf{M}^{*-1} \cdot)\},$$

where $\mathbf{M}^*$ is a random matrix with distribution G .

This result is applied to:

- **Extremes of Gaussian Fields:** It is used to derive the uniform tail asymptotics for a homogeneous functional Θ applied to a family of Gaussian fields $\{\xi_{u,\tau_u}(t)\}$. This result generalizes the classical Pickands and Piterbarg lemmas to a wider class of functionals and sets that depend on the threshold u (Corollaries 5.3.2 and 5.3.3).
- **Self-Standardized Processes:** Analyze the tail behavior of the supremum of self-standardized Gaussian processes, such as $\{Y(t)/Y(T)\}$, which are relevant in modeling random variance (Theorem 5.3.5 for a process over a fixed interval, Theorem 5.3.6 for a process over a growing interval).

The proof of the main theorem adapts the methodology of Fougères & Mercadier, extending it to ensure uniform convergence. The key steps involve decomposing the probability based on the norm of the random matrix and showing that the contribution from matrices with large norms is negligible. The proofs of the applications then consist of casting the specific problems into the abstract product form of the main theorem and verifying its assumptions.

Summary: The thesis presents new and significant results that are well-organized and clearly articulated. Given the rigorous nature of the underlying theory, I am confident that the material provided is of the highest quality, successfully bridging theoretical probability with practical applications. These findings are likely to have important uses in modern stochastic modeling, particularly in areas such as insurance mathematics, network theory, and systems reliability. Therefore, I highly recommend accepting Kępczyński's thesis and awarding the degree with distinction.

Podgorica, 25th August 2025

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Minor suggestions:

1. Page 10, 4th row from below, please change "ans" to "and".
2. Page 10, 5th row from below, please use larger brackets for events by employing the commands " $\left$ " & " $\right$ ". Additionally, please apply this change to all other similar instances.