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Report on the PhD Thesis by Matteo Bonforte

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Name of the Ph.D. Candidate:	**Milosz Krupski, Wroclaw University**
Title of the Ph.D. Thesis:	**Non-conventional, non-linear, non-local evolution PDEs. A two-case study**
Ph.D. Advisor:	**Professor Grzegorz Karch, Wroclaw University**

This Ph.D. thesis is organized in 3 parts: an Introduction and two chapters that deal with two different problems. This thesis is based on 3 publications: Two of them have been written by the candidate alone, and are already published or accepted for publication in prestigious journals. The third is in final stage of preparation and my feeling is that it will be published in a prestigious/top journal, as it deserves.

The first part of this thesis consists of a nice introduction which contains the motivation of the models and a brief history of the more classical models (local) that in this thesis are extended to a different, more general, setting. The Introduction is split in three sections, one about nonlocal operators, the second about two models environments, random fields and diffusions, while the third is about two model equations, Burgers and Porous Medium equations. This introductory chapter is concise, well written, carefully explained, and quite pleasant to read, and it sets the basis for understanding the results presented in the next chapters.

The second and third chapters contains the main contributions and results that I will briefly discuss below. Besides the main results, whose importance is unquestionable, this thesis contains deep and interesting results of high technical level. I find remarkable that the candidate masters a wide variety of techniques which range from Partial Differential Equations (PDE), Spectral and Functional Analysis, to Stochastic Calculus and Probability. On one hand, such techniques are extremely useful tools and are essential in the proof of the main results. On the other hand, they have their own interest and can possibly be used for other purposes, that may motivate future work, that I strongly encourage.

In my opinion, the work done in this thesis is excellent, the topics and the problems considered are of current interest in the field of PDEs, to be more specific, of parabolic type, possibly nonlinear and nonlocal and of degenerate/singular type and/or with viscous or convective terms. The results are undoubtedly both original and strong. Let me also stress that the results obtained in this thesis are more than sufficient to grant to the Candidate the Ph.D. title, and this thesis represents a significative advance in the field.

The candidate shows a certain maturity and independence: this is evident by the writing stile, but also confirmed by the fact that he has written two papers alone. On the other hand, he has collaborated with well-known experts in the field, like Prof. M. Kassmann and his Ph.D. Advisor, Prof. G. Karch.

Let us now briefly discuss the two main chapters together with the main results.

The second Chapter investigates solutions to a nonlocal version of the viscous Burger equation of the form $\partial_t u + (-\Delta)^s u = \nabla_z f(u)$, where $(-\Delta)^s$ is the Fractional Laplacian with $s \in (1/2, 1]$, ∇_z is the directional derivative and the function f is a smooth function with polynomial growth (which is allowed to be nonlinear). In the deterministic case, this equation has been widely studied and has several important applications: for instance, when $s = 1$ it is the famous/classical viscous Burger equation. The real novelty here is represented by the fact that the candidate carefully analyzes the concept and the existence of solutions corresponding to random initial conditions, which significantly complicates the problem and is the source of many mathematical difficulties. The chapter starts with a nice introduction of the basic concepts and definitions needed in the rest of this part, like Isometric invariant random fields, spectral moments, etc. Then the study is focussed on the linear equation, which is the core and the starting point to tackle the nonlinear case. For the linear case, existence of semigroup solutions is proven. Next, the candidate considers the case of Lipschitz nonlinearities, and the solutions are obtained through approximations via Picard-type iterations, making a strong use of the linear results previously obtained. Finally the chapter concludes by proving existence of strong solutions in the case in which f has polynomial growth. Regularity of the solutions is also analyzed especially through moments estimates: these are the main tools to justify in which sense the strong solution of the latter case can actually be considered a solution to the problem. Indeed, the existence of solutions in the latter case is proven through a careful approximation by means of solutions of problems with Lipschitz nonlinearity, and in order to justify the limiting process the regularity estimates are essential. The chapter also contains a useful appendix about orthogonal random measures and their connection to homogeneous/Gaussian and related random fields.

The third Chapter deals with the delicate task of finding an abstract setup for nonlocal and nonlinear diffusions, whose prototype are the fractional Porous Medium equation and the Fractional p-Laplacian equation. The equation can be put in the form $\partial_t + \mathcal{L}_u = 0$ where $\mathcal{L}_v u(x) = \int_{\mathbb{R}^d}[u(x) - u(y)]\varrho(v(x), v(y); x, y)dy$, with ϱ a suitable admissible jump kernel. The problem is posed on the whole space. The first subsection consists of the main definitions and assumptions, especially on the jump kernel, which allow to include the above mentioned important model equations. The second subsection contains the statement of the main result, in the form of a Theorem about existence and uniqueness of entropy solution (for admissible jump kernels) which has the properties of mass conservation, non-expansivity of the L^p-norms, L^1 contractivity and preservation of positivity (non-negativity). The next subsection contains a brief explanation of the scope of the theory and contains examples of admissible kernels which allow to include nonlocal (fractional) version of the Porous Medium and p-Laplace type equations, as well as many other possible generalizations, called here Nonlinear Levy operators, studied in Subsection 3.2. The rest of the chapter contains the proof of the main result, which is obtained through priori estimates mixed with an approximation of the admissible jump kernel by means of the so-called regular jump kernels. The final section contains explicit and relevant examples to which the theorem applies, some of them already mentioned above, but also other interesting examples of jump kernels.

As a final comment, the thesis is very well written, and reveals a certain maturity of the candidate, which has certainly mastered the difficult techniques and deeply understood the mathematical subtleties. This work also evidence the elevated research skills of the Candidate, which I encourage in continuing doing research.

Summing up, I strongly recommend to award the Ph.D. title to Milosz Krupski without any hesitation: in this work he has proven a strong mathematical profile and the remarkable ability to work at a high level both alone and with high level mathematicians. Let me conclude by stating that this excellent work undoubtedly fulfills all the requirements of a Ph. D. thesis in any top-rank university worldwide.

Madrid, September 5, 2018,

Matteo Bonforte

(Matteo Bonforte)