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Report on the doctoral thesis

On boundaries of bicombable spaces

by Daniel Danielski

This doctoral thesis contributes to the growing literature on spaces of generalized non-positive curvature from the perspective of geometric group theory. For a long time, researchers in the area have focused on the CAT(0) triangle comparison condition. An important feature of non-positively curved spaces in this sense is that any two points x, y are connected by a unique geodesic σ_{xy}, parametrized on $[0, 1]$ with constant speed, and that for any two such geodesics $\sigma_{xy}, \sigma_{x'y'}$ the distance function $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex. Already in the late 1940es, Busemann had adopted this property as the *definition* of a weak notion of non-positive curvature in the large. The resulting class of metric spaces, now called *Busemann spaces*, comprises in particular all CAT(0) spaces and all normed real vector spaces with strictly convex norm. By contrast, non-strictly convex norms are excluded, and therefore this class is apparently not closed under limit constructions. This motivates the study of a more flexible notion of global non-positive curvature that retains the convexity condition merely for a suitable family $\{\sigma_{xy}\}_{(x,y)\in X\times X}$ of geodesics, referred to as a *convex (geodesic) bicombing*. Spaces with such a generalized convex structure need no longer be uniquely geodesic, but still share many properties with CAT(0) or Busemann spaces, and their utility in geometric group theory has been recognized in recent years. An important source of examples comes from *injective metric spaces*, that is, the injective objects in the category of metric spaces with 1-Lipschitz mappings as morphisms, which are also known as *hyperconvex spaces* or *absolute 1-Lipschitz retracts*. Namely, every proper injective metric space admits a convex bicombing. For spaces of finite dimension, this bicombing is furthermore unique, hence equivariant with respect to all isometries, and *consistent* in the sense that σ_{pq} is a subsegment of σ_{xy} whenever p, q are points lying in this order on the trace of σ_{xy}. A result of Isbell from 1964 shows that every metric space X has an *injective hull* $E(X)$. His construction was rediscovered twenty years later under the name *tight span* by Dress. If X is finite, then $E(X)$ is a finite polyhedral complex with l_∞-metrics on the cells. This generalizes to some extent to infinite discrete metric spaces, despite the fact the injective hull of infinite spaces tend to be huge in general. Most notably, the injective hull of a Gromov (word) hyperbolic group is still a proper finite-dimensional polyhedral complex with only finitely many isometry types of cells. The group acts geometrically on this complex, which by the aforementioned result possesses an equivariant convex and consistent bicombing. This has furthermore led the study of *Helly groups*, groups acting geometrically on Helly graphs, the discrete analogues of injective metric spaces.

The doctoral thesis of Daniel Danielski comprises two parts. Part A, which constitutes the bulk of the thesis, provides a detailed investigation of boundaries (at infinity) of metric spaces with a convex and consistent bicombing and of EZ-structures of groups acting on them. The relatively short Part B is based on a joint paper with Michael Kapovich and Jacek Świątkowski and gives a complete characterization of those word hyperbolic groups whose boundary is homeomorphic to the Sierpiński curve or the Menger curve.

Part A starts with a concise introduction (Sect. 0), presenting the main results and outlining the structure of this part of the thesis. This is followed by Sect. 1, which collects some basic notions and properties regarding coarse geometry, geodesic bicombings, Euclidean and absolute retracts, Z-sets and EZ-structures.

Sect. 2 is devoted to the boundary defined as the set of asymptote classes of geodesic rays compatible with the given convex and consistent bicombing. This construction is similar as for CAT(0) spaces and was already carried out in work of Descombes and myself (2015), where it was also shown that the boundary is a Z-set in the compactification. Danielski completes this by showing that if G is a group acting geometrically on a proper, finite-dimensional space with a (unique) convex, consistent, and G-equivariant bicombing, then G admits an EZ-structure (Theorem I).

In a recent paper, Engel and Wulff (2023) have produced coronas (boundaries) in a more general setting for so called properly combable coarse spaces X, via the Gelfand dual of some C^*-algebra of functions on X. In Sect. 3, Danielski gives first a more elementary description of this construction and of the resulting EZ-structure in the case of proper metric spaces, and then shows that this structure is G-equivariantly equivalent to the one from Sect. 2 (Corollary 3.7).

Sect. 4 discusses the non-uniqueness of boundaries. An example due to Croke and Kleiner (2000) shows that two CAT(0) spaces with geometric actions of the same group may have non-homeomorphic boundaries. Danielski adapts this example by replacing the original piecewise Euclidean metrics with piecewise l_∞ metrics. A detailed analysis of the geodesics, partly relying on work of Miesch (2015/17), shows that the new metrics are injective and the convex bicombings have the same trajectories (non-parametrized geodesics) as the corresponding CAT(0) metrics. This allows to conclude that the two injective metric spaces thus obtained have non-homeomorphic boundaries (Theorem IV).

Sect. 5 pursues these ideas further and shows that if a locally finite, two-dimensional CAT(0) space is remetrized with the injective piecewise l_∞ metric, then the new convex bicombing still has the same trajectories. In combination with a result from Sect. 4, it follows that the two compactifications are homeomorphic (Theorem V). The short Sect. 6 adapts the known quasisymmetric structure on the boundary of a CAT(0) space to any complete metric space with a convex and consistent bicombing.

Sect. 7 contains two propositions relating algebraic properties of a group acting geometrically on a space with an equivariant, convex and consistent bicombing to the geometry of the boundary. Sect. 8 shows that (non-compact, finite-dimensional) such spaces are almost geodesically complete (Theorem X) and that the top-dimensional reduced Alexander-Spanier cohomology group of their boundary is non-zero (Theorem IX).

Altogether this is a solid thesis on a timely topic at the intersection of metric geometry and geometric group theory. The author is well informed about the existing results and techniques in this very active area, and combines them with his own ideas to advance the knowledge of spaces with bicombings and groups acting on them. The results of the thesis are not unexpected, nevertheless they constitute a valuable addition to the literature. For example, the existence of an EZ-structure for Helly groups was stated in the recent paper on the topic by Chalopin et al., but a detailed proof has been missing up to now. The thesis is very well written, in a fluent and efficient style, providing the right amount of details. I'm happy to recommend that the thesis be accepted as a doctoral dissertation.

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