UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

CENTRO DE CIENCIAS MATEMÁTICAS

Morelia, Mexico, August 14, 2026

To whom it may concern:

This is an evaluation of the work *On sequences of measures and ideals in Boolean algebras written* by **Tomasz Żuchowski** who submits it as a dissertation to obtain the doctoral degree at the Faculty of Mathematics and Computer Science of the University of Wroclaw.

The thesis is a non-trivial and solid contribution to set theory, topology and functional analysis. It studies several combinatorial phenomena on the boundary between the three areas that involve sequences of measures. In particular, the thesis deals with the *Grothendieck* and *Nikodym* properties of Banach spaces and/or Boolean algebras, and with extensions of the so-called *Rosenthal* lemma and with the corresponding cardinal invariants of the continuum. Let us first mention the original results:

Theorem (Grothendieck '53) Every weak* convergent sequence of Borel measures on an extremally disconnected topological space is weakly convergent.

Theorem (Nikodym '33) Every point-wise bounded sequence of measures on a σ-complete Boolean algebra is norm bounded.

Theorem (Rosenthal '68) Given a countable antichain (a_n) in a Boolean algebra **B** and a sequence (φ_n) of non-negative measures on **B** such that the set $\{\sum_n \varphi_k(a_n)\colon k\in\omega\}$ is bounded, there is for every $\varepsilon>0$ an infinite $B\subseteq\omega$ such that for all $k\in\omega$ $\sum_{n\in B\setminus\{k\}} \varphi_k(a_n)<\varepsilon$.

The thesis contains five chapters. The very short first chapter – Introduction – motivates and summarizes the results contained in the work which are based on three separate research articles (two of them published and the third submitted). On of them is a single authored paper and the other two are co-authored by the candidate with each of the co-advisors of the thesis, respectively.

The second, preliminary, chapter briefly introduces the set theoretic notation used, as well as the basic concepts and facts of set theory, measure theory, topology and functional analysis used in the text. Close attention is paid to ideals on ω given by submeasures and the Katětov order on them. It also introduces the fundamental definitions of Nikodym and Grothendieck properties:

A Boolean algebra **B** has the *Nikodym property* if every pointwise bounded sequence of measures on **B** is norm bounded.

A Banach space X has the *Grothendieck property* if every weak* convergent sequence in X* is weakly convergent. Stone duality and the Riesz representation theorem permit to directly translate this property for spaces of continuous function on zero-dimensional compact spaces as follows:

A Boolean algebra **B** has the *Grothendieck property* if every norm bounded sequence of measures on **B** pointwise convergent to 0 is weakly convergent to 0.

The, above mentioned, results of Nikodym and Grothendieck, respectively, show that complete Boolean algebras satisfy these properties.

The third chapter introduces a variant of the Nikodym property relevant to the study of filters and ideals on ω. It deals with a particular instance of the Nikodym property restricted to the algebras of clopen subsets of zero-dimensional spaces with isolated points and to *finitely supported* measures on them, talking hence about the *fs-Nikodym property*. Of particular interest are countable spaces with unique non-isolated point, in natural correspondence with filters (and ideals) on ω. **Corollary 3.2.5** identifies those ideals which fail to satisfy the fs-Nikodym property as exactly those ideals which are contained in non-trivial summable ideals. It is then showed **Theorem 3.3.2** that containment can be replaced by the Katětov order. As a corollary it was shown the structure of ideals which are not fs-Nikodym under the Katětov order is *cofinally similar* to the standard order of *eventual dominance* $\leq^*$ of functions from ω to ω (**Corollary 3.4.4**). Relation with versions of the bounded version of the *Josefson-Nissenzweig* property of ideals is briefly mentioned (**Proposition 3.5.1**) and examples are given.

Chapter four is devoted to the simultaneous study of the Nikodym and Grothendieck properties of Boolean algebras and the relation between the two, in particular expressing the negation of the Nikodym property and strengthenings thereof in terms of existence of sequences of measures with strong concentration points and showing that a Boolean algebra which admits one of those is, in particular, not Grothendieck (**Corollary 4.1.21**). We omit the rather technical definitions here and refer to the content of the thesis itself. Section 4.2. links these inquiries with the content of chapter 3. Perhaps the most interesting and impressive of these results being the fact that Boolean subalgebra of $P(\omega)$ generated by the ideal Z of subsets of ω of *asymptotic density zero* has the Nikodym property but not the Grothendieck property. Elaborating on the theme, **c**-many definable (Borel) such Boolean algebras (**Theorem 4.4.26**) and $2^{\mathfrak{c}}$-many non-definable ones (**Theorem 4.4.3**) are constructed

Chapter studies cardinal invariant related to *Rosenthal families.* The cardinal invariants can be succinctly expresses as follows. Given a family F of functions from ω to ω without fixed points let

$\mathbf{ros}_F$ denote the minimal size of a family A of infinite subsets of ω such that for every $f\in F$ there is an $A\in A$ such that the $A\cap f[A]$ is finite,

and, *dually,* let

Δ_F denote the minimal size of a family $F'\subseteq F$ of infinite subsets of ω such that for every infinite $A\subseteq\omega$ there is an $f\in F'$ such that the $A\cap f[A]$ is infinite.

It is proved (**Corollary 5.2.6**) that if F is the family of *all* functions then $\mathbf{s}\leq\Delta_F\leq\mathbf{s}_{\max}$ where $\mathbf{s}$ is the standard *splitting* number and $\mathbf{s}_{\max}$ its variant not known to be consistently distinct from $\mathbf{s}$, and that $\Delta_{\text{fin-to-one}}=\max\{\mathbf{b}, \Delta_F\}$ (**Theorem 5.2.10**), here b denotes the *bounding* number. The cardinal invariants $\mathbf{ros}_{1\text{-}1}$ and $\Delta_{1\text{-}1}$ corresponding to the family of *one-to-one* functions were the main subject of stude of the rest of chapter 5. It was first shown that the cardinal invariants for one-to-one functions, permutations and involutions coincide (**Theorem 5.3.3**), that they also coincide with cardinal invariant Δ introduced by T. Banach in relation to *large-scale-topology* (**Corollary 5.3.4**). Further lower and upper bounds on these invariants were abtained (**Theorem 5.3.5** and **Theorem 5.4.3**) plus a forcing model showing that these bounds are consistently strict was constructed (**Theorem 5.5.4** and **Theorem 5.5.5**).

To conclude, the student has clearly exhibited that he has learned and dominated large parts of the methods of modern set theory including the theory of cardinal invariants of the continuum, modern treatment of filters and ideals based on the study of the Katětov order, and the method of forcing. The applications to topology and functional analysis produced are both interesting and deep and leave room for further interesting research. The thesis is impeccably written and opens a new and fruitful connections between the study of ideals on countable sets and the theory of Banach spaces. I am fully convinced that the work fulfills the necessary requirements to constitute a doctoral thesis and **I recommend that the student is awarded** the corresponding title**.**

Sincerely

Michael Hrusak

Investigador Titular "C" de Tiempo Completo

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