Evaluation Report on the PhD thesis

Rational approximation of surface group representations

by Aleksander Antasik

The thesis addresses the problem of rational approximation of real Hitchin representations of the fundamental group of an oriented closed surface of genus greater than one. The classical solution to this problem, due to Takeuchi, states that every Fuchsian representation can be rationally approximated. Since Fuchsian representations naturally generalize to higher-dimensional Hitchin representations in the context of higher Teichmüller theory, it is natural to ask whether Hitchin representations also admit rational approximation.

The present thesis approaches this problem in two different ways and provides an affirmative (or partially affirmative) answer. The first approach is based on results of Goldman concerning canonical Hamiltonian dynamics on character varieties generated by appropriate length functions. The idea is to use Hamiltonian twist flows associated with length functions of simple curves to move special rational representations into arbitrary open subsets of the Hitchin component of the representation variety. The corresponding Goldman twists allow one to deform rational representations, and the problem of rational approximation is reduced to analyzing how large a portion of the representation variety can be covered by lifts of Goldman twists of particular rational representations. After strengthening the classical result of Takeuchi in the case of Teichmüller space, this method, when applied to three-dimensional Hitchin representations, yields rational approximation at least on an open neighborhood of a curve of Hitchin representations obtained from an interesting curve introduced by Long, Reid, and Thistlethwaite.

In the second approach, inspired by Takeuchi's original method and based on a genericity result of Labourie, a rational approximation of a Hitchin representation is constructed through explicit manipulations of matrices and their eigenvectors. This construction uses the fact that $n \times n$ matrices with n pairwise distinct eigenvalues form an open subset of all $n \times n$ matrices, and that the centralizer in matrices of determinant one with rational entries of such a matrix with rational entries is dense in its centralizer in matrices of determinant one with real entries. The main result of this approach is the following:

Theorem 3.1.4 (Higher Takeuchi theorem). *Any Hitchin* $\mathrm{SL}_n(\mathbb{R})$*-representation can be approximated by a* $\mathrm{SL}_n(\mathbb{Q})$*-representation arbitrarily well.*

The thesis consists of five chapters and an appendix.

In Chapter 1, the author develops a linear-algebraic toolkit to be applied later. In particular, he proves a well-known result that diagonalization of an $n \times n$ matrix with n pairwise distinct eigenvalues can be smoothly extended to a diagonalization of matrices in its neighborhood. He also proves a lemma concerning the density of the centralizer of such a matrix (with rational entries and determinant one) in the group of rational points of the corresponding special linear group scheme, inside its centralizer in the group of real points of this scheme. This lemma is then used to twist a representation by a rational matrix. The chapter concludes with a basic recollection of curves on surfaces, introducing terminology used in the remainder of the thesis.

In Chapter 2, the author recalls the theory of higher Teichmüller theory and Hitchin representations. The chapter ends with a lemma combining results of Bonahon–Dreyer and Labourie concerning the pairwise distinct eigenvalues of images of homotopy classes of loops under Hitchin representations. This lemma connects higher Teichmüller theory with the aforementioned linear-algebraic toolkit and is crucial to the thesis.

Chapter 3 introduces the main problem of extending Takeuchi's result about Fuchsian representations to the class of Hitchin representations and contains an affirmative solution.

In Chapter 4, the author introduces Goldman twists on the character variety, which (for simple curves) lift to the representation variety, and applies them to obtain rational approximations of Hitchin representations. Here, the aforementioned density result for centralizers is applied. In this way, the problem of rational approximation is reduced to deforming a given rational representation so as to cover as large an open subset of the representation variety as possible. For two-dimensional Hitchin representations, this open subset coincides with the entire Hitchin component. Since simple Goldman twists preserve the so-called Witt class (introduced by Nekovář), this approximation preserves the Witt class of the representation. By strengthening a result of Farb and Margalit, the author finds an appropriate set of twists to approximate two-dimensional Hitchin representations.

Finally, Chapter 5 shows that a neighborhood of the aforementioned Long–Reid–Thistlethwaite curve can be covered by simple Goldman twists. It is established—with the help of a computer program, whose code is provided in the appendix—that the derivatives of these twists span the tangent space of the Hitchin component. The tangent spaces at explicit rational representations are investigated using the FriCAS computer algebra system. Therefore, in dimension three and genus two, some open neighborhood of this curve can be densely filled with rational images of simple Goldman twists of certain rational representations.

There is some overlap between ideas in this thesis and previously published or announced results of other authors. However, in the referee's opinion, the slightly stronger results of the present thesis were obtained independently and using different arguments. The author openly discusses comparisons with similar results previously circulated by Audibert and Zshornack.

As a final remark, the thesis is well written in terms of logic and clarity. However, it is not entirely free of minor imperfections, if one expects fully impeccable exposition. These are mostly probable typos, typographical issues, or occasional imprecise language tolerable in a journal article directed at readers familiar with the field and its occasional oversimplifications. Below is a list of examples.

p. 5. The introduction to the lemma is misplaced after the lemma, or "following" should be replaced by "above".

Lemma 1.1.4. *For a simple spectrum matrix* $M \in \mathrm{SL}_n(\mathbb{Q})$ *the group centralizer* $Z_{\mathrm{SL}_n(\mathbb{Q})}(M)$ *is dense in* $Z_{\mathrm{SL}_n(\mathbb{R})}(M)$.

The following Lemma will allow us to twist a representation by a rational matrix later.

p. 9. The definition of the Teichmüler space:

Let us define the *Teichmüller space* $\mathcal{T}(\Sigma_g)$ of Σ_g as

$$\{\text{orientation-compatible hyperbolic metrics on } \Sigma_g\ \}/\operatorname{Diff}_0(\Sigma_g),$$

doesn't make sense. There is no compatibility condition for a Riemannian metric and orientation. Given an orientation and a Riemannian metric there is a construction of the volume form.

p. 10. According to preceding definitions, the notation:

$$\partial_\infty(\pi_1(\Sigma_g))$$

conflates a group and its Cayley graph. Different generating sets can produce non-isomorphic graphs for the same group (in the notation a reference to the standard set of generators is missing), and non-isomorphic groups can sometimes share the same Cayley graph.

pp. 10-11. The character variety as a GIT quotient is denoted by

$$\operatorname{Hom}(\pi_1(\Sigma_g), G)//G$$

but next an unspecified quotient denoted differently

$$\operatorname{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_n(\mathbb{R}))/\,\mathrm{PSL}_n(\mathbb{R})$$

is used. The relation between them is not explained. Possibly a typo.

p. 11. The identification in:

Take any identification of $\mathbb{RP}^{n-1}$ with the set of real homogeneous polynomials of degree $n-1$ in variables X, Y.

doesn't make sense. When equipped with canonical Euclidean topology, one space is compact, another one is not, and they are of different dimensions. Projectivization of the second space is missing, although the symbol of a class of a homogeneous polynomial in the notation is used..

p. 12. The word "contragredient" is misspelled as "contragradient".

Despite these minor shortcomings, the author has demonstrated a good understanding of the mathematical subtleties of the problem. His solution—especially his use of elementary linear-algebraic methods and computer-assisted explicit computations in a concrete example—is sufficiently independent of approaches proposed by other authors. This achievement demonstrates the scientific potential of Aleksander Antasik, which I strongly encourage him to develop in his further research.

In summary, this work undoubtedly fulfills all the requirements for a Ph.D. thesis.

I therefore recommend, without hesitation, awarding the Ph.D. degree to Aleksander Antasik.

Warsaw, February 25, 2026

(Tomasz Maszczyk)