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Report on the thesis of Agnieszka Hejna
'Harmonic Analysis and Hardy spaces
in the rational Dunkl setting'

At the very outset let me say that this is an excellent thesis making a substantial contribution to various aspects of Dunkl Harmonic Analysis and hence we recommend this very strongly for the award of Ph D degree from Wroclaw University.

This well written thesis deals with several problems of Dunkl Analysis which are of current interest, the techniques used are varied and modern and there is a vast literature behind every problem. This is certainly a very technical thesis and it is difficult to describe the results and put them in perspective without going through a brief history of known results. In fact, there are more results than necessary to make a very good thesis, and hence commenting on all the results presented here is really challenging. We therefore will be very choosy and only go through the most important contributions of the author in every chapter.

By Dunkl Harmonic Analysis we mean Fourier Analysis performed in the Dunkl setting where the Dunkl transform $\mathcal{F}_\kappa$ and the Dunkl-Laplacian Δ_κ play the roles of Fourier transform and Laplacian respectively. The Dunkl-Laplacian is defined as $\Delta_\kappa = \sum_{j=1}^{N} T_j^2$ where T_j are certain differential-difference operators introduced by Dunkl way back in 1975. Here is the setting that allows us introduce these characters: (i) a root system R in $\mathbb{R}^N$ and a finite subgroup $G \subset O(N)$, called the Coxeter group, generated by the reflections defined by the roots (ii) a multiplicity function κ which is a G-invariant function on R and (iii) a G-invariant weight function $w(x) = w_\kappa(x)$ defined in terms of the roots and κ. The operators T_j possess a family of joint eigenfunctions $E_\kappa(\xi, \cdot), \xi \in \mathbb{R}^N$ playing the role of the exponential: $e^{\langle \xi, \cdot \rangle}$. The Dunkl transform is then defined by

$$\mathcal{F}f(\xi) = c_\kappa^{-1} \int_{\mathbb{R}^N} E(-i\xi, x) f(x) w_\kappa(x) dx$$

which turns out to be a unitary operator on $L^2(\mathbb{R}^N, w_\kappa)$ sharing several other properties with the Fourier transform (which is indeed the special case $\kappa = 0$).

Having set the stage, we can embark on developing a Dunkl theory closely following the classical Fourier analysis which in a nutshell means studying singular integrals, maximal functions and Hardy spaces. But unfortunately, the sailing is not smooth since the measure $w(x)dx$ is not translation invariant and more importantly, the generalised exponentials $E_\kappa(\xi, \cdot)$ are not characters of the group $\mathbb{R}^N$. However, it is possible to define an analogue of translations (generalised/Dunkl translations) by the formula $\mathcal{F}(\tau_x f)(\xi) = \mathcal{F}f(\xi)E(ix, \xi)$ assuming that $f \in L^2(\mathbb{R}^N, wdx)$. Using these translations we are led to define the Dunkl convolution between two functions by the formula

$$f * g(x) = \int_{\mathbb{R}^N} \tau_x f(-y) g(y) w(y) dy.$$

This generalised translation is clearly bounded on $L^2(\mathbb{R}^N, wdx)$ and the convolution has the expected property, namely $\mathcal{F}(f * g)(\xi) = \mathcal{F}f(\xi)\mathcal{F}g(\xi)$. But the analogy stops here: the boundedness properties of τ_x and hence the convolution, on other L^p spaces remain largely unknown.

This poses a major hurdle when we try to prove analogues of classical theorems in the Dunkl setting. There is no explicit formula for $\tau_x f$ except when $N = 1$ or when f is radial in which case there is an interesting and useful formula due to Rösler. Thanks to this, we have an analogue of Young's inequality for the generalised convolution at least when one of the factors is radial. This limited knowledge has been used by several authors in the study of certain convolution operators such as Bochner-Riesz means and heat semigroup associated to Δ_κ. Since the heat semigroup plays an important role in several problems of classical Fourier analysis, we would like to have a good understanding of the heat kernel $h_t(x, y)$ associate to the semigroup $e^{t\Delta_\kappa}$. However, the formula

$$h_t(x, y) = c_\kappa^{-1} t^{-\mathbf{N}/2} e^{-\frac{1}{4t}(|x|^2+|y|^2)} E_\kappa\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right)$$

involves $E_\kappa(x, y)$ which is not known explicitly. Therefore, a lack of precise estimate on the heat kernel is another problem to tackle.

Therefore, any improvement on the known estimates on the heat kernel or any extra information on the generalised translation τ_x are extremely important in Dunkl Harmonic Analysis. In her JFAA paper, written in collaboration with J. Dziubanski and J-P. Anker, Agnieszka has proved good upper and lower bounds for the heat kernel $h_t(x, y)$. In Chapter 3 of this thesis she has further improved the estimates by proving the following, see Theorem 3.1:

$$h_t(x, y) \leq C \left(1 + t^{-1}|x - y|^2\right)^{-1} \frac{1}{\max\{w(B(x, \sqrt{t}), w(B(y, \sqrt{t})\}} e^{-\frac{c}{t} d(x,y)^2}$$

where $d(x, y)$ is the orbital distance between x and y. As the generalised Poisson kernels $p_t(x, y)$ and Bessel potentials $J^{\{s\}}(x)$ are expressible in terms of the heat kernel, sharp estimates for them are also proved in the same chapter. This chapter also contains an improved estimate for translates of compactly supported radial functions and these estimates are used in later chapters.

As a consequence of the Rösler's formula for $\tau_x f$ for radial functions, it can be proved that $\operatorname{supp} \tau_x f \subset \mathcal{O}(B(x, r))$ whenever $\operatorname{supp} f \subset B(0, r)$. A very important result proved in Chapter 4 of this thesis is Theorem 4.1 where the above mentioned support property is proved for all $f \in L^2(\mathbb{R}^N, wdx)$. The idea of the proof is very ingenious- it is based on the observation that for any polynomial p of degree d on $\mathbb{R}^N$ one has

$$p(x) g_L(x) = \sum_{l=0}^{d} \sum_{|\beta| \leq l} c_{l,\beta} T^\beta (g_{L+l})(x)$$

where $g_L(x) = (1 - |x|^2)_+^L$ and $T = (T_1, T_2, ..., T_N)$. As functions of the form $p(x)g_L(x)$ are dense in $L^2(B(0, 1), wdx)$ the above formula combined with the fact that τ_x commutes with T^β allows one to prove Theorem 4.1. Let us mention two important consequences of this support theorem: (i) the inequality

$$\|\tau_y(f*\varphi)\|_{L^1(w)} \le C(r_1(r_1+r_2))^{\mathbf{N}/2}\|\varphi\|_\infty\|f\|_{L^1(w)}$$

valid for all $f\in L^1(\mathbb{R}^N, wdx)$ with $\operatorname{supp} f\subset B(0,r_2)$ and for all continuous radial functions φ with $\operatorname{supp}\varphi\subset B(0,r_1)$ (see Theorem 4.8) and (ii) improved estimates for the translations of a Schwartz class function φ (see Proposition 4.23).

The estimates of various kernels established in Part I of this thesis are put to good use in Part II where several problems of Fourier analysis are studied in the Dunkl setting. To start with, in Chapter 5 the boundedness properties of the maximal function

$$M_\varphi f(x) = \sup_{t>0}|f*\varphi_t(x)|,\ \ \varphi_t(x) = t^{-\mathbf{N}}\varphi(\frac{x}{t})$$

associated to the Dunkl convolution have been investigated. Observe that when φ is the characteristic function of the ball, M_φ reduces to the Hardy-Littlewood maximal function which was studied by this reviewer in a joint work with Y. Xu using a long winding argument. Thanks to the estimates proved in Part I, Agnieszka has given a nice proof that the general maximal function M_φ is bounded on $L^p(w), 1<p\le\infty$ and is of weak type (1,1) under some assumptions on the decay of φ and some of its derivatives, see Theorem 5.4.

In Chapter 6, an analogue of Hörmander-Mihlin multiplier theorem for the Dunkl transform has been established. The problem is to find sufficient conditions on the function m called the multiplier, so that the operator T_m defined via $\mathcal{F}(T_m f)(\xi) = m(\xi)\mathcal{F}f(\xi)$ is bounded on $L^p(w)$. In the context of Fourier transform this problem has a very long history with an extensive literature. If W_2^s stands for the Sobolev space (based on Fourier transform) the Hörmander's theorem asserts that T_m is bounded on $L^p(\mathbb{R}^N)$ whenever m satisfies the (uniform local Sobolev) condition

$$\sup_{t>0}\|\psi(\cdot)m(t\cdot)\|_{W_2^s}<\infty,\ \ s>N/2$$

where ψ is any radial bump function. Earlier works on this problem dealt only with $N=1$ or with radial multipliers. In this thesis, the author has proved that T_m is bounded on $L^p(w), 1<p<\infty$ and is of weak type (1,1) under the stronger assumption that m satisfies the above condition for some $s>\mathbf{N}$ (Theorem 6.4). The conjecture that the result is true under the weaker assumption on m, namely the condition holds only for $s>\mathbf{N}/2$ is still open. However, in Theorem 6.12, the author has shown that the conjecture follows once we have the following estimate: for all $f\in L^1(w)$,

$$\sup_{y\in\mathbb{R}^N}\|\tau_y f\|_{L^1(w)} \le C\|f\|_{L^1(w)}.$$

This brings out the importance of the boundedness of the translation operator τ_y on $L^1(w)$.

Fourier multipliers are known to be examples of singular integrals and the same is true of multipliers for the Dunkl transform. It is therefore natural to study singular integral operators of convolution type defined as

$$Kf(x) = \text{ p.v}\int_{\mathbb{R}^N}\tau_x K(-y)f(y)w(y)dy$$

where the kernel is assumed to satisfy the standard Calderon-Zygmund conditions with extra degree of smoothness. Once again, the main difficulty in carrying out the classical proof is the lack of boundedness of the translation operator. However, making use of extra information (proved in Part I of the thesis) on $\tau_x f$ when f has some regularity and decay, the author has proved weak type (1,1) and $L^p(w)$ boundedness for $1 < p < \infty$, see Theorem 7.5. By establishing Cotlar type estimates, she has also studied the maximal function associated to the singular integral operator. In Chapter 8, boundedness properties of various square functions are studied. As they can be realised as singular integral operators whose kernels are taking values in Hilbert spaces, the author has developed a unified approach to study such vector valued operators.

In the context of graded homogeneous groups, Dziubanski and his collaborators have a developed a method to study kernels of semigroups generated positive Rockland operators. It is natural to sudy such operators and associated semigroups in the Dunkl setting. In Chapter 9 of this thesis, the author has considered operators of the form

$$L = (-1)^{2l+1} \sum_{j=1}^{m} T_{\xi_j}^{2l}$$

where $\xi_j, j = 1, 2, ..., , m$ spans $\mathbb{R}^N$. If u_t stands for the kernel associated to the semigroup $U_t = e^{-tL}$ then by homogeneity arguments it is easy to see that $u_t(x) = t^{-\mathbf{N}/(2l)} u(t^{-1/(2l)} x)$. Following the ideas used by Dziubanski et al she has proved the estimate $|u(x)| \leq C e^{-c\,|x|^{\frac{2l}{2l-1}}}$ (Theorem 9.2) and the improved estimate

$$|u_t(x, y)| \leq C\, V(x, y, t^{1/(2l)})^{-1} e^{-c\, \frac{d(x,y)^{2l/(2l-1)}}{t^{1/(2l-1)}}}$$

in Theorem 9.3. In order to achieve these estimates she studied certain sesquilinear forms on suitable weighted Sobolev spaces leading to an analogue of Garding's inequality. A theorem of Lions on semigroups generated by operators has played an important role in proving the kernel estimates. This chapter has an interesting blend of techniques from partial differential equations, operator theory and harmonic analysis.

In a joint work with J.-P. Anker and J. Dziubanski, Agnieszka has introduced and studied the Hardy space $\mathcal{H}^1$ as the space of all systems $\mathbf{u} = (u_0, u_1, ..., u_N)$ on $\mathbb{R}^+ \times \mathbb{R}^N$ satisfying generalised Cauchy-Riemann equations and a uniform integrability condition. This work extends the classical theory of H^p spaces developed by Stein and Weiss to the Dunkl setting. The real Hardy space $H^1_{\Delta_\kappa}$ is then defined as the space of all boundary values of the first component $u_0(x_0, x_1, x_2, ..., x_N)$ of $\mathbf{u} \in \mathcal{H}^1$ as $x_0 \to 0$. In the same work, she has proved various characterisations of $H^1_{\Delta_\kappa}$ in terms of non-tangential maximal functions associated to Poisson and heat semigroups, and also in terms of Dunkl-Riesz transforms. In Chapter 10 of this thesis she has obtained two more characterisations- (i) in terms of radial and tangential maximal functions and (ii) an atomic decomposition. Of these two, the latter one in terms of atoms in the sense of Coifman and Weiss is more involved. The proof has necessitated the study of tent spaces and Calderon's reproducing formula in the Dunkl setting. In the following chapter, she has also introduced local Hardy spaces in terms of local maximal function associated to the heat semigroup and proved an atomic decomposition and a characterisation in terms of local Riesz transforms.

Part IV of the thesis deals with Dunkl-Schrödinger operators $L = -\Delta_\kappa + V$ where V is a non-negative potential from the reverse Hölder class $RH^q(w)$. One of the main goals was to prove an analogue of the Fefferman-Phong inequality which involves the auxiliary function m defined by

$$m(x)^{-1} = \sup\{r > 0 : \frac{r^2}{w(B(x,r)} \int_{B(x,r)} V(y)w(y)dy \leq 1\}.$$

Inspired by the works of Dziubanski and Zienkiewicz, she has proved the inequality

$$\int_{\mathbb{R}^N} |f(x)|^2 m(x)w(x)dx \leq CQ(f,f)$$

where Q is the quadratic form associated to the operator L. Another interesting problem studied is related to the eigenvalue counting function $N(L, \lambda)$ which is the number of eigenvalues of L less than or equal to λ. In Theorem 13.1 she has proved both upper and lower bounds for this function. The final chapter of this thesis deals with an atomic decomposition of the Hardy space defined in terms of the heat semigroup maximal function associated to the Schrödinger operator L.

The candidate has published five papers, some of them in very good journals, and has five more posted in the arxiv. Three of her papers are single authored and based on the 'statement of contributions concerning the joint works' by her advisor, it is clear that her contribution is over 50 per cent in these joint works. This is quite an achievement for a young graduate student!

This impressive thesis is a piece of serious mathematical work making a substantial contribution to Fourier analysis in the Dunkl setting. The problems investigated here are all very well motivated, with solid roots in the existing literature. The author has certainly read through a large number of papers and mastered several techniques employed by various authors. The proofs of most of the results proved here are elaborate, very technical and demand a solid background in several aspects of Fourier analysis. Moreover, the author has taken pains to present the results and proofs as clearly as possible which makes the well written thesis quite readable.

In conclusion, we are of the opinion that this is an outstanding thesis which deserves to be nominated for the prestigious doctoral dissertations award and hence we strongly recommend for the same.

With warm regards,

Yours sincerely,

(S. Thangavelu)