Report on the Doctoral Thesis **Ruin probability in multidimensional self-similar Gaussian risk models** by **Konrad Krystecki**

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The dissertation titled "Ruin Probability in Multidimensional Self-Similar Gaussian Risk Models" by Konrad Krystecki is structured into five chapters, starting with an Introduction. This introductory chapter outlines the motivation behind the research, contains the key notation, and provides a concise overview of the four main chapters that follow.

Chapter 2 is titled "Ruin Probability of Two-Dimensional Brownian Risk Model with Drift Dependent on Initial Capital" and consists of four sections. The introductory part introduces the problem formulation, which will be further explored in later sections. This extends the research from [1] to the two-dimensional case with drift dependence on u and finds the exact asymptotic behavior of

$$\mathbb{P}\left\{\exists t \in [0,T] : W_1(t) - c_1 u^\alpha t > a_1 u^\beta, W_2(t) - c_2 u^\alpha t > a_2 u^\beta\right\}, \quad u \to \infty, \tag{1}$$

where a_1, a_2, c_1, c_2 are positive constants, and $\alpha, \beta \geq 0$. Here,

$$W_1(t) = B_1(t), \quad W_2(t) = \rho B_1(t) + \sqrt{1-\rho^2} B_2(t),$$

$\rho \in (-1,1)$ and $B_1(t)$, $B_2(t)$ are standard independent Brownian motions. The practical use of (1) is in applied probability problems, such as ruin theory (simultaneous ruin problems), the model of junctions of three independent Brownian motions, and so on.

The key findings in this chapter are Theorems 2.2.1, 2.3.2, 2.3.3, and 2.4.1. Proving these theorems required the enhancement of existing techniques. The relationship between α and β leads to three scenarios ($\alpha < \beta$, $\alpha = \beta$, and $\alpha > \beta$), each requiring different approaches.

When $\alpha < \beta$ (subsection 2.2), the asymptotic behavior is dominated by u^β, while for $\alpha \geq \beta$, both u^α and u^β have a significant impact on the asymptotics. New Pickands-type constants are derived in the asymptotic analysis. Theorem 2.2.1 examines three cases: $a_2 < \rho$, $a_2 = \rho$, and $a_2 > \rho$. The proof involves analyzing the behavior of the function

$$q^*_{\boldsymbol{a}_{u^\gamma}}(t) = \min_{\boldsymbol{x} \geq a_{u^\gamma}} q_{\boldsymbol{x}}(t),$$

on $[0,T]$, where $\gamma = \alpha - \beta$, $\boldsymbol{a}_{u^\gamma}(t) = (1 + c_1 t u^\gamma, a_2 + c_2 t u^\gamma)$,

$$q_{\boldsymbol{x}}(t) = \boldsymbol{x} \Sigma_t^{-1} \boldsymbol{x}^\top,$$

and Σ_t is covariance matrix of $(W_1(t), W_2(t))$.

Lemma 2.2.2 provides a solution for the quadratic programming problem $\min_{t\in[0,T]} q^*_{\boldsymbol{a}_{u^\gamma}}(t)$, and Lemma 2.2.3 gives an upper bound for the probability of type (1) within the interval $[0, T - f(u)]$, where $f(u) = o(1)$ as $u \to \infty$. Lemma 2.2.4 is a Pickands-type lemma, derived for a small interval $E_{u_k} = [(k+1)_u, k_u]$, where $k_u = T - \frac{(k-1)\Delta}{u^{2\beta}}$, k is a nonnegative integer, and $\Delta > 0$ is fixed. If $a_2 < \rho$, it is established that the probability in (1) behaves as

$$\mathbb{P}\left\{\exists t \in [0,T] : B_1(t) - c_1 u^\alpha t > u^\beta\right\}, \text{ as } u \to \infty.$$

In the case where $a_2 = \rho$, it is found that the probability in equation (1) behaves as follows as $u \to \infty$:

$$\mathbb{P}\left\{\exists t \in [0,T] : B_1(t) - c_1 u^\alpha t > u^\beta\right\} \cdot \mathbb{P}\left\{B_2(T) + \frac{c_1\rho - c_2}{\sqrt{1-\rho^2}} u^\alpha T > 0\right\}.$$

The second factor behaves differently depending on whether $\rho c_1 > c_2$, $\rho c_1 = c_2$, or $\rho c_1 < c_2$. The final case $a_2 > \rho$ utilizes Lemmas 2.2.3 and 2.2.4.

If $\alpha = \beta$, then the main results are Theorems 2.3.2 (Dimension-reduction case, where one coordinate of $(W_1(t), W_2(t))$ in (1) asymptotically dominates the other) and 2.3.3 (Full-dimensional case, where both components of $(W_1(t), W_2(t))$ in (1) affect the asymptotics), which are also valid for the infinite time interval.

The first step of the proof involves finding the minimum of $q(t) = \min_{\boldsymbol{x} > \boldsymbol{a} + \boldsymbol{c}t} \boldsymbol{x} \Sigma_t^{-1} \boldsymbol{x}^\top$ in the interval $[0, T]$ (Lemma 2.3.5). Lemma 2.3.6 contains the upper bound for (1) outside the neighborhood of the optimal point of $q(t)$, (point T), in the case of finite and infinite time intervals. Lemma 2.3.7 obtains the exact asymptotics of (1) within $E_{u,1}$ in the dimension-reduction case and $E_{u,k}$ in the full-dimensional case. Lemma 2.3.8 proves that the constants introduced in Lemma 2.3.7 (iii) are in $(0, \infty)$. Lemma 2.3.9 shows that if the optimal point is the interior point of $(0, T)$, then the asymptotics of (1) is equivalent to

$$\mathbb{P}\left\{\exists t \in [0,\infty) : W_1(t) - c_1 t > u^{2\alpha}, W_2(t) - c_2 t > a_2 u^{2\alpha}\right\}, \quad u \to \infty.$$

The proof of Theorem 2.3.3 follows by combining Lemmas 2.3.5-2.3.9 and Theorem 3.1 of [1].

If $\alpha > \beta$, the drift increases more rapidly than the initial capital, suggesting that it ultimately becomes dominant for sufficiently large u. However, the intuitive conclusion is not correct. Here, using the self-similarity of Brownian motion, it is proven that this case simplifies the case $\alpha = \beta$, with the speed parameter $\frac{\alpha+\beta}{2}$ and $T = \infty$ as considered in Theorems 2.3.2 and 2.3.3.

In Chapter 3, titled "Finite Time Ruin Probability for Subordinated Fractional Brownian Motion," there are three sections: Introduction, Main Results, and Proofs. The main result of this chapter (Theorem 3.2.2) provides the exact asymptotic behavior of:

$$\mathbb{P}\left\{\sup_{i \geq 0, X_i \in [0,T]} (B_H(X_i) - cX_i) > u\right\}, \text{ as } u \to \infty. \tag{2}$$

Here, $B_H(t)$ denotes fractional Brownian motion with Hurst index $H \in (0,1)$, $X_i = \sum_{j=1}^{i} Z_j$, where $Z_j, j \geq 1$, are nonnegative, independent, identically distributed random variables, and $T > 0$ is a constant.

The probability given by equation (2) describes the behavior of the risk process at random points within a finite interval. If $X_1 > T$, then there are no random points within the interval $[0, T]$, and the value of equation (2) is 0. The concept of subordination of Gaussian processes is introduced in references [2–4]. In reference [5], it was shown that for sufficiently large u, the probability of ruin in the continuous time setting is affected by the variability at the end of the interval $[0, T]$. Similar behaviour was demonstrated here. Since the density of X_τ is not dependent on u (where $\tau = \sup\{i : X_i \leq T\}$), the most significant factor is the variance of the fractional Brownian motion itself, while the random inspection times only contribute to the constant. Finally, an application of Theorem 3.2.2 was illustrated, assuming Z_i are exponentially distributed with parameter $\lambda > 0$ (Corollary 3.2.3).

Assuming that Z_1 has a continuous density function $f_Z(\cdot)$ and $f_Z(0) \in (0, \infty)$, Proposition 3.2.1 concludes that X_τ has a continuous density function, that is both finite and positive in any positive T.

The properties of fractional Brownian motion are significantly different for $H \geq \frac{1}{2}$ and $H < \frac{1}{2}$, therefore, the proof of Theorem 3.2.2 is divided into two parts. The approach needed for the case of $H < \frac{1}{2}$ differs from the approach used for $H \geq \frac{1}{2}$, due to the negative correlation of increments of fractional Brownian motion for $H < \frac{1}{2}$, and in the both parts was used Proposition 3.2.1 of this chapter and Proposition 3.1 of [6].

The title of Chapter 4 is "Logarithmic Asymptotics of Parisian Ruin Probability for Positively Correlated Brownian Motions." This chapter is divided into five sections: Introduction, Notation and Preliminaries, Main Results, Proofs, and Simulations.

In this chapter, the author investigates the non-simultaneous Parisian ruin probability for two-dimensional time spent over the barrier $\boldsymbol{H}(u) = (H_1(u), H_2(u))$. Specifically, the focus is on the following probability:

$$\mathbb{P}\left\{\exists_{(s',t')\in[0,T]^2} \forall_{s\in[s',s'+H_1(u)]} \forall_{t\in[t',t'+H_2(u)]} : W_1^*(s) > u, W_2^*(t) > au\right\}, \tag{3}$$

with the results centering on the logarithmic asymptotics of this expression as $u \to \infty$.

In the equation above, the terms are defined as follows:

$$W_1^*(s) = B_1(s) - c_1 s, \quad W_2^*(t) = \rho B_1(t) + \sqrt{1-\rho^2} B_2(t) - c_2 t,$$

where $\rho \in [-1, 1]$, and $B_1(s)$ and $B_2(t)$ represent standard independent Brownian motions. The functions $H_1(u)$ and $H_2(u)$ are defined as some non-negative functions. Due to the self-similarity of Brownian motion, it can be assumed, without loss of generality, that $T = 1$. The function $q(\cdot)$ is essential for calculating logarithmic asymptotics.

The selection of $\boldsymbol{H}(u)$ in references [8, 9] was based on the variance-covariance structure of the problem, which enabled exact asymptotic calculations. Konrad Krystecki is the author of papers [8, 9], published in the esteemed journals. In the thesis, he expands these findings to consider different forms of $\boldsymbol{H}(u)$. The results are divided into two cases based on the behavior of the function $\boldsymbol{H}(u)$.

In the first case, described in Theorem 4.3.1, $\boldsymbol{H}(u) \to (0, 0)$. In the second case, outlined in Theorem 4.3.2, $\boldsymbol{H}(u) \to (H_1, H_2) > (0, 0)$ as $u \to \infty$, where $\rho > 0$.

The proof of Theorem 4.3.1 is divided into two scenarios: when $t^* = 1$ or when $t^* < 1$. For Theorem 4.3.2, the proof first considers the case of $H_1 = H_2 = H$ and then examines the scenario where $H_1 > H_2$ (the case where $H_1 < H_2$ is similar and thus omitted). The assertions from reference [7], coauthored by Konrad Krystecki and published in a prestigious journal, were utilized in the proofs.

Section 4.5 presents a study of simulations of multivariate Brownian motion for practical applications, with the following parameter sets: (1) $a = 1$, $\rho = 0.75$, $c_1 = -0.5$, $c_2 = 0.25$; (2) $a = 0.25$, $\rho = -0.25$, $c_1 = -0.5$, $c_2 = 0.25$; (3) $a = 0.1$, $\rho = -0.9$, $c_1 = -1$, $c_2 = -1$.

The title of Chapter 5 is "Non-simultaneous Ruin Probability for a Positively Correlated Brownian Risk Model," which comprises three sections: Introduction, Main Result, and Proofs.

The main result is Theorem 5.2.2, which provides the exact asymptotic behavior of the following probability:

$$\mathbb{P}\left\{\exists \boldsymbol{t} \in [0,1]^d : \boldsymbol{W}^*(\boldsymbol{t}) > \boldsymbol{\alpha} u\right\}, \quad \text{as } u \to \infty, \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\boldsymbol{c} = (c_1, \ldots, c_d) \in \mathbb{R}^d$, $\boldsymbol{W}^*(\boldsymbol{t}) = A\boldsymbol{B}(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}$, and A is a $d \times d$ matrix. Here, $\boldsymbol{B}(\boldsymbol{t})$ represents a d-dimensional Brownian motion with independent components. The proof utilizes Lemmas 5.3.1 (which describes the asymptotic behavior of $q_\alpha(\boldsymbol{s}) - q_\alpha(\boldsymbol{t})$ as $\boldsymbol{s}$ approaches $\boldsymbol{t}$), 5.3.3 (the Pickands-type lemma), and 5.3.4 (the positivity and finiteness of constants).

Theorem 5.2.1 provides both upper and lower bounds for the probability in (4). The results align with the two-dimensional findings reported in [7] for $\rho > 0$.

Summary: The thesis presents novel and significant results. Its organization and clarity are commendable and engaging. The theory behind the thesis results is demanding, so I have no doubts that the presented text is very high quality. Based on these observations, along with the fact that Krystecki has published three papers [7–9] in esteemed journals, I highly recommend accepting Krystecki's thesis and awarding the degree with distinction.

Podgorica, 30th October 2024

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Minor Typos:

1. Page 2, Contents: The title of Chapter 2 differs from the title used in the chapter itself.
2. Page 5, 6th line from the top: Please replace B_0 with $B(0)$.
3. Page 9, 6th line from the top: Change "dependant" to "dependent."
4. Page 9, 11th line from the top: Please add the equation label referenced on Page 8, 7th line from the below.
5. Page 11, 5th line from the top: Add $i = 1, 2$ after "$C_i \in \mathbb{R}$."

6. Page 22, 10th line from the top: Remove "(i)" from "From Lemma 2.2.3 (i)" and also from the 2nd line below.

7. Page 24, 9th line from the top: Separate "and" from "$I := I(t_0^*)$."

8. Page 36: In the first formula, remove the ":" after inf, and similarly, remove it after sup in the last formula (3.1).

9. Page 41, 2nd line: The second subscript of π should be "H", not "B_H."

10. Page 42, Case $H < \frac{1}{2}$: Replace "than" with "than" (likely a typo), and add a comma in $l = 1, \ldots, N_u$.

11. Page 49, first formula: Change "$W(t) =$" to "$W(s,t) =$" .

12. Page 50, 5th line from the top: Change "dependant" to "dependent."

13. Page 51, 1st line from the top: Replace "onto" with "into."

14. Page 51, 4th line from the top: Change "$(H_1, H_2) \geq (0,0)$" to "$(H_1, H_2) > (0,0)$."

15. Page 52, 4th line from the top: Change "logarythmic" to "logarithmic."

16. Page 66, 6th line from the top: Change "d-dimensional" to "d-dimensional." Apply the same change in line 8.

17. Page 66: Revise the formula $\Sigma_{\boldsymbol{t}} = \ldots$, as there are some redundant Σ at the end, or add an explanation.

18. Page 67, 8th line from the top: Add $\boldsymbol{c} = (c_1, c_2, \ldots, c_d)$ before $\in \mathbb{R}^d$.

19. Page 67, 16th line from the top: Use bold α instead of the regular one. Additionally, check the rest of this chapter for instances where 0 is used as a number instead of vector notation.

20. Page 80, reference [18]: Add "2021(10): 890-915" as the volume(issue) number and page range.

21. Page 84, reference [57]: Include the article number 109327.

22. Page 83, reference [46]: Add the volume(issue) number and correct page range.

23. Page 84, reference [58]: Change "parisian" to "Parisian."

Bibliography

[1] Hashorva, E. and Hüsler, J. (2000). Extremes of Gaussian processes with maximal variance near the boundary points. Methodology and Computing in Applied Probability, 2(3):255-269.

[2] Grigelionis, B. (2007). On subordinated multivariate Gaussian Lévy processes. Acta Applicandae Mathematicae, 96(1):233-246.

[3] Gajda, J. and Magdziarz, M. (2014). Large deviations for subordinated Brownian motion and applications. Statistics & Probability Letters, 88:149-156.

[4] Wang, W. and Chen, Z. (2018). Large deviations for subordinated fractional Brownian motion and applications. Journal of Mathematical Analysis and Applications, 458(2):1678–1692.

[5] Dębicki, K., Michna, Z., and Rolski, T. (1998). On the supremum from Gaussian processes over infinite horizon. Probability and Mathematical Statistics, 18:83–100.

[6] Dębicki, K. and Rolski, T. (2002). A note on transient Gaussian fluid models. Queueing Systems, 41(4):321–342.

[7] Dębicki, K., Hashorva, E., and Krystecki, K. (2021). Finite-time ruin probability for correlated Brownian motions. Scandinavian Actuarial Journal, 2021(10): 890-915.

[8] Krystecki, K. (2022). Parisian ruin probability for two-dimensional Brownian risk model. Statistics & Probability Letters, 182 (article number 109327).

[9] Krystecki, K. (2023). Cumulative Parisian ruin probability for two-dimensional Brownian risk model. Probability and Mathematical Statistics, pages 63–81.