NONLOCAL DIFFUSION AND APPLICATIONS

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Abstract. We consider the fractional Laplace framework and provide models and theorems related to nonlocal diffusion phenomena. Some applications are presented, including: a simple probabilistic interpretation, water waves, crystal dislocations, nonlocal phase transitions, nonlocal minimal surfaces and Schrödinger equations. Furthermore, an example of an s-harmonic function, the harmonic extension and some insight on a fractional version of a classical conjecture formulated by De Giorgi are presented. Although this paper aims at gathering some introductory material on the applications of the fractional Laplacian, some proofs and results are original. Also, the work is self contained, and the reader is invited to consult the rich bibliography for further details, whenever a subject is of interest.

Contents

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1. Introduction

In the recent years the fractional Laplace operator has received much attention both in pure and in applied mathematics.

The purpose of these pages is to collect a set of notes that are a result of several talks and minicourses delivered here and there in the world (Milan, Cortona, Pisa, Roma, Santiago del Chile, Madrid, Bologna, Porquerolles, Catania to name a few). We will present here some mathematical models related to nonlocal equations, providing some introductory material and examples.

Starting from the basics of the nonlocal equations, we will discuss in detail some recent developments in four topics of research on which we focused our attention, namely:

- a problem arising in crystal dislocation (which is related to a classical model introduced by Peierls and Nabarro),
- a problem arising in phase transitions (which is related to a nonlocal version of the classical Allen–Cahn equation),
- the limit interfaces arising in the above nonlocal phase transitions (which turn out to be nonlocal minimal surfaces, as introduced by Caffarelli, Roquejoffre and Savin), and
- a nonlocal version of the Schrödinger equation for standing waves (as introduced by Laskin).

This set of notes is organized as follows. To start with, in Section 2, we will give a motivation for the fractional Laplacian (which is the typical nonlocal operator for our framework), that originates from probabilistic considerations. As a matter of fact, no advanced knowledge of probability theory is assumed from the reader, and the topic is dealt with at an elementary level.

In Section 3, we will recall some basic properties of the fractional Laplacian, discuss some explicit examples in detail and point out some structural inequalities, that are due to a fractional comparison principle. This part ends with a quite surprising result, which states that every function can be locally approximated by functions with vanishing fractional Laplacian (in sharp contrast with the rigidity of the classical harmonic functions).

In Section 4 we deal with extended problems. It is indeed a quite remarkable fact that in many occasions nonlocal operators can be equivalently represented as local (though possibly degenerate or singular) operators in one dimension more. Moreover, as a counterpart, several models arising in a local framework give rise to nonlocal equations, due to boundary effects. So, to introduce the extension problem and give a concrete intuition of it, we will present some models in physics that are naturally set on an extended space to start with, and will show their relation to the fractional Laplacian on a trace space. We will also give a detailed justification of this extension procedure by means of the Fourier transform.

As a special example of problems arising in physics that produce a nonlocal equation, we consider a problem related to crystal dislocation, present some mathematical results that have been recently obtained on this topic, and discuss the relation between these results and the observable phenomena.

Sections 5, 6 and 7 present topics of contemporary research. We will discuss in particular: some phase transition equations of nonlocal type, their limit interfaces, which (below a critical threshold of the fractional parameter) are surfaces that minimize a nonlocal perimeter functional, and some nonlocal equations arising in quantum mechanics.

We remark that the introductory part of these notes is intended not to be separated from the one which is more research oriented: namely, even the sections whose main goal is to develop the basics of the theory contain some parts related to contemporary research trends.

Of course, these notes and the results presented do not aim to be comprehensive and cannot take into account all the material that would deserve to be included. Even a thorough introduction to nonlocal (or even just fractional) equations goes way beyond the purpose of this paper.

Many fundamental topics slipped completely out of these notes: just to name a few, the topological methods and the fine regularity theory in the fractional cases are not presented here, the fully nonlinear or singular/degenerate equations are not taken into account, only very few applications are discussed briefly, important models such as the quasi-geostrophic equation and the fractional porous media equation are not covered in these notes, we will not consider models arising in game theory such as the nonlocal tug-of-war, the parabolic equations are not taken into account in detail, unique continuation and overdetermined problems will not be studied here and the link to probability theory that we consider here is not rigorous and only superficial (the reader interested in these important topics may look, for instance, at [71, 66, 20, 65, 25, 10, 36, 11, 84, 6, 48, 49, 89]). Also, a complete discussion of the nonlocal equations in bounded domains is not available here (for this, we refer to the recent survey [78]). In terms of surveys, collections of results and open problems, we also mention the very nice website [1], which gets[1] constantly updated.

Using a metaphor with fine arts, we could say that the picture that we painted here is not even impressionistic, it is just naïf. Nevertheless, we hope that these pages may be of some help to the young researchers of all ages who are willing to have a look at the exciting nonlocal scenario (and who are willing to tolerate the partial and incomplete point of view offered by this modest observation point).

2. A probabilistic motivation

The fractional Laplacian will be the main operator studied in this paper. We consider a function $u\colon \mathbb{R}^n \to \mathbb{R}$ (which is supposed[2] to be regular enough) and a fractional parameter $s \in (0,1)$. Then, the fractional Laplacian of u is given by

$$(-\Delta)^s u(x) = \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy, \tag{2.1}$$

[1] It seems to be known that Plato did not like books because they cannot respond to questions. He might have liked websites.

[2] To write (2.1) it is sufficient, for simplicity, to take here u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of smooth and rapidly decaying functions, or in $C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We refer to [87] for a refinement of the space of definition.

where $C(n,s)$ is a dimensional[3] constant.

One sees from (2.1) that $(-\Delta)^s$ is an operator of order $2s$, namely, it arises from a differential quotient of order $2s$ weighted in the whole space. Different fractional operators have been considered in literature (see e.g. [26, 85, 73]), and all of them come from interesting problems in pure or/and applied mathematics. We will focus here on the operator in (2.1) and we will motivate it by probabilistic considerations (as a matter of fact, many other motivations are possible).

The probabilistic model under consideration is a random process that allows long jumps (in further generality, it is known that the fractional Laplacian is an infinitesimal generator of Lèvy processes, see e.g. [9] for further details). A more detailed mathematical introduction to the fractional Laplacian is then presented in the subsequent Subsection 3.1.

2.1. **The random walk with arbitrarily long jumps.** We will show here that the fractional heat equation (i.e. the "typical" equation that drives the fractional diffusion and that can be written, up to dimensional constants, as $\partial_t u+(-\Delta)^s u=0$) naturally arises from a probabilistic process in which a particle moves randomly in the space subject to a probability that allows long jumps with a polynomial tail.

For this scope, we introduce a probability distribution on the natural numbers $\mathbb{N}^*:=\{1,2,3,\cdots\}$ as follows. If $I\subseteq\mathbb{N}^*$, then the probability of I is defined to be

$$P(I):=c_s\sum_{k\in I}\frac{1}{|k|^{1+2s}}.$$

The constant c_s is taken in order to normalize P to be a probability measure. Namely, we take

$$c_s:=\left(\sum_{k\in\mathbb{N}*}\frac{1}{|k|^{1+2s}}\right)^{-1},$$

so that we have $P(\mathbb{N}^*)=1$.

Now we consider a particle that moves in $\mathbb{R}^n$ according to a probabilistic process. The process will be discrete both in time and space (in the end, we will formally take the limit when these time and space steps are small). We denote by τ the discrete time step, and by h the discrete space step. We will take the scaling $\tau=h^{2s}$ and we denote by $u(x,t)$ the probability of finding the particle at the point x at time t.

The particle in $\mathbb{R}^n$ is supposed to move according to the following probabilistic law: at each time step τ, the particle selects randomly both a direction $v\in\partial B_1$, according to the uniform distribution on ∂B_1, and a natural number $k\in\mathbb{N}^*$, according to the probability law P, and it moves by a discrete space step khv. Notice that long jumps are allowed with small probability.

Then, if the particle is at time t at the point x_0 and, following the probability law, it picks up a direction $v\in\partial B_1$ and a natural number $k\in\mathbb{N}^*$, then the particle at time $t+\tau$ will lie in x_0+khv.

Now, the probability $u(x,t+\tau)$ of finding the particle at x at time $t+\tau$ is the sum of the probabilities of finding the particle somewhere else, say at $x+khv$, for some direction $v\in\partial B_1$ and some natural number $k\in\mathbb{N}^*$, times the probability of having selected such a direction and such a natural number.

[3]The explicit value of $C(n,s)$ is usually unimportant. Nevertheless, we will compute its value explicitly in formulas (3.9) and (3.14). The reason for which it is convenient to divide $C(n,s)$ by a factor 2 in (2.1) will be clear later on, in formula (3.5).

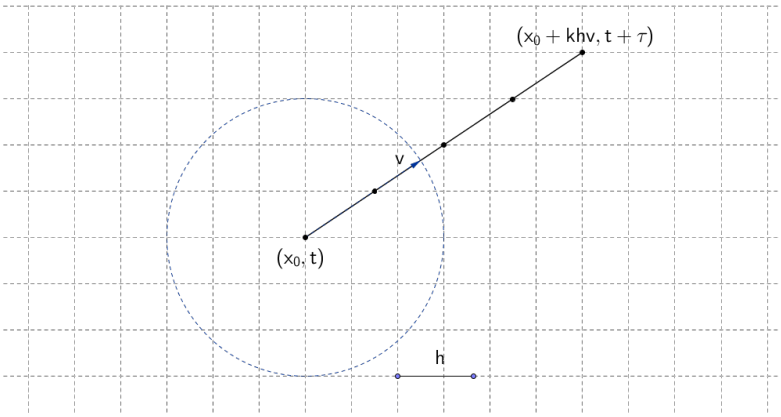

FIGURE 1. The random walk with jumps

This translates into

$$u(x,t+\tau) = \frac{c_s}{|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x+khv,t)}{|k|^{1+2s}}\, d\mathcal{H}^{n-1}(v).$$

Notice that the factor $c_s/|\partial B_1|$ is a normalizing probability constant, hence we subtract $u(x,t)$ and we obtain

$$\begin{aligned} u(x,t+\tau) - u(x,t) &= \frac{c_s}{|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x+khv,t)}{|k|^{1+2s}}\, d\mathcal{H}^{n-1}(v) - u(x,t) \\ &= \frac{c_s}{|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x+khv,t) - u(x,t)}{|k|^{1+2s}}\, d\mathcal{H}^{n-1}(v). \end{aligned}$$

As a matter of fact, by symmetry, we can change v to $-v$ in the integral above, so we find that

$$u(x,t+\tau) - u(x,t) = \frac{c_s}{|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x-khv,t) - u(x,t)}{|k|^{1+2s}}\, d\mathcal{H}^{n-1}(v).$$

Then we can sum up these two expressions (and divide by 2) and obtain that

$$\begin{aligned} &u(x,t+\tau) - u(x,t) \\ &\quad = \frac{c_s}{2\,|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x+khv,t) + u(x-khv,t) - 2u(x,t)}{|k|^{1+2s}}\, d\mathcal{H}^{n-1}(v). \end{aligned}$$

Now we divide by $\tau = h^{2s}$, we recognize a Riemann sum, we take a formal limit and we use polar coordinates, thus obtaining:

$$\begin{aligned}\partial_t u(x,t) \simeq & \frac{u(x,t+\tau)-u(x,t)}{\tau}\\ = & \frac{c_s\, h}{2\,|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x+khv,t)+u(x-khv,t)-2u(x,t)}{|hk|^{1+2s}}d\mathcal{H}^{n-1}(v)\\ \simeq & \frac{c_s}{2\,|\partial B_1|}\int_0^{+\infty}\int_{\partial B_1}\frac{u(x+rv,t)+u(x-rv,t)-2u(x,t)}{|r|^{1+2s}}d\mathcal{H}^{n-1}(v)\,dr\\ = & \frac{c_s}{2\,|\partial B_1|}\int_{\mathbb{R}^n}\frac{u(x+y,t)+u(x-y,t)-2u(x,t)}{|y|^{n+2s}}\,dy\\ = & -c_{n,s}\,(-\Delta)^s u(x,t)\end{aligned}$$

for a suitable $c_{n,s} > 0$.

This shows that, at least formally, for small time and space steps, the above probabilistic process approaches a fractional heat equation.

We observe that processes of this type occur in nature quite often, see in particular the biological observations in [91, 62] and the mathematical discussions in [63, 57, 72, 69].

Roughly speaking, let us say that it is not unreasonable that a predator may decide to use a nonlocal dispersive strategy to hunt its preys more efficiently (or, equivalently, that the natural selection may favor some kind of nonlocal diffusion): small fishes will not wait to be eaten by a big fish once they have seen it, so it may be more convenient for the big fish just to pick up a random direction, move rapidly in that direction, stop quickly and eat the small fishes there (if any) and then go on with the hunt. And this "hit-and-run" hunting procedure seems quite related to that described in Figure 1.

2.2. **A payoff model.** Another probabilistic motivation for the fractional Laplacian arises from a payoff approach. Suppose to move in a domain Ω according to a random walk with jumps as discussed in Subsection 2.1. Suppose also that exiting the domain Ω for the first time by jumping to an outside point $y \in \mathbb{R}^n\backslash\Omega$, means earning $u_0(y)$ sestertii. A relevant question is, of course, how rich we expect to become in this way. That is, if we start at a given point $x \in \Omega$ and we denote by $u(x)$ the amount of sestertii that we expect to gain, is there a way to obtain information on u?

The answer is that (in the right scale limit of the random walk with jumps presented in Subsection 2.1) the expected payoff u is determined by the equation

$$\begin{cases}(-\Delta)^s u = 0 & \text{in } \Omega,\\ u = u_0 & \text{in } \mathbb{R}^n\backslash\Omega.\end{cases} \tag{2.2}$$

To better explain this, let us fix a point $x \in \Omega$. The expected value of the payoff at x is the average of all the payoffs at the points $\tilde{x}$ from which one can reach x, weighted by the probability of the jumps. That is, by writing $\tilde{x} = x + khv$, with $v \in \partial B_1$, $k \in \mathbb{N}^*$ and $h > 0$, as in the previous Section 2.1, we have that the probability of

jump is $\frac{c_s}{|\partial B_1|\,|k|^{1+2s}}$. This leads to the formula

$$u(x)=\frac{c_s}{|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x+khv)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v).$$

By changing v into $-v$ we obtain

$$u(x)=\frac{c_s}{|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x-khv)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v)$$

and so, by summing up,

$$2u(x)=\frac{c_s}{|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x-khv)+u(x-khv)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v).$$

Since the total probability is 1, we can subtract $2u(x)$ to both sides and obtain that

$$0=\frac{c_s}{|\partial B_1|}\sum_{k\in\mathbb{N}^*}\int_{\partial B_1}\frac{u(x+khv)+u(x-khv)-2u(x)}{|k|^{1+2s}}\,d\mathcal{H}^{n-1}(v).$$

We can now divide by h^{1+2s} and recognize a Riemann sum, which, after passing to the limit as $h\searrow 0$, gives $0=-(-\Delta)^s u(x)$, that is (2.2).

3. An introduction to the fractional Laplacian

We introduce here some preliminary notions on the fractional Laplacian and on fractional Sobolev spaces. Moreover, we present an explicit example of an s-harmonic function on the positive half-line $\mathbb{R}_+$, discuss some maximum principles and a Harnack inequality, and present a quite surprising local density property of s-harmonic functions into the space of smooth functions.

3.1. **Preliminary notions.**

We introduce here the fractional Laplace operator, the fractional Sobolev spaces and give some useful pieces of notation. We also refer to [40] for a further introduction to the topic.

We consider the Schwartz space of rapidly decaying functions defined as

$$\mathcal{S}(\mathbb{R}^n):=\left\{f\in C^\infty(\mathbb{R}^n)\;\middle|\;\forall\alpha,\,\beta\in\mathbf{N}_0^n,\;\sup_{x\in\mathbb{R}^n}|x^\alpha\partial_\beta f(x)|<\infty\right\}.$$

For any $f\in\mathcal{S}(\mathbb{R}^n)$, denoting the space variable $x\in\mathbb{R}^n$ and the frequency variable $\xi\in\mathbb{R}^n$, the Fourier transform and the inverse Fourier transform are defined, respectively, as

$$\widehat{f}(\xi):=\mathcal{F}f(\xi):=\int_{\mathbb{R}^n}f(x)e^{-2\pi ix\cdot\xi}\,dx \tag{3.1}$$

and

$$f(x)=\mathcal{F}^{-1}\widehat{f}(x)=\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{2\pi ix\cdot\xi}\,d\xi. \tag{3.2}$$

Another useful notion is the one of principal value, namely we consider the definition

$$\text{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy:=\lim_{\varepsilon\to0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. \tag{3.3}$$

Notice indeed that the integrand above is singular when y is in a neighborhood of x, and this singularity is, in general, not integrable (in the sense of Lebesgue):

indeed notice that, near x, we have that $u(x)-u(y)$ behaves at the first order like $\nabla u(x)\cdot(x-y)$, hence the integral above behaves at the first order like

$$\frac{\nabla u(x)\cdot(x-y)}{|x-y|^{n+2s}} \tag{3.4}$$

whose absolute value gives an infinite integral near x (unless either $\nabla u(x)=0$ or $s<1/2$).

The idea of the definition in (3.3) is that the term in (3.4) averages out in a neighborhood of x by symmetry, since the term is odd with respect to x, and so it does not contribute to the integral if we perform it in a symmetric way. In a sense, the principal value in (3.3) kills the first order of the function at the numerator, which produces a linear growth, and focuses on the second order remainders.

The notation in (3.3) allows us to write (2.1) in the following more compact form:

$$\begin{aligned}
(-\Delta)^s u(x) &= \frac{C(n,s)}{2}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy\\
&= \frac{C(n,s)}{2}\lim_{\varepsilon\to0}\int_{\mathbb{R}^n\setminus B_\varepsilon}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy\\
&= \frac{C(n,s)}{2}\lim_{\varepsilon\to0}\left[\int_{\mathbb{R}^n\setminus B_\varepsilon}\frac{u(x)-u(x+y)}{|y|^{n+2s}}\,dy+\int_{\mathbb{R}^n\setminus B_\varepsilon}\frac{u(x)-u(x-y)}{|y|^{n+2s}}\,dy\right]\\
&= \frac{C(n,s)}{2}\lim_{\varepsilon\to0}\left[\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(\eta)}{|x-\eta|^{n+2s}}\,d\eta+\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(\zeta)}{|x-\zeta|^{n+2s}}\,d\zeta\right]\\
&= C(n,s)\lim_{\varepsilon\to0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(\eta)}{|x-\eta|^{n+2s}}\,d\eta,
\end{aligned}$$

where the changes of variable $\eta:=x+y$ and $\zeta:=x-y$ were used, i.e.

$$(-\Delta)^s u(x)=C(n,s)\,\text{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. \tag{3.5}$$

The simplification above also explains why it was convenient to write (2.1) with the factor 2 dividing $C(n,s)$. Notice that the expression in (2.1) does not require the P.V. formulation since, for instance, taking $u\in L^\infty(\mathbb{R}^n)$ and locally C^2, using a Taylor expansion of u in B_1, one observes that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{|2u(x)-u(x+y)-u(x-y)|}{|y|^{n+2s}}\,dy\\
&\leqslant \|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_1}|y|^{-n-2s}\,dy+\int_{B_1}\frac{|D^2u(x)||y|^2}{|y|^{n+2s}}\,dy\\
&\leqslant \|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_1}|y|^{-n-2s}\,dy+\|D^2u\|_{L^\infty(\mathbb{R}^n)}\int_{B_1}|y|^{-n-2s+2}\,dy,
\end{aligned}$$

and the integrals above provide a finite quantity.

Furthermore, for $u\in\mathcal{S}(\mathbb{R}^n)$ the fractional Laplace operator can be expressed as an inverse Fourier transform, as stated in the following lemma.

Lemma 3.1. *We have that*

$$(-\Delta)^s u(x)=\mathcal{F}^{-1}\big((2\pi|\xi|)^{2s}\widehat{u}(\xi)\big). \tag{3.6}$$

Roughly speaking, formula (3.6) characterizes the fractional Laplace operator in the Fourier space, by taking the s-power of the multiplier associated to the classical Laplacian operator. Indeed, by inverse Fourier transform, one has that

$$\begin{aligned}-\Delta u(x) &= -\Delta(\mathcal{F}^{-1}(\widehat{u}))(x) = -\Delta\int_{\mathbb{R}^n}\widehat{u}(\xi)e^{2\pi ix\cdot\xi}\,d\xi\\ &= \int_{\mathbb{R}^n}(2\pi|\xi|)^2\widehat{u}(\xi)e^{2\pi ix\cdot\xi}\,d\xi = \mathcal{F}^{-1}\big((2\pi|\xi|)^2\widehat{u}(\xi)\big),\end{aligned}$$

which gives that the classical Laplacian acts in a Fourier space as a multiplier of $(2\pi|\xi|)^2$. From this and Lemma 3.1 it also follows that the classical Laplacian is the limit case of the fractional one, namely for any $u\in\mathcal{S}(\mathbb{R}^n)$

$$\lim_{s\to1}(-\Delta)^s u = -\Delta u \qquad \text{and also} \qquad \lim_{s\to0}(-\Delta)^s u = -u.$$

Let us now prove that indeed the two formulations (2.1) and (3.6) are equivalent.

Proof of Lemma 3.1. Consider identity (2.1) and apply the Fourier transform to obtain

$$\begin{aligned}\mathcal{F}\Big((-\Delta)^s u(x)\Big) &= \frac{C(n,s)}{2}\int_{\mathbb{R}^n}\frac{\mathcal{F}\Big(2u(x)-u(x+y)-u(x-y)\Big)}{|y|^{n+2s}}\,dy\\ &= \frac{C(n,s)}{2}\int_{\mathbb{R}^n}\widehat{u}(\xi)\frac{2-e^{2\pi i\xi\cdot y}-e^{-2\pi i\xi\cdot y}}{|y|^{n+2s}}\,dy\\ &= C(n,s)\,\widehat{u}(\xi)\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy.\end{aligned}\tag{3.7}$$

Now, we use the change of variable $z=|\xi|y$ and obtain that

$$\begin{aligned}J(\xi) &:= \int_{\mathbb{R}^n}\frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy\\ &= |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-\cos\frac{2\pi\xi}{|\xi|}\cdot z}{|z|^{n+2s}}\,dz.\end{aligned}$$

Now we use that J is rotationally invariant. More precisely, we consider a rotation R that sends $e_1=(1,0,\dots,0)$ into $\xi/|\xi|$, that is $Re_1=\xi/|\xi|$, and we call R^T its transpose. Then, by using the change of variables $\omega=R^Tz$ we have that

$$\begin{aligned}J(\xi) &= |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-\cos(2\pi Re_1\cdot z)}{|z|^{n+2s}}\,dz\\ &= |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-\cos(2\pi R^Tz\cdot e_1)}{|R^Tz|^{n+2s}}\,dz\\ &= |\xi|^{2s}\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\omega_1)}{|\omega|^{n+2s}}\,d\omega.\end{aligned}$$

Changing variables $\tilde{\omega}=2\pi\omega$ (we still write ω as a variable of integration), we obtain that

$$J(\xi) = (2\pi|\xi|)^{2s}\int_{\mathbb{R}^n}\frac{1-\cos\omega_1}{|\omega|^{n+2s}}\,d\omega.\tag{3.8}$$

Notice that this latter integral is finite. Indeed, integrating outside the ball B_1 we have that

$$\int_{\mathbb{R}^n\setminus B_1}\frac{|1-\cos\omega_1|}{|\omega|^{n+2s}}\,d\omega \leqslant \int_{\mathbb{R}^n\setminus B_1}\frac{2}{|\omega|^{n+2s}} < \infty,$$

while inside the ball we can use the Taylor expansion of the cosine function and observe that

$$\int_{B_1} \frac{|1-\cos\omega_1|}{|\omega|^{n+2s}}\,d\omega \leqslant \int_{B_1} \frac{|\omega|^2}{|\omega|^{n+2s}}\,d\omega \leqslant \int_{B_1} \frac{d\omega}{|\omega|^{n+2s-2}} < \infty.$$

Therefore, by taking

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1-\cos\omega_1}{|\omega|^{n+2s}}\,d\omega\right)^{-1} \tag{3.9}$$

it follows from (3.8) that

$$J(\xi) = \frac{(2\pi|\xi|)^{2s}}{C(n,s)}.$$

By inserting this into (3.7), we obtain that

$$\mathcal{F}\Big((-\Delta)^s u(x)\Big) = C(n,s)\,\widehat{u}(\xi)\,J(\xi) = (2\pi|\xi|)^{2s}\widehat{u}(\xi),$$

which concludes the proof. □

It is worth to point out that the renormalization constant $C(n,s)$ introduced in (2.1) has now been explicitly computed in (3.9).

Another approach to the fractional Laplacian comes from the theory of semigroups (or, equivalently from the fractional calculus arising in subordination identities). This technique is classical (see [92]), but it has also been efficiently used in recent research papers (see for instance [34, 90, 22]). Roughly speaking, the main idea underneath the semigroup approach comes from the following explicit formulas for the Euler's function: for any $\lambda > 0$, one uses an integration by parts and the substitution $\tau = \lambda t$ to see that

$$\begin{aligned}
-s\Gamma(-s) &= \Gamma(1-s)\\
&= \int_0^{+\infty} \tau^{-s} e^{-\tau}\,d\tau\\
&= -\int_0^{+\infty} \tau^{-s}\frac{d}{d\tau}(e^{-\tau}-1)\,d\tau\\
&= -s\int_0^{+\infty} \tau^{-s-1}(e^{-\tau}-1)\,d\tau\\
&= -s\lambda^{-s}\int_0^{+\infty} t^{-s-1}(e^{-\lambda t}-1)\,dt,
\end{aligned}$$

that is

$$\lambda^s = \frac{1}{\Gamma(-s)}\int_0^{+\infty} t^{-s-1}(e^{-\lambda t}-1)\,dt. \tag{3.10}$$

Then one applies formally this identity to $\lambda := -\Delta$. Of course, this formal step needs to be justified, but if things go well one obtains

$$(-\Delta)^s = \frac{1}{\Gamma(-s)}\int_0^{+\infty} t^{-s-1}(e^{t\Delta}-1)\,dt,$$

that is (interpreting 1 as the identity operator)

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)}\int_0^{+\infty} t^{-s-1}(e^{t\Delta}u(x)-u(x))\,dt. \tag{3.11}$$

Formally, if $U(x,t) := e^{t\Delta}u(x)$, we have that $U(x,0) = u(x)$ and

$$\partial_t U = \frac{\partial}{\partial t}(e^{t\Delta}u(x)) = \Delta e^{t\Delta}u(x) = \Delta U,$$

that is $U(x,t) = e^{t\Delta}u(x)$ can be interpreted as the solution of the heat equation with initial datum u. We indeed point out that these formal computations can be justified:

Lemma 3.2. *Formula (3.11) holds true. That is, if $u \in \mathcal{S}(\mathbb{R}^n)$ and $U = U(x,t)$ is the solution of the heat equation*

$$\begin{cases} \partial_t U = \Delta U & \text{in } t > 0, \\ U\big|_{t=0} = u, \end{cases}$$

then

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{+\infty} t^{-s-1}(U(x,t) - u(x))\, dt. \tag{3.12}$$

Proof. From Theorem 1 on page 47 in [47] we know that U is obtained by Gaussian convolution with unit mass, i.e.

$$\begin{aligned} U(x,t) = \int_{\mathbb{R}^n} G(x-y,t)\, u(y)\, dy = \int_{\mathbb{R}^n} G(y,t)\, u(x-y)\, dy, \\ \text{where } \ G(x,t) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}. \end{aligned} \tag{3.13}$$

As a consequence, using the substitution $\tau := |y|^2/(4t)$,

$$\begin{aligned} &\int_0^{+\infty} t^{-s-1}(U(x,t) - u(x))\, dt \\ &= \int_0^{+\infty} \left[\int_{\mathbb{R}^n} t^{-s-1} G(y,t) \big(u(x-y) - u(x)\big)\, dy \right] dt \\ &= \int_0^{+\infty} \left[\int_{\mathbb{R}^n} (4\pi t)^{-n/2} t^{-s-1} e^{-|y|^2/(4t)} \big(u(x-y) - u(x)\big)\, dy \right] dt \\ &= \int_0^{+\infty} \left[\int_{\mathbb{R}^n} \tau^{n/2} (\pi |y|^2)^{-n/2} |y|^{-2s} (4\tau)^{s+1} e^{-\tau} \big(u(x-y) - u(x)\big)\, dy \right] \frac{d\tau}{4\tau^2} \\ &= 2^{2s-1}\pi^{-n/2} \int_0^{+\infty} \left[\int_{\mathbb{R}^n} \tau^{\frac{n}{2}+s-1} e^{-\tau} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy \right] d\tau. \end{aligned}$$

Now we notice that

$$\int_0^{+\infty} \tau^{\frac{n}{2}+s-1} e^{-\tau}\, d\tau = \Gamma\left(\frac{n}{2} + s\right),$$

so we obtain that

$$\begin{aligned} &\int_0^{+\infty} t^{-s-1}(U(x,t) - u(x))\, dt \\ &= 2^{2s-1}\pi^{-n/2}\Gamma\left(\frac{n}{2}+s\right) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy \\ &= -\frac{2^{2s}\, \pi^{-n/2}\, \Gamma\left(\frac{n}{2}+s\right)}{C(n,s)} (-\Delta)^s u(x). \end{aligned}$$

This proves (3.12), by choosing $C(n,s)$ appropriately. And, as a matter of fact, gives the explicit value of the constant $C(n,s)$ as

$$C(n,s) = -\frac{2^{2s}\,\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\Gamma(-s)} = \frac{2^{2s}\,s\,\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\Gamma(1-s)}, \tag{3.14}$$

where we have used again that $\Gamma(1-s) = -s\Gamma(-s)$, for any $s \in (0,1)$. □

It is worth to point out that the renormalization constant $C(n,s)$ introduced in (2.1) has now been explicitly computed in (3.14). Notice that the choices of $C(n,s)$ in (3.9) and (3.14) must agree (since we have computed the fractional Laplacian in two different ways): for a computation that shows that the quantity in (3.9) coincides with the one in (3.14), see Theorem 3.9 in [12]. For completeness, we give below a direct proof that the settings in (3.9) and (3.14) are the same, by using Fourier methods and (3.10):

Lemma 3.3. *For any $n \in \mathbb{N}$, $n \geqslant 1$, and $s \in (0,1)$, we have that*

$$\int_{\mathbb{R}^n} \frac{1-\cos(2\pi\omega_1)}{|\omega|^{n+2s}}\,d\omega = \frac{\pi^{\frac{n}{2}+2s}\,\Gamma(1-s)}{s\,\Gamma\left(\frac{n}{2}+s\right)}. \tag{3.15}$$

Equivalently, we have that

$$\int_{\mathbb{R}^n} \frac{1-\cos\omega_1}{|\omega|^{n+2s}}\,d\omega = \frac{\pi^{\frac{n}{2}}\,\Gamma(1-s)}{2^{2s}s\,\Gamma\left(\frac{n}{2}+s\right)}. \tag{3.16}$$

Proof. Of course, formula (3.15) is equivalent to (3.16) (after the substitution $\tilde{\omega} := 2\pi\omega$). Strictly speaking, in Lemma 3.1 (compare (2.1), (3.6), and (3.9)) we have proved that

$$\frac{1}{2\displaystyle\int_{\mathbb{R}^n} \frac{1-\cos\omega_1}{|\omega|^{n+2s}}\,d\omega} \int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy = \mathcal{F}^{-1}\big((2\pi|\xi|)^{2s}\widehat{u}(\xi)\big). \tag{3.17}$$

Similarly, by means of Lemma 3.2 (compare (2.1), (3.12) and (3.14)) we know that

$$\begin{aligned} &\frac{2^{2s-1}\,s\,\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\Gamma(1-s)} \int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy \\ &\qquad = \frac{1}{\Gamma(-s)} \int_0^{+\infty} t^{-s-1}(U(x,t)-u(x))\,dt. \end{aligned} \tag{3.18}$$

Moreover (see (3.13)), we have that $U(x,t) := \Gamma_t * u(x)$, where

$$\Gamma_t(x) := G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}.$$

We recall that the Fourier transform of a Gaussian is a Gaussian itself, namely

$$\mathcal{F}(e^{-\pi|x|^2}) = e^{-\pi|\xi|^2},$$

therefore, for any fixed $t > 0$, using the substitution $y := x/\sqrt{4\pi t}$,

$$\begin{aligned} \mathcal{F}\,\Gamma_t(\xi) &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2/(4t)} e^{-2\pi i x\cdot\xi}\,dx \\ &= \int_{\mathbb{R}^n} e^{-\pi|y|^2} e^{-2\pi i y\cdot(\sqrt{4\pi t}\xi)}\,dy \\ &= e^{-4\pi^2 t|\xi|^2}. \end{aligned}$$

As a consequence

$$\begin{aligned}\mathcal{F}\big(U(x,t)-u(x)\big) &= \mathcal{F}\big(\Gamma_t * u(x) - u(x)\big)\\ &= \mathcal{F}(\Gamma_t * u)(\xi) - \widehat{u}(\xi) = \big(\mathcal{F}\,\Gamma_t(\xi) - 1\big)\widehat{u}(\xi)\\ &= (e^{-4\pi^2 t|\xi|^2} - 1)\widehat{u}(\xi).\end{aligned}$$

We multiply by t^{-s-1} and integrate over $t>0$, and we obtain

$$\begin{aligned}\mathcal{F}\int_0^{+\infty} t^{-s-1}\big(U(x,t)-u(x)\big)\,dt &= \int_0^{+\infty} t^{-s-1}(e^{-4\pi^2 t|\xi|^2}-1)\,dt\,\widehat{u}(\xi)\\ &= \Gamma(-s)\,(4\pi^2|\xi|^2)^s\,\widehat{u}(\xi),\end{aligned}$$

thanks to (3.10) (used here with $\lambda := 4\pi^2|\xi|^2$). By taking the inverse Fourier transform, we have

$$\int_0^{+\infty} t^{-s-1}\big(U(x,t)-u(x)\big)\,dt = \Gamma(-s)\,(2\pi)^{2s}\mathcal{F}^{-1}\big(|\xi|^{2s}\,\widehat{u}(\xi)\big).$$

We insert this information into (3.18) and we get

$$\frac{2^{2s-1}\,s\,\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\Gamma(1-s)}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy = (2\pi)^{2s}\mathcal{F}^{-1}\big(|\xi|^{2s}\,\widehat{u}(\xi)\big).$$

Hence, recalling (3.17),

$$\begin{aligned}&\frac{2^{2s-1}\,s\,\Gamma\left(\frac{n}{2}+s\right)}{\pi^{n/2}\Gamma(1-s)}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy\\ &\quad= \frac{1}{2\displaystyle\int_{\mathbb{R}^n}\frac{1-\cos\omega_1}{|\omega|^{n+2s}}\,d\omega}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy,\end{aligned}$$

which gives the desired result. □

For the sake of completeness, a different proof of Lemma 3.3 will be given in Appendix A. There, to prove Lemma 3.3, we will use the theory of special functions rather than the fractional Laplacian.

3.2. **An s-harmonic function.** We provide here an explicit example of a function that is s-harmonic on the positive line $\mathbb{R}_+ := (0,+\infty)$. Namely, we prove the following result:

Theorem 3.4. *For any $x\in\mathbb{R}$, let $w_s(x) := x_+^s = \max\{x,0\}^s$. Then*

$$(-\Delta)^s w_s(x) = \begin{cases} -c_s|x|^{-s} & \text{if } x<0,\\ 0 & \text{if } x>0,\end{cases}$$

for a suitable constant $c_s>0$.

At a first glance, it may be quite surprising that the function x_+^s is s-harmonic in $(0,+\infty)$, since such function is not smooth (but only continuous) uniformly up to the boundary, so let us try to give some heuristic explanations for it.

We try to understand why the function x_+^s is s-harmonic in, say, the interval $(0,1)$ when $s\in(0,1]$. From the discussion in Subsection 2.2, we know that the s-harmonic function in $(0,1)$ that agrees with x_+^s outside $(0,1)$ coincides with the expected value of a payoff, subject to a random walk (the random walk is classical when $s=1$ and it presents jumps when $s\in(0,1)$). If $s=1$ and we start from the middle of the interval, we have the same probability of being moved randomly to the left and to

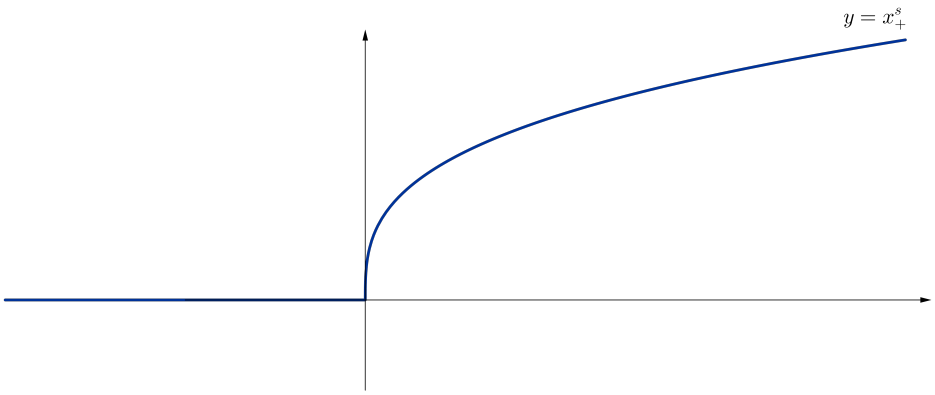

FIGURE 2. An s-harmonic function

the right. This means that we have the same probability of exiting the interval $(0,1)$ to the right (and so ending the process at $x = 1$, which gives 1 as payoff) or to the left (and so ending the process at $x = 0$, which gives 0 as payoff). Therefore the expected value starting at $x = 1/2$ is exactly the average between 0 and 1, which is $1/2$. Similarly, if we start the process at the point $x = 1/4$, we have the same probability of reaching the point 0 on the left and the point $1/2$ to the right. Since we know that the payoff at $x = 0$ is 0 and the expected value of the payoff at $x = 1/2$ is $1/2$, we deduce in this case that the expected value for the process starting at $1/4$ is the average between 0 and $1/2$, that is exactly $1/4$. We can repeat this argument over and over, and obtain the (rather obvious) fact that the linear function is indeed harmonic in the classical sense.

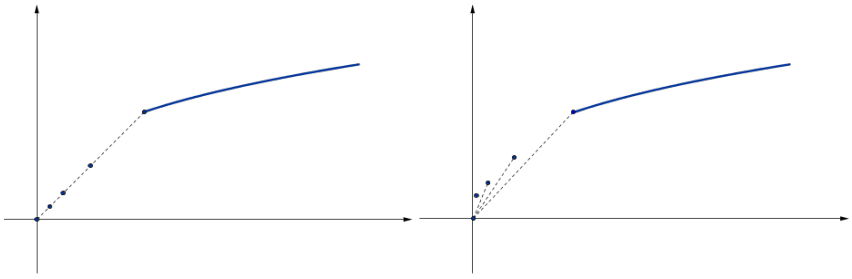

FIGURE 3. A payoff model: case $s = 1$ and $s \in (0,1)$

The argument above, which seems either trivial or unnecessarily complicated in the classical case, can be adapted when $s \in (0,1)$ and it can give a qualitative picture of the corresponding s-harmonic function. Let us start again the random walk, this time with jumps, at $x = 1/2$: in presence of jumps, we have the same probability of reaching the left interval $(-\infty, 0]$ and the right interval $[1, +\infty)$. Now, the payoff at $(-\infty, 0]$ is 0, while the payoff at $[1, +\infty)$ is *bigger* than 1. This implies

that the expected value at $x = 1/2$ is the average between 0 and something bigger than 1, which produces a value larger than $1/2$. When repeating this argument over and over, we obtain a concavity property enjoyed by the s-harmonic functions in this case (the exact values prescribed in $[1, +\infty)$ are not essential here, it would be enough that these values were monotone increasing and larger than 1).

In a sense, therefore, this concavity properties and loss of Lipschitz regularity near the minimal boundary values is typical of nonlocal diffusion and it is due to the possibility of "reaching far away points", which may increase the expected payoff.

Now we present a complete, elementary proof of Theorem 3.4. The proof originated from some pleasant discussions with Fernando Soria and it is based on some rather surprising integral cancellations. The reader who wishes to skip this proof can go directly to Subsection 3.3 on page 18.

We start with some preliminary computations.

Lemma 3.5. *For any $s \in (0,1)$*

$$\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\,dt + \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}} = \frac{1}{s}.$$

Proof. Fixed $\varepsilon > 0$, we integrate by parts:

$$\begin{aligned}
&\int_\varepsilon^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\,dt \\
&= -\frac{1}{2s}\int_\varepsilon^1 \Big[(1+t)^s + (1-t)^s - 2\Big]\frac{d}{dt}t^{-2s}\,dt \\
&= \frac{1}{2s}\left[\frac{(1+\varepsilon)^s + (1-\varepsilon)^s - 2}{\varepsilon^{2s}} - 2^s + 2\right] + \frac{1}{2}\int_\varepsilon^1 \frac{(1+t)^{s-1} - (1-t)^{s-1}}{t^{2s}}\,dt \\
&= \frac{1}{2s}\left[o(1) - 2^s + 2\right] + \frac{1}{2}\left(\int_\varepsilon^1 (1+t)^{s-1}t^{-2s}\,dt - \int_\varepsilon^1 (1-t)^{s-1}t^{-2s}\,dt\right),
\end{aligned} \tag{3.19}$$

with $o(1)$ infinitesimal as $\varepsilon \searrow 0$. Moreover, by changing variable $\tilde t := t/(1-t)$, that is $t := \tilde t/(1+\tilde t)$, we have that

$$\int_\varepsilon^1 (1-t)^{s-1}t^{-2s}\,dt = \int_{\varepsilon/(1-\varepsilon)}^{+\infty} (1+\tilde t)^{s-1}\tilde t^{-2s}\,d\tilde t.$$

Inserting this into (3.19) (and writing t instead of $\tilde t$ as variable of integration), we obtain

$$\begin{aligned}
&\int_\varepsilon^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\,dt \\
&\quad= \frac{1}{2s}\big[o(1) - 2^s + 2\big] + \frac{1}{2}\left[\int_\varepsilon^1 (1+t)^{s-1}t^{-2s}\,dt - \int_{\varepsilon/(1-\varepsilon)}^{+\infty} (1+t)^{s-1}t^{-2s}\,dt\right] \\
&\quad= \frac{1}{2s}\big[o(1) - 2^s + 2\big] + \frac{1}{2}\left[\int_\varepsilon^{\varepsilon/(1-\varepsilon)} (1+t)^{s-1}t^{-2s}\,dt - \int_1^{+\infty} (1+t)^{s-1}t^{-2s}\,dt\right].
\end{aligned} \tag{3.20}$$

Now we remark that

$$\int_\varepsilon^{\varepsilon/(1-\varepsilon)} (1+t)^{s-1}t^{-2s}\,dt \leqslant \int_\varepsilon^{\varepsilon/(1-\varepsilon)} (1+\varepsilon)^{s-1}\varepsilon^{-2s}\,dt = \varepsilon^{2-s}(1-\varepsilon)^{-1}(1+\varepsilon)^{s-1},$$

therefore

$$\lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\varepsilon/(1-\varepsilon)} (1+t)^{s-1} t^{-2s}\, dt = 0.$$

So, by passing to the limit in (3.20), we get

$$\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt = \frac{-2^s+2}{2s} - \frac{1}{2}\int_1^{+\infty} (1+t)^{s-1} t^{-2s}\, dt. \tag{3.21}$$

Now, integrating by parts we see that

$$\begin{aligned} \frac{1}{2}\int_1^{+\infty} (1+t)^{s-1} t^{-2s}\, dt &= \frac{1}{2s}\int_1^{+\infty} t^{-2s} \frac{d}{dt}(1+t)^s\, dt \\ &= -\frac{2^s}{2s} + \int_1^{+\infty} t^{-1-2s}(1+t)^s\, dt. \end{aligned}$$

By plugging this into (3.21) we obtain that

$$\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt = \frac{-2^s+2}{2s} + \frac{2^s}{2s} - \int_1^{+\infty} t^{-1-2s}(1+t)^s\, dt,$$

which gives the desired result. □

From Lemma 3.5 we deduce the following (somehow unexpected) cancellation property:

Corolary 3.6. *Let w_s be as in the statement of Theorem 3.4. Then*

$$(-\Delta)^s w_s(1) = 0.$$

Proof. The function $t \mapsto (1+t)^s + (1-t)^s - 2$ is even, therefore

$$\int_{-1}^1 \frac{(1+t)^s + (1-t)^s - 2}{|t|^{1+2s}}\, dt = 2\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt.$$

Moreover, by changing variable $\tilde{t} := -t$, we have that

$$\int_{-\infty}^{-1} \frac{(1-t)-2}{|t|^{1+2s}}\, dt = \int_1^{+\infty} \frac{(1+\tilde{t})-2}{\tilde{t}^{1+2s}}\, d\tilde{t}.$$

Therefore

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{w_s(1+t) + w_s(1-t) - 2w_s(1)}{|t|^{1+2s}}\, dt \\ = &\int_{-\infty}^{-1} \frac{(1-t)^s - 2}{|t|^{1+2s}}\, dt + \int_{-1}^1 \frac{(1+t)^s + (1-t)^s - 2}{|t|^{1+2s}}\, dt + \int_1^{+\infty} \frac{(1+t)^s - 2}{|t|^{1+2s}}\, dt \\ = &\ 2\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt + 2\int_1^{+\infty} \frac{(1+t)^s - 2}{t^{1+2s}}\, dt \\ = &\ 2\left[\int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt + \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt - 2\int_1^{+\infty} \frac{dt}{t^{1+2s}}\right] \\ = &\ 2\left[\frac{1}{s} - 2\int_1^{+\infty} \frac{dt}{t^{1+2s}}\right], \end{aligned}$$

where Lemma 3.5 was used in the last line. Since

$$\int_1^{+\infty} \frac{dt}{t^{1+2s}} = \frac{1}{2s},$$

we obtain that

$$\int_{-\infty}^{+\infty} \frac{w_s(1+t)+w_s(1-t)-2w_s(1)}{|t|^{1+2s}}\,dt = 0,$$

that proves the desired claim. □

The counterpart of Corollary 3.6 is given by the following simple observation:

Lemma 3.7. *Let w_s be as in the statement of Theorem 3.4. Then*

$$-(-\Delta)^s w_s(-1) > 0.$$

Proof. We have that

$$w_s(-1+t)+w_s(-1-t)-2w_s(-1) = (-1+t)_+^s + (-1-t)_+^s \geqslant 0$$

and not identically zero, which implies the desired result. □

We have now all the elements to proceed to the proof of Theorem 3.4.

Proof of Theorem 3.4. We let $\sigma \in \{+1,-1\}$ denote the sign of a fixed $x \in \mathbb{R}\setminus\{0\}$. We claim that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{w_s(\sigma(1+t))+w_s(\sigma(1-t))-2w_s(\sigma)}{|t|^{1+2s}}\,dt \\ =\ &\int_{-\infty}^{+\infty} \frac{w_s(\sigma+t)+w_s(\sigma-t)-2w_s(\sigma)}{|t|^{1+2s}}\,dt. \end{aligned} \tag{3.22}$$

Indeed, the formula above is obvious when $x>0$ (i.e. $\sigma=1$), so we suppose $x<0$ (i.e. $\sigma=-1$) and we change variable $\tau := -t$, to see that, in this case,

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{w_s(\sigma(1+t))+w_s(\sigma(1-t))-2w_s(\sigma)}{|t|^{1+2s}}\,dt \\ =\ &\int_{-\infty}^{+\infty} \frac{w_s(-1-t)+w_s(-1+t)-2w_s(\sigma)}{|t|^{1+2s}}\,dt \\ =\ &\int_{-\infty}^{+\infty} \frac{w_s(-1+\tau)+w_s(-1-\tau)-2w_s(\sigma)}{|\tau|^{1+2s}}\,d\tau \\ =\ &\int_{-\infty}^{+\infty} \frac{w_s(\sigma+\tau)+w_s(\sigma-\tau)-2w_s(\sigma)}{|\tau|^{1+2s}}\,d\tau, \end{aligned}$$

thus checking (3.22).

Now we observe that, for any $r \in \mathbb{R}$,

$$w_s(|x|r) = (|x|r)_+^s = |x|^s r_+^s = |x|^s w_s(r).$$

That is

$$w_s(xr) = w_s(\sigma|x|r) = |x|^s w_s(\sigma r).$$

So we change variable $y = tx$ and we obtain that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{w_s(x+y)+w_s(x-y)-2w_s(x)}{|y|^{1+2s}}\,dy\\
&= \int_{-\infty}^{+\infty} \frac{w_s(x(1+t))+w_s(x(1-t))-2w_s(x)}{|x|^{2s}|t|^{1+2s}}\,dt\\
&= |x|^{-s}\int_{-\infty}^{+\infty} \frac{w_s(\sigma(1+t))+w_s(\sigma(1-t))-2w_s(\sigma)}{|t|^{1+2s}}\,dt\\
&= |x|^{-s}\int_{-\infty}^{+\infty} \frac{w_s(\sigma+t)+w_s(\sigma-t)-2w_s(\sigma)}{|t|^{1+2s}}\,dt,
\end{aligned}$$

where (3.22) was used in the last line. This says that

$$(-\Delta)^s w_s(x) = \begin{cases} |x|^{-s}\,(-\Delta)^s w_s(-1) & \text{if } x<0,\\ |x|^{-s}\,(-\Delta)^s w_s(1) & \text{if } x>0, \end{cases}$$

hence the result in Theorem 3.4 follows from Corollary 3.6 and Lemma 3.7. □

3.3. **Maximum Principle and Harnack Inequality.** The Harnack Inequality and the Maximum Principle for harmonic functions are classical topics in elliptic regularity theory. Namely, in the classical case, if a nonnegative function is harmonic in B_1 and $r \in (0,1)$, then its minimum and maximum in B_r must always be comparable (in particular, the function cannot touch the level zero in B_r).

It is worth pointing out that the fractional counterpart of these facts is, in general, false, as this next simple result shows (see [64]):

Theorem 3.8. *There exists a bounded function u which is s-harmonic in B_1, nonnegative in B_1, but such that $\inf_{B_1} u = 0$.*

Sketch of the proof. The main idea is that we are able to take the datum of u outside B_1 in a suitable way as to "bend down" the function inside B_1 until it reaches the level zero. Namely, let $M \geqslant 0$ and we take u_M to be the function satisfying

$$\begin{cases} (-\Delta)^s u_M = 0 & \text{in } B_1,\\ u_M = 1-M & \text{in } B_3\backslash B_2,\\ u_M = 1 & \text{in } \mathbb{R}^n\backslash\big((B_3\backslash B_2)\cup B_1\big). \end{cases} \tag{3.23}$$

When $M=0$, the function u_M is identically 1. When $M>0$, we expect u_M to bend down, since the fact that the fractional Laplacian vanishes in B_1 forces the second order quotient to vanish in average (recall (2.1), or the equivalent formulation in (3.5)). Indeed, we claim that there exists $M_\star > 0$ such that $u_{M_\star} \geqslant 0$ in B_1 with $\inf_{B_1} u_{M_\star} = 0$. Then, the result of Theorem 3.8 would be reached by taking $u := u_{M_\star}$.

To check the existence of such $M_\star$, we show that $\inf_{B_1} u_M \to -\infty$ as $M \to +\infty$. Indeed, we argue by contradiction and suppose this cannot happen. Then, for any $M \geqslant 0$, we would have that

$$\inf_{B_1} u_M \geqslant -a, \tag{3.24}$$

for some fixed $a \in \mathbb{R}$. We set

$$v_M := \frac{u_M + M - 1}{M}.$$

Then, by (3.23),

$$\begin{cases} (-\Delta)^s v_M = 0 & \text{in } B_1, \\ v_M = 0 & \text{in } B_3\backslash B_2, \\ v_M = 1 & \text{in } \mathbb{R}^n\backslash\big((B_3\backslash B_2)\cup B_1\big). \end{cases}$$

Also, by (3.24), for any $x \in B_1$,

$$v_M(x) \geqslant \frac{-a+M-1}{M}.$$

By taking limits, one obtains that v_M approaches a function v_∞ that satisfies

$$\begin{cases} (-\Delta)^s v_\infty = 0 & \text{in } B_1, \\ v_\infty = 0 & \text{in } B_3\backslash B_2, \\ v_\infty = 1 & \text{in } \mathbb{R}^n\backslash\big((B_3\backslash B_2)\cup B_1\big) \end{cases}$$

and, for any $x \in B_1$,

$$v_\infty(x) \geqslant 1.$$

In particular the maximum of v_∞ is attained at some point $x_\star \in B_1$, with $v_\infty(x_\star) \geqslant 1$. Accordingly,

$$\begin{aligned} 0 = P.V.\int_{\mathbb{R}^n} \frac{v_\infty(x_\star)-v_\infty(y)}{|x_\star-y|^{n+2s}}\,dy &\geqslant P.V.\int_{B_3\backslash B_2} \frac{v_\infty(x_\star)-v_\infty(y)}{|x_\star-y|^{n+2s}}\,dy \\ &\geqslant P.V.\int_{B_3\backslash B_2} \frac{1-0}{|x_\star-y|^{n+2s}}\,dy > 0, \end{aligned}$$

which is a contradiction. □

The example provided by Theorem 3.8 is not the end of the story concerning the Harnack Inequality in the fractional setting. On the one hand, Theorem 3.8 is just a particular case of the very dramatic effect that the datum at infinity may have on the fractional Laplacian (a striking example of this phenomenon will be given in Section 3.4). On the other hand, the Harnack Inequality and the Maximum Principle hold true if, for instance, the sign of the function u is controlled in the whole of $\mathbb{R}^n$.

We refer to [7, 86, 64] and to the references therein for a detailed introduction to the fractional Harnack Inequality, and to [38] for general estimates of this type.

Just to point out the validity of a global Maximum Principle, we state in detail the following simple result:

Theorem 3.9. *If $(-\Delta)^s u \geqslant 0$ in B_1 and $u \geqslant 0$ in $\mathbb{R}^n\backslash B_1$, then $u \geqslant 0$ in B_1.*

Proof. Suppose, by contradiction, that the minimal point $x_\star \in B_1$ satisfies $u(x_\star) < 0$. Since $u(x_*)$ is a minimum, if $y \in B_2$ we have that $2u(x_\star)-u(x_\star+y)-u(x_\star-y) \leqslant 0$. On the other hand, in $\mathbb{R}^n\backslash B_2$ we have that $x_\star \pm y \in B_1$, hence $u(x_\star \pm y) \geqslant 0$. We thus have

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^n} \frac{2u(x_\star)-u(x_\star+y)-u(x_\star-y)}{|y|^{n+2s}}\,dy \\ &\leqslant \int_{\mathbb{R}^n\backslash B_2} \frac{2u(x_\star)-u(x_\star+y)-u(x_\star-y)}{|y|^{n+2s}}\,dy \\ &\leqslant \int_{\mathbb{R}^n\backslash B_2} \frac{2u(x_\star)}{|y|^{n+2s}}\,dy < 0. \end{aligned}$$

This leads to a contradiction. □

Similarly to Theorem 3.9, one can prove a Strong Maximum Principle, such as:

Theorem 3.10. *If $(-\Delta)^s u \geqslant 0$ in B_1 and $u \geqslant 0$ in $\mathbb{R}^n \setminus B_1$, then $u > 0$ in B_1, unless u vanishes identically.*

Proof. We observe that we already know that $u \geqslant 0$ in the whole of $\mathbb{R}^n$, thanks to Theorem 3.9. Hence, if u is not strictly positive, there exists $x_0 \in B_1$ such that $u(x_0) = 0$. This gives that

$$0 \leqslant \int_{\mathbb{R}^n} \frac{2u(x_0) - u(x_0+y) - u(x_0-y)}{|y|^{n+2s}}\,dy = -\int_{\mathbb{R}^n} \frac{u(x_0+y) + u(x_0-y)}{|y|^{n+2s}}\,dy.$$

Now both $u(x_0+y)$ and $u(x_0-y)$ are non-negative, hence the latter integral is less than or equal to zero, and so it must vanish identically, proving that u also vanishes identically. □

A simple version of a Harnack-type inequality in the fractional setting can be also obtained as follows:

Proposition 3.11. *Assume that $(-\Delta)^s u \geqslant 0$ in B_2, with $u \geqslant 0$ in the whole of $\mathbb{R}^n$. Then*

$$u(0) \geqslant c \int_{B_1} u(x)\,dx,$$

for a suitable $c > 0$.

Proof. Let $\Gamma \in C_0^\infty(B_{1/2})$, with $\Gamma(x) \in [0,1]$ for any $x \in \mathbb{R}^n$, and $\Gamma(0) = 1$. We fix $\epsilon > 0$, to be taken arbitrarily small at the end of this proof and set

$$\eta := u(0) + \epsilon > 0. \tag{3.25}$$

We define $\Gamma_a(x) := 2\eta\,\Gamma(x) - a$. Notice that if $a > 2\eta$, then $\Gamma_a(x) \leqslant 2\eta - a < 0 \leqslant u(x)$ in the whole of $\mathbb{R}^n$, hence the set $\{\Gamma_a < u \text{ in } \mathbb{R}^n\}$ is not empty, and we can define

$$a_* := \inf_{a\in\mathbb{R}} \{\Gamma_a < u \text{ in } \mathbb{R}^n\}.$$

By construction

$$a_* \leqslant 2\eta. \tag{3.26}$$

If $a < \eta$ then $\Gamma_a(0) = 2\eta - a > \eta > u(0)$, therefore

$$a_* \geqslant \eta. \tag{3.27}$$

Notice that

$$\Gamma_{a_*} \leqslant u \text{ in the whole of } \mathbb{R}^n. \tag{3.28}$$

We claim that

$$\text{there exists } x_0 \in \overline{B_{1/2}} \text{ such that } \Gamma_{a_*}(x_0) = u(x_0). \tag{3.29}$$

To prove this, we suppose by contradiction that $u > \Gamma_{a_*}$ in $\overline{B_{1/2}}$, i.e.

$$\mu := \min_{\overline{B_{1/2}}} (u - \Gamma_{a_*}) > 0.$$

Also, if $x \in \mathbb{R}^n \setminus \overline{B_{1/2}}$, we have that

$$u(x) - \Gamma_{a_*}(x) = u(x) - 2\eta\,\Gamma(x) + a_* = u(x) + a_* \geqslant a_* \geqslant \eta,$$

thanks to (3.27). As a consequence, for any $x \in \mathbb{R}^n$,

$$u(x) - \Gamma_{a_*}(x) \geqslant \min\{\mu, \eta\} =: \mu_* > 0.$$

So, if we define $a_\sharp := a_* - (\mu_*/2)$, we have that $a_\sharp < a_*$ and

$$u(x) - \Gamma_{a_\sharp}(x) = u(x) - \Gamma_{a_*}(x) - \frac{\mu_*}{2} \geqslant \frac{\mu_*}{2} > 0.$$

This is in contradiction with the definition of a_* and so it proves (3.29).

From (3.29) we have that $x_0 \in \overline{B_{1/2}}$, hence $(-\Delta)^s u(x_0) \geqslant 0$. Also $|(-\Delta)^s \Gamma_{a_*}(x)| = 2\eta\,|(-\Delta)^s \Gamma(x)| \leqslant C\eta$, for any $x \in \mathbb{R}^n$, therefore, recalling (3.28) and (3.29),

$$\begin{aligned} C\eta &\geqslant (-\Delta)^s \Gamma_{a_*}(x_0) - (-\Delta)^s u(x_0) \\ &= C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\big[\Gamma_{a_*}(x_0) - \Gamma_{a_*}(x_0+y)\big] - \big[u(x_0) - u(x_0+y)\big]}{|y|^{n+2s}}\,dy \\ &= C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x_0+y) - \Gamma_{a_*}(x_0+y)}{|y|^{n+2s}}\,dy \\ &\geqslant C(n,s)\,\text{P.V.} \int_{B_1(-x_0)} \frac{u(x_0+y) - \Gamma_{a_*}(x_0+y)}{|y|^{n+2s}}\,dy. \end{aligned}$$

Notice now that if $y \in B_1(-x_0)$, then $|y| \leqslant |x_0| + 1 < 2$, thus we obtain

$$C\eta \geqslant \frac{C(n,s)}{2^{n+2s}} \int_{B_1(-x_0)} \big[u(x_0+y) - \Gamma_{a_*}(x_0+y)\big]\,dy.$$

Notice now that $\Gamma_{a_*}(x) = 2\eta\Gamma(x) - a_* \leqslant \eta$, due to (3.27), therefore we conclude that

$$C\eta \geqslant \frac{C(n,s)}{2^{n+2s}} \left(\int_{B_1(-x_0)} u(x_0+y)\,dy - \eta |B_1| \right).$$

That is, using the change of variable $x := x_0 + y$, recalling (3.25) and renaming the constants, we have

$$C\big(u(0) + \epsilon\big) = C\eta \geqslant \int_{B_1} u(x)\,dx,$$

hence the desired result follows by sending $\epsilon \to 0$. □

3.4. **All functions are locally s-harmonic up to a small error.** Here we will show that s-harmonic functions can locally approximate any given function, without any geometric constraints. This fact is rather surprising and it is a purely nonlocal feature, in the sense that it has no classical counterpart. Indeed, in the classical setting, harmonic functions are quite rigid, for instance they cannot have a strict local maximum, and therefore cannot approximate a function with a strict local maximum. The nonlocal picture is, conversely, completely different, as the oscillation of a function "from far" can make the function locally harmonic, almost independently from its local behavior.

We want to give here some hints on the proof of this approximation result:

Theorem 3.12. *Let $k \in \mathbb{N}$ be fixed. Then for any $f \in C^k(\overline{B_1})$ and any $\varepsilon > 0$ there exists $R > 0$ and $u \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that*

$$\begin{cases} (-\Delta)^s u(x) = 0 & \text{in } B_1 \\ u = 0 & \text{in } \mathbb{R}^n \backslash B_R \end{cases} \tag{3.30}$$

and

$$\|f - u\|_{C^k(\overline{B_1})} \leqslant \varepsilon.$$

Sketch of the proof. For the sake of convenience, we divide the proof is into three steps. Also, for simplicity, we give the sketch of the proof in the one-dimensional case. See [43] for the entire and more general proof.

Step 1. Reducing to monomials

Let $k \in \mathbb{N}$ be fixed. We use first of all the Stone-Weierstrass Theorem and we have that for any $\varepsilon > 0$ and any $f \in C^k\big([0,1]\big)$ there exists a polynomial P such that

$$\|f - P\|_{C^k(\overline{B_1})} \leqslant \varepsilon.$$

Hence it is enough to prove Theorem 3.12 for polynomials. Then, by linearity, it is enough to prove it for monomials. Indeed, if $P(x) = \sum_{m=0}^{N} c_m x^m$ and one finds an s-harmonic function u_m such that

$$\|u_m - x^m\|_{C^k(\overline{B_1})} \leqslant \frac{\varepsilon}{|c_m|},$$

then by taking $u := \sum_{m=1}^{N} c_m u_m$ we have that

$$\|u - P\|_{C^k(\overline{B_1})} \leqslant \sum_{m=1}^{N} |c_m| \|u_m - x^m\|_{C^k(\overline{B_1})} \leqslant \varepsilon.$$

Notice that the function u is still s-harmonic, since the fractional Laplacian is a linear operator.

Step 2. Spanning the derivatives

We prove the existence of an s-harmonic function in B_1, vanishing outside a compact set and with arbitrarily large number of derivatives prescribed. That is, we show that for any $m \in \mathbb{N}$ there exist $R > r > 0$, a point $x \in \mathbb{R}$ and a function u such that

$$\begin{aligned} &(-\Delta)^s u = 0 \text{ in } (x-r, x+r), \\ &u = 0 \text{ outside } (x-R, x+R), \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} &D^j u(x) = 0 \text{ for any } j \in \{0, \dots, m-1\}, \\ &D^m u(x) = 1. \end{aligned} \tag{3.32}$$

To prove this, we argue by contradiction.

We consider $\mathcal{Z}$ to be the set of all pairs (u, x) of s-harmonic functions and points $x \in \mathbb{R}$ satisfying (3.31). To any pair, we associate the vector

$$\big(u(x), Du(x), \dots, D^m u(x)\big) \in \mathbb{R}^{m+1}$$

and take V to be the vector space spanned by this construction. Suppose by contradiction that a pair $(u, x) \in \mathcal{Z}$ with the property (3.32) does not exist. Then the vector $(0, \dots, 0, 1) \in \mathbb{R}^{m+1}$ does not belong to V. This means that the considered spanned space must be contained in a hyperplane. Hence there exists a vector $(c_1, \dots, c_m) \in \mathbb{R}^{m+1} \setminus \{0\}$ that is orthogonal to any vector $\big(u(x), Du(x), \dots, D^m u(x)\big)$ with $(u, x) \in \mathcal{Z}$, namely

$$\sum_{j \leqslant m} c_j D^j u(x) = 0.$$

A good candidate for the s-harmonic function is x_+^s, as we know from Theorem 3.4: strictly speaking, this function is not allowed here, since it is not compactly supported, but let us say that one can construct a compactly supported s-harmonic

function with the same behavior near the origin. With this slight caveat set aside, we compute in $(0,1)$

$$D^j x^s = s(s-1)\dots(s-j+1)x^{s-j}$$

and multiplying the sum with x^{m-s} (for $x \neq 0$) we have that

$$\sum_{j\leqslant m} c_j s(s-1)\dots(s-j+1)x^{m-j} = 0.$$

But since $s \in (0,1)$ the product $s(s-1)\dots(s-j+1)$ never vanishes. Hence the polynomial is identically null if and only if $c_j = 0$ for any j, and we reach a contradiction. This completes the proof of the existence of a function u that satisfies (3.31) and (3.32).

Step 3. Rescaling argument and completion of the proof
By Step 2, for any $m \in \mathbb{N}$ we are able to construct a locally s-harmonic function u such that $u(x) = x^m + \mathcal{O}(x^{m+1})$ near the origin (up to a translation). By considering the blow-up

$$u_\lambda(x) = \frac{u(\lambda x)}{\lambda^m} = x^m + \lambda\mathcal{O}(x^{m+1})$$

we have that for λ small, u_λ is arbitrarily close to the monomial x^m. As stated in Step 1, this concludes the proof of Theorem 3.12. □

It is worth pointing out that the flexibility of s-harmonic functions given by Theorem 3.12 may have concrete consequences. For instance, as a byproduct of Theorem 3.12, one has that a biological population with nonlocal dispersive attitudes can better locally adapt to a given distribution of resources (see e.g. Theorem 1.2 in [69]). Namely, nonlocal biological species may efficiently use distant resources and they can fit to the resources available nearby by consuming them (almost) completely, thus making more difficult for a different competing species to come into place.

4. Extension problems

We dedicate this part of the paper to the harmonic extension of the fractional Laplacian. We present at first two applications, the water wave model and the Peierls-Nabarro model related to crystal dislocations, making clear how the extension problem appears in these models. We conclude this part by discussing[4] in detail the extension problem via the Fourier transform.

The harmonic extension of the fractional Laplacian in the framework considered here is due to Luis Caffarelli and Luis Silvestre (we refer to [19] for details). The idea of this extension procedure is that the nonlocal operator $(-\Delta)^s$ acting on functions defined on $\mathbb{R}^n$ may be reduced to a local operator, acting on functions defined in the higher-dimensional half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,+\infty)$. Indeed, take $U\colon \mathbb{R}^{n+1}_+ \to \mathbb{R}$ such that $U(x,0) = u(x)$ in $\mathbb{R}^n$, solution to the equation

$$\operatorname{div}\Big(y^{1-2s}\nabla U(x,y)\Big) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+.$$

[4]Though we do not develop this approach here, it is worth mentioning that extended problems arise naturally also from the probabilistic interpretation described in Section 2. Roughly speaking, a stochastic process with jumps in $\mathbb{R}^n$ can often been seen as the "trace" of a classical stochastic process in $\mathbb{R}^n \times [0,+\infty)$ (i.e., each time that the classical stochastic process in $\mathbb{R}^n \times [0,+\infty)$ hits $\mathbb{R}^n \times \{0\}$ it induces a jump process over $\mathbb{R}^n$). Similarly, stochastic process with jumps may also be seen as classical processes at discrete, random, time steps.

Then up to constants one has that

$$\lim_{y\to 0}\Big(y^{1-2s}\partial_y U(x,y)\Big) = (-\Delta)^s u(x).$$

4.1. **Water wave model.** Let us consider the half space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,+\infty)$ endowed with the coordinates $x \in \mathbb{R}^n$ and $y \in (0,+\infty)$. We show that the half-Laplacian (namely when $s = 1/2$) arises when looking for a harmonic function in $\mathbb{R}^{n+1}_+$ with given data on $\mathbb{R}^n \times \{y = 0\}$. Thus, let us consider the following local Dirichlet-to-Neumann problem:

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) = u(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

The function U is the harmonic extension of u, we write $U = Eu$, and define the operator $\mathcal{L}$ as

$$\mathcal{L}u(x) := -\partial_y U(x,0). \tag{4.1}$$

We claim that

$$\mathcal{L} = \sqrt{-\Delta_x}, \tag{4.2}$$

in other words

$$\mathcal{L}^2 = -\Delta_x.$$

Indeed, by using the fact that $E(\mathcal{L}u) = -\partial_y U$ (that can be proved, for instance, by using the Poisson kernel representation for the solution), we obtain that

$$\begin{aligned} \mathcal{L}^2 u(x) &= \mathcal{L}\big(\mathcal{L}u\big)(x) \\ &= -\partial_y E\big(\mathcal{L}u\big)(x,0) \\ &= -\partial_y\big(-\partial_y U\big)(x,0) \\ &= \big(\partial_{yy}U + \Delta_x U - \Delta_x U\big)(x,0) \\ &= \Delta U(x,0) - \Delta u(x) \\ &= -\Delta u(x), \end{aligned}$$

which concludes the proof of (4.2).

One remark in the above calculation lies in the choice of the sign of the square root of the operator. Namely, if we set $\tilde{\mathcal{L}}u(x) := \partial_y U(x,0)$, the same computation as above would give that $\tilde{\mathcal{L}}^2 = -\Delta$. In a sense, there is no surprise that a quadratic equation offers indeed two possible solutions. But a natural question is how to choose the "right" one.

There are several reasons to justify the sign convention in (4.1). One reason is given by spectral theory, that makes the (fractional) Laplacian a negative definite operator. Let us discuss a purely geometric justification, in the simpler $n = 1$-dimensional case. We wonder how the solution of problem

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } (-1,1), \\ u = 0 & \text{in } \mathbb{R}\backslash(-1,1) \end{cases} \tag{4.3}$$

should look like in the extended variable y. First of all, by Maximum Principle (recall Theorems 3.9 and 3.10), we have that u is positive[5] when $x \in (-1,1)$ (since this is an s-superharmonic function, with zero data outside).

[5]As a matter of fact, the solution of (4.3) is explicit and it is given by $(1-x^2)^s$, up to dimensional constants. See [46] for a list of functions whose fractional Laplacian can be explicitly computed (unfortunately, differently from the classical cases, explicit computations in the fractional setting are available only for very few functions).

Then the harmonic extension U in $y > 0$ of a function u which is positive in $(-1,1)$ and vanishes outside $(-1,1)$ should have the shape of an elastic membrane over the halfplane $\mathbb{R}^2_+$ that is constrained to the graph of u on the trace $\{y=0\}$.

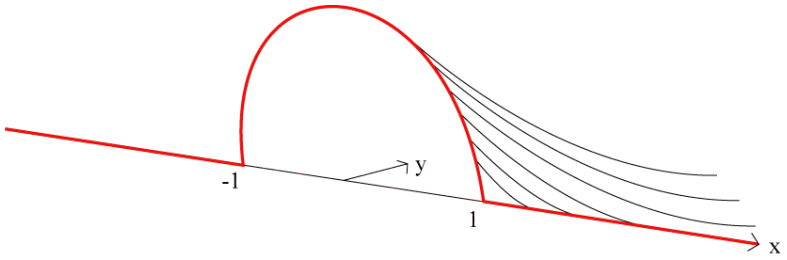

FIGURE 4. The harmonic extension

We give a picture of this function U in Figure 4. Notice from the picture that $\partial_y U(x,0)$ is negative, for any $x \in (-1,1)$. Since $(-\Delta)^s(x)$ is positive, we deduce that, to make our picture consistent with the maximum principle, we need to take the sign of $\mathcal{L}$ opposite to that of $\partial_y U(x,0)$. This gives a geometric justification of (4.1), which is only based on maximum principles (and on "how classical harmonic functions look like").

Application to the water waves.

We show now that the operator $\mathcal{L}$ arises in the theory of water waves of irrotational, incompressible, inviscid fluids in the small amplitude, long wave regime.

Consider a particle moving in the sea, which is, for us, the space $\mathbb{R}^n \times (0,1)$, where the bottom of the sea is set at level 1 and the surface at level 0 (see Figure 5). The velocity of the particle is $v\colon \mathbb{R}^n \times (0,1) \to \mathbb{R}^{n+1}$ and we write $v(x,y) = \big(v_x(x,y), v_y(x,y)\big)$, where $v_x\colon \mathbb{R}^n \times (0,1) \to \mathbb{R}^n$ is the horizontal component and $v_y\colon \mathbb{R}^n \times (0,1) \to \mathbb{R}$ is the vertical component. We are interested in the vertical velocity of the water at the surface of the sea which we call $u(x)$, namely $u(x) := v_y(x,0)$.

In our model, the water is incompressible, thus div $v = 0$ in $\mathbb{R}^n \times (0,1)$. Furthermore, on the bottom of sea (since water cannot penetrate into the sand), the velocity has only a non-null horizontal component, hence $v_y(x,1) = 0$. Also, in our model we assume that there are no vortices: at a mathematical level, this gives that v is irrotational, thus we may write it as the gradient of a function $U\colon \mathbb{R}^{n+1} \to \mathbb{R}$. We are led to the problem

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_y U(x,1) = 0 & \text{in } \mathbb{R}^n, \\ U(x,0) = u(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{4.4}$$

Let $\mathcal{L}$ be, as before, the operator $\mathcal{L}u(x) := -\partial_y U(x,0)$. We solve the problem (4.4) by using the Fourier transform and, up to a normalization factor, we obtain that

$$\mathcal{L}u = \mathcal{F}^{-1}\left(|\xi| \frac{e^{|\xi|} - e^{-|\xi|}}{e^{|\xi|} + e^{-|\xi|}} \widehat{u}(\xi) \right).$$

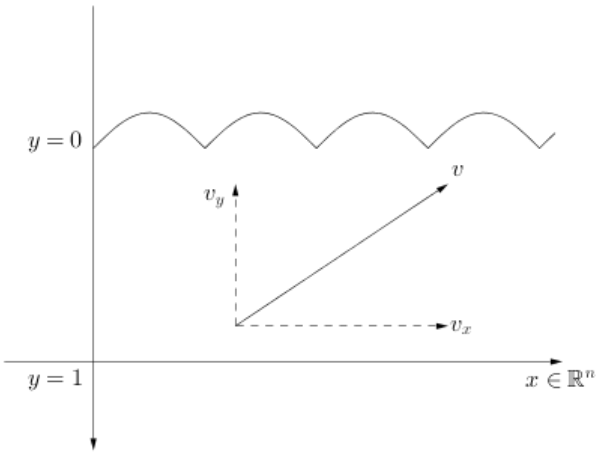

FIGURE 5. The water waves model

Notice that for large frequencies ξ, this operator is asymptotic to the square root of the Laplacian:

$$\mathcal{L}u \simeq \mathcal{F}^{-1}\Big(|\xi|\widehat{u}(\xi)\Big) = \sqrt{-\Delta}u.$$

The operator $\mathcal{L}$ in the two-dimensional case has an interesting property, that is in analogy to a conjecture of De Giorgi (the forthcoming Section 5.2 will give further details about it): more precisely, one considers entire, bounded, smooth, monotone solutions of the equation $\mathcal{L}u = f(u)$ for given f, and proves that the solution only depends on one variable. More precisely:

Theorem 4.1. *Let $f \in C^1(\mathbb{R})$ and u be a bounded smooth solution of*

$$\begin{cases} \mathcal{L}u = f(u) & \text{in } \mathbb{R}^2, \\ \partial_{x_2} u > 0 & \text{in } \mathbb{R}^2. \end{cases}$$

Then there exist a direction $\omega \in S^1$ and a function $u_0 \colon \mathbb{R} \to \mathbb{R}$ such that, for any $x \in \mathbb{R}^2$,

$$u(x) = u_0(x \cdot \omega).$$

See Corollary 2 in [35] for a proof of Theorem 4.1 and to Theorem 1 in [35] for a more general result (in higher dimension).

4.2. **Crystal dislocation.** A crystal is a material whose atoms are displayed in a regular way. Due to some impurities in the material or to an external stress, some atoms may move from their rest positions. The system reacts to small modifications by pushing back towards the equilibrium. Nevertheless, slightly larger modifications may lead to plastic deformations. Indeed, if an atom dislocation is of the order of the periodicity size of the crystal, it can be perfectly compatible with the behavior of the material at a large scale, and it can lead to a permanent modification.

Suitably superposed atom dislocations may also produce macroscopic deformations of the material, and the atom dislocations may be moved by a suitable external force, which may be more effective if it happens to be compatible with the periodic structure of the crystal.

These simple considerations may be framed into a mathematical setting, and they also have concrete applications in many industrial branches (for instance, in the production of a soda can, in order to change the shape of an aluminium sheet, it is reasonable to believe that applying the right force to it can be simpler and less expensive than melting the metal).

It is also quite popular (see e.g. [68]) to describe the atom dislocation motion in crystals in analogy with the movement of caterpillar (roughly speaking, it is less expensive for the caterpillar to produce a defect in the alignment of its body and to dislocate this displacement, rather then rigidly translating his body on the ground).

The mathematical framework of crystal dislocation presented here is related to the Peierls-Nabarro model, that is a hybrid model in which a discrete dislocation occurring along a slide line is incorporated in a continuum medium. The total energy in the Peierls-Nabarro model combines the elastic energy of the material in reaction to the single dislocations, and the potential energy of the misfit along the glide plane. The main result is that, at a macroscopic level, dislocations tend to concentrate at single points, following the natural periodicity of the crystal.

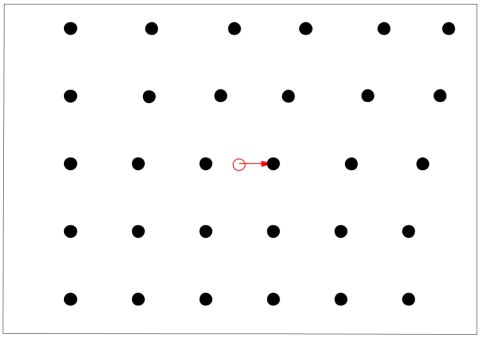

FIGURE 6. Crystal dislocation

To introduce a mathematical framework for crystal dislocation, first, we "slice" the crystal with a plane. The mathematical setting will be then, by symmetry arguments, the half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \text{ s.t. } y \geqslant 0\}$ and the glide line will be the x-axis. In a crystalline structure, the atoms display periodically. Namely, the atoms on the x-axis have the preference of occupying integer sites. If atoms move out of their rest position due to a misfit, the material will have an elastic reaction, trying to restore the crystalline configuration. The tendency is to move back the atoms to their original positions, or to recreate, by translations, the natural periodic configuration. This effect may be modeled by defining $v^0(x) := v(x,0)$ to be the discrepancy between the position of the atom x and its rest position. Then, the misfit energy is

$$\mathcal{M}(v^0) := \int_{\mathbb{R}} W\Big(v^0(x)\Big)\,dx, \tag{4.5}$$

where W is a smooth periodic potential, normalized in such a way that $W(u+1) = W(u)$ for any $u \in \mathbb{R}$ and $0 = W(0) < W(u)$ for any $u \in (0,1)$. We also assume that the minimum of W is nondegenerate, i.e. $W''(0) > 0$.

We consider the dislocation function $v(x,y)$ on the half-plane $\mathbb{R}^2_+$. The elastic energy of this model is given by

$$\mathcal{E}(v) := \frac{1}{2}\int_{\mathbb{R}^2_+} \Big|\nabla v(x,y)\Big|^2\,dx\,dy. \tag{4.6}$$

The total energy of the system is therefore

$$\mathcal{F}(v) := \mathcal{E}(v) + \mathcal{M}(v^0) = \frac{1}{2}\int_{\mathbb{R}^2_+} \Big|\nabla v(x,y)\Big|^2\,dx\,dy + \int_{\mathbb{R}} W\Big(v(x,0)\Big)\,dx. \tag{4.7}$$

Namely, the total energy of the system is the superposition of the energy in (4.5), which tends to settle all the atoms in their rest position (or in another position equivalent to it from the point of view of the periodic crystal), and the energy in (4.6), which is the elastic energy of the material itself.

Notice that some approximations have been performed in this construction. For instance, the atom dislocation changes the structure of the crystal itself: to write (4.5), one is making the assumption that the dislocations of the single atoms do not destroy the periodicity of the crystal at a large scale, and it is indeed this "permanent" periodic structure that produces the potential W.

Moreover, in (4.6), we are supposing that a "horizontal" atom displacement along the line $\{y = 0\}$ causes a horizontal displacement at $\{y = \epsilon\}$ as well. Of course, in real life, if an atom at $\{y = 0\}$ moves, say, to the right, an atom at level $\{y = \epsilon\}$ is dragged to the right as well, but also slightly downwards towards the slip line $\{y = 0\}$. Thus, in (4.6) we are neglecting this "vertical" displacement. This approximation is nevertheless reasonable since, on the one hand, one expects the vertical displacement to be negligible with respect to the horizontal one and, on the other hand, the vertical periodic structure of the crystal tends to avoid vertical displacements of the atoms outside the periodicity range (from the mathematical point of view, we notice that taking into account vertical displacements would make the dislocation function vectorial, which would produce a system of equations, rather than one single equation for the system).

Also, the initial assumption of slicing the crystal is based on some degree of simplification, since this comes to studying dislocation curves in spaces which are "transversal" to the slice plane.

In any case, we will take these (reasonable, after all) simplifying assumptions for granted, we will study their mathematical consequences and see how the results obtained agree with the physical experience.

To find the Euler-Lagrange equation associated to (4.7), let us consider a perturbation $\phi \in C_0^\infty(\mathbb{R}^2)$, with $\varphi(x) := \phi(x,0)$ and let v be a minimizer. Then

$$\frac{d}{d\varepsilon}\mathcal{F}(v+\varepsilon\phi)\Big|_{\varepsilon=0} = 0,$$

which gives

$$\int_{\mathbb{R}^2_+} \nabla v\cdot\nabla\phi\,dx\,dy + \int_{\mathbb{R}} W'(v^0)\varphi\,dx = 0.$$

Consider at first the case in which $\mathrm{supp}\phi\cap\partial\mathbb{R}^2_+ = \varnothing$, thus $\varphi = 0$. By the Divergence Theorem we obtain that

$$\int_{\mathbb{R}^2_+} \phi\,\Delta v\,dx\,dy = 0 \quad \text{ for any } \phi \in C_0^\infty(\mathbb{R}^2),$$

thus $\Delta v = 0$ in $\mathbb{R}^2_+$. If $\mathrm{supp}\phi \cap \partial\mathbb{R}^2_+ \neq \varnothing$ then we have that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2_+} \mathrm{div}(\phi\nabla v)\,dx\,dy + \int_{\mathbb{R}} W'(v^0)\varphi\,dx \\ &= \int_{\partial\mathbb{R}^2_+} \phi\frac{\partial v}{\partial \nu}\,dx + \int_{\mathbb{R}} W'(v^0)\varphi\,dx \\ &= -\int_{\mathbb{R}} \varphi\frac{\partial v}{\partial y}\,dx + \int_{\mathbb{R}} W'(v^0)\varphi\,dx \end{aligned}$$

for an arbitrary $\varphi \in C^\infty_0(\mathbb{R})$ therefore $\dfrac{\partial v}{\partial y}(x,0) = W'(v^0(x))$ for $x \in \mathbb{R}$. Hence the critical points of $\mathcal{F}$ are solutions of the problem

$$\begin{cases} \Delta v(x,y) = 0 & \text{for } x \in \mathbb{R} \text{ and } y > 0, \\ v(x,0) = v^0(x) & \text{for } x \in \mathbb{R}, \\ \partial_y v(x,0) = W'\Big(v(x,0)\Big) & \text{for } x \in \mathbb{R} \end{cases}$$

and up to a normalization constant, recalling (4.1) and (4.2), we have that

$$-\sqrt{-\Delta}v(x,0) = W'\big(v(x,0)\big), \text{ for any } x \in \mathbb{R}.$$

The corresponding parabolic evolution equation is $\partial_t v(x,0) = -\sqrt{-\Delta}v(x,0) - W'\big(v(x,0)\big)$.

After this discussion, one is lead to consider the more general case of the fractional Laplacian $(-\Delta)^s$ for any $s \in (0,1)$ (not only the half Laplacian), and the corresponding parabolic equation

$$\partial_t v = -(-\Delta)^s v - W'(v) + \sigma,$$

where σ is a (small) external stress.

If we take the lattice of size ϵ and rescale v and σ as

$$v_\epsilon(t,x) = v\left(\frac{t}{\epsilon^{1+2s}}, \frac{x}{\epsilon}\right) \quad \text{and} \quad \sigma = \varepsilon^{2s}\sigma\left(\frac{t}{\epsilon^{1+2s}}, \frac{x}{\epsilon}\right),$$

then the rescaled function satisfies

$$\partial_t v_\epsilon = \frac{1}{\epsilon}\Big(-(-\Delta)^s v_\epsilon - \frac{1}{\epsilon^{2s}}W'(v_\epsilon) + \sigma\Big) \text{ in } (0,+\infty)\times\mathbb{R} \tag{4.8}$$

with the initial condition

$$v_\epsilon(0,x) = v^0_\epsilon(x) \text{ for } x \in \mathbb{R}.$$

To suitably choose the initial condition v^0_ϵ, we introduce the basic layer[6] solution u, that is, the unique solution of the problem

$$\begin{cases} -(-\Delta)^s u(x) = W'(u) & \text{in } \mathbb{R}, \\ u' > 0 \text{ and } u(-\infty) = 0, u(0) = 1/2, u(+\infty) = 1. \end{cases} \tag{4.9}$$

For the existence of such solution and its main properties see [75] and [15]. Furthermore, the solution decays polynomially at $\pm\infty$ (see [42] and [41]), namely

$$\left| u(x) - H(x) + \frac{1}{2sW''(0)}\frac{x}{|x|^{1+2s}} \right| \leqslant \frac{C}{|x|^\vartheta} \quad \text{for any } x \in \mathbb{R}^n, \tag{4.10}$$

[6] As a matter of fact, the solution of (4.9) coincides with the one of a one-dimensional fractional Allen-Cahn equation, that will be discussed in further detail in the forthcoming Section 5.1.

where $\vartheta > 2s$ and H is the Heaviside step function

$$H(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0. \end{cases}$$

We take the initial condition of the solution of (4.8) to be the superposition of transitions all occurring with the same orientation, i.e. we set

$$v_\epsilon(x,0) := \frac{\epsilon^{2s}}{W''(0)}\sigma(0,x) + \sum_{i=1}^{N} u\left(\frac{x - x_i^0}{\epsilon}\right), \tag{4.11}$$

where $x_1^0, \dots, x_N^0$ are N fixed points.

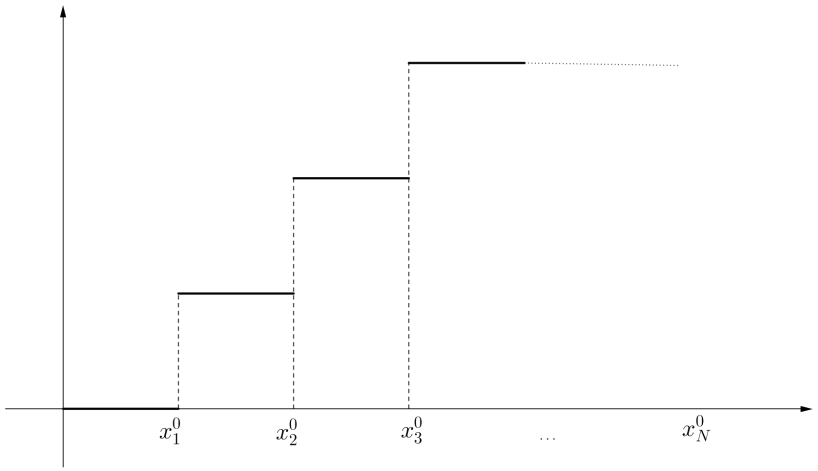

FIGURE 7. The initial datum when $\varepsilon \to 0$

The main result in this setting is that the solution v_ϵ approaches, as $\epsilon \to 0$, the superposition of steps functions. The discontinuities of the limit function occur at some points $\big(x_i(t)\big)_{i=1,\dots,N}$ which move accordingly to the following[7] dynamical system

$$\begin{cases} \dot{x_i} = \gamma\Big(- \sigma(t,x_i) + \displaystyle\sum_{j\neq i} \frac{x_i - x_j}{2s|x_i - x_j|^{2s+1}} \Big) & \text{in } (0,+\infty), \\ x_i(0) = x_i^0, \end{cases} \tag{4.12}$$

where

$$\gamma = \left(\int_{\mathbb{R}} (u')^2 \right)^{-1}. \tag{4.13}$$

More precisely, the main result obtained here is the following.

Theorem 4.2. *There exists a unique viscosity solution of*

$$\begin{cases} \partial_t v_\epsilon = \dfrac{1}{\epsilon}\Big(- (-\Delta)^s v_\epsilon - \dfrac{1}{\epsilon^{2s}} W'(v_\epsilon) + \sigma \Big) & \text{in } (0,+\infty)\times\mathbb{R}, \\ v_\epsilon(0,x) = \dfrac{\epsilon^{2s}}{W''(0)}\sigma(0,x) + \displaystyle\sum_{i=1}^{N} u\left(\frac{x - x_i^0}{\epsilon}\right) & \text{for } x \in \mathbb{R} \end{cases}$$

[7] The system of ordinary differential equations in (4.12) has been extensively studied in [55].

such that

$$\lim_{\epsilon\to 0} v_\epsilon(t,x) = \sum_{i=1}^{N} H\big(x - x_i(t)\big), \tag{4.14}$$

where $\big(x_i(t)\big)_{i=1,\dots,N}$ *is solution to* (4.12).

We refer to [60] for the case $s = \frac{1}{2}$, to [42] for the case $s > \frac{1}{2}$, and [41] for the case $s < \frac{1}{2}$ (in these papers, it is also carefully stated in which sense the limit in (4.14) holds true).

We would like to give now a formal (not rigorous) justification of the ODE system in (4.12) that drives the motion of the transition layers.

Justification of ODE system (4.12). We assume for simplicity that the external stress σ is null. We use the notation $\simeq$ to denote the equality up to negligible terms in ϵ. Also, we denote

$$u_i(t,x) := u\left(\frac{x - x_i(t)}{\epsilon}\right)$$

and, with a slight abuse of notation

$$u_i'(t,x) := u'\left(\frac{x - x_i(t)}{\epsilon}\right).$$

By (4.10) we have that the layer solution is approximated by

$$u_i(t,x) \simeq H\left(\frac{x - x_i(t)}{\epsilon}\right) - \frac{\epsilon^{2s}\big(x - x_i(t)\big)}{2sW''(0)\big|x - x_i(t)\big|^{1+2s}}. \tag{4.15}$$

We use the assumption that the solution v_ϵ is well approximated by the sum of N transitions and write

$$v_\epsilon(t,x) \simeq \sum_{i=1}^{N} u_i(t,x) = \sum_{i=1}^{N} u\Big(\frac{x - x_i(t)}{\epsilon}\Big).$$

For that

$$\partial_t v_\epsilon(t,x) = -\frac{1}{\epsilon}\sum_{i=1}^{N} u_i'(t,x)\dot{x}_i(t)$$

and, since the basic layer solution u is the solution of (4.9), we have that

$$\begin{aligned}
-(-\Delta)^s v_\epsilon \simeq & -\sum_{i=1}^{N}(-\Delta)^s u_i(t,x) \\
= & -\frac{1}{\epsilon^{2s}}\sum_{i=1}^{N}(-\Delta)^s u\Big(\frac{x - x_i(t)}{\epsilon}\Big) \\
= & \frac{1}{\epsilon^{2s}}\sum_{i=1}^{N} W'\left(u\Big(\frac{x - x_i(t)}{\epsilon}\Big)\right) \\
= & \frac{1}{\epsilon^{2s}}\sum_{i=1}^{N} W'\big(u_i(t,x)\big).
\end{aligned}$$

Now, returning to the parabolic equation (4.8) we have that

$$-\frac{1}{\epsilon}\sum_{i=1}^{N} u_i'(t,x)\dot{x_i}(t) = \frac{1}{\epsilon^{2s+1}}\Big(\sum_{i=1}^{N} W'\big(u_i(t,x)\big) - W'\Big(\sum_{i=1}^{N} u_i(t,x)\Big)\Big). \tag{4.16}$$

Fix an integer k between 1 and N, multiply (4.16) by $u_k'(t,x)$ and integrate over $\mathbb{R}$. We obtain

$$\begin{aligned} &-\frac{1}{\epsilon}\sum_{i=1}^{N}\dot{x_i}(t)\int_{\mathbb{R}} u_i'(t,x)u_k'(t,x)\,dx \\ &\quad= \frac{1}{\epsilon^{2s+1}}\Big(\sum_{i=1}^{N}\int_{\mathbb{R}} W'\big(u_i(t,x)\big)u_k'(t,x)\,dx - \int_{\mathbb{R}} W'\Big(\sum_{i=1}^{N} u_i(t,x)\Big)u_k'(t,x)\,dx\Big). \end{aligned} \tag{4.17}$$

We compute the left hand side of (4.17). First, we take the k^{th} term of the sum (i.e. we consider the case $i=k$). By using the change of variables

$$y := \frac{x - x_k(t)}{\epsilon} \tag{4.18}$$

we have that

$$\begin{aligned} -\frac{1}{\epsilon}\dot{x_k}(t)\int_{\mathbb{R}}(u_k')^2(t,x)\,dx &= -\frac{1}{\epsilon}\dot{x_k}(t)\int_{\mathbb{R}}(u')^2\Big(\frac{x-x_k(t)}{\epsilon}\Big)\,dx \\ &= -\dot{x_k}(t)\int_{\mathbb{R}}(u')^2(y)\,dy \\ &= -\frac{\dot{x_k}(t)}{\gamma}, \end{aligned} \tag{4.19}$$

where γ is defined by (4.13).

Then, we consider the i^{th} term of the sum on the left hand side of (4.17). By performing again the substitution (4.18), we see that this term is

$$\begin{aligned} -\frac{1}{\epsilon}\dot{x_i}(t)\int_{\mathbb{R}} u_i'(t,x)u_k'(t,x)\,dx &= -\frac{1}{\epsilon}\dot{x_i}(t)\int_{\mathbb{R}} u'\Big(\frac{x-x_i(t)}{\epsilon}\Big)u'\Big(\frac{x-x_k(t)}{\epsilon}\Big)\,dx \\ &= -\dot{x_i}(t)\int_{\mathbb{R}} u'\Big(y+\frac{x_k(t)-x_i(t)}{\epsilon}\Big)u'(y)\,dy \\ &\simeq 0, \end{aligned}$$

where, for the last equivalence we have used that for ϵ small, $u'\Big(y+\dfrac{x_k(t)-x_i(t)}{\epsilon}\Big)$ is asymptotic to $u'(\pm\infty)=0$.

We consider the first member on the right hand side of the identity (4.17), and, as before, take the k^{th} term of the sum. We do the substitution (4.18) and have that

$$\begin{aligned} \frac{1}{\epsilon}\int_{\mathbb{R}} W'\big(u_k(t,x)\big)u_k'(t,x)\,dx &= \int_{\mathbb{R}} W'\big(u(y)\big)u'(y)\,dy \\ &= W\big(u(y)\big)\Big|_{-\infty}^{+\infty} \\ &= W(1) - W(0) = 0 \end{aligned}$$

by the periodicity of W. Now we use (4.15), the periodicity of W' and we perform a Taylor expansion, noticing that $W'(0) = 0$. We see that

$$\begin{aligned} W'\big(u_i(t,x)\big) &\simeq W'\left(H\left(\frac{x-x_i(t)}{\epsilon}\right) - \frac{\epsilon^{2s}\big(x-x_i(t)\big)}{2sW''(0)\big|x-x_i(t)\big|^{1+2s}}\right) \\ &\simeq W'\Big(- \frac{\epsilon^{2s}\big(x-x_i(t)\big)}{2sW''(0)\big|x-x_i(t)\big|^{1+2s}}\Big) \\ &\simeq \frac{-\epsilon^{2s}\big(x-x_i(t)\big)}{2s\big|x-x_i(t)\big|^{1+2s}}. \end{aligned}$$

Therefore, the i^{th} term of the sum on the right hand side of the identity (4.17) for $i \neq k$, by using the above approximation and doing one more time the substitution (4.18), for ϵ small becomes

$$\begin{aligned} \frac{1}{\epsilon}\int_{\mathbb{R}} W'\big(u_i(t,x)\big)u_k'(t,x)\,dx &= -\frac{1}{\epsilon}\int_{\mathbb{R}} \frac{\epsilon^{2s}\big(x-x_i(t)\big)}{2s\big|x-x_i(t)\big|^{1+2s}} u'\left(\frac{x-x_k(t)}{\epsilon}\right) dx \\ &= -\int_{\mathbb{R}} \frac{\epsilon^{2s}\big(\epsilon y + x_k(t) - x_i(t)\big)}{2s\big|\epsilon y + x_k(t) - x_i(t)\big|^{1+2s}} u'(y)\,dy \\ &\simeq -\frac{\epsilon^{2s}\big(x_k(t)-x_i(t)\big)}{2s\big|x_k(t)-x_i(t)\big|^{1+2s}}\int_{\mathbb{R}} u'(y)\,dy \\ &= -\frac{\epsilon^{2s}\big(x_k(t)-x_i(t)\big)}{2s\big|x_k(t)-x_i(t)\big|^{1+2s}}. \end{aligned} \tag{4.20}$$

We also observe that, for ϵ small, the second member on the right hand side of the identity (4.17), by using the change of variables (4.18), reads

$$\begin{aligned} &\frac{1}{\epsilon}\int_{\mathbb{R}} W'\Big(\sum_{i=1}^N u_i(t,x)\Big)u_k'(t,x)\,dx \\ &\qquad = \frac{1}{\epsilon}\int_{\mathbb{R}} W'\Big(u_k(t,x) + \sum_{i\neq k} u_i(t,x)\Big)u_k'(t,x)\,dx \\ &\qquad = \int_{\mathbb{R}} W'\left(u(y) + \sum_{i\neq k} u\Big(y + \frac{x_k(t)-x_i(t)}{\epsilon}\Big)\right)u'(y)\,dy. \end{aligned}$$

For ϵ small, $u\left(y + \dfrac{x_k(t)-x_i(t)}{\epsilon}\right)$ is asymptotic either to $u(+\infty) = 1$ for $x_k > x_i$, or to $u(-\infty) = 0$ for $x_k < x_i$. By using the periodicity of W, it follows that

$$\frac{1}{\epsilon}\int_{\mathbb{R}} W'\Big(\sum_{i=1}^N u_i(t,x)\Big)u_k'(t,x)\,dx = \int_{\mathbb{R}} W'\Big(u(y)\Big)u'(y)\,dy = W(1) - W(0) = 0,$$

again by the asymptotic behavior of u. Concluding, by inserting the results (4.19) and (4.20) into (4.17) we get that

$$\frac{\dot{x_k}(t)}{\gamma} = \sum_{i\neq k} \frac{x_k(t)-x_i(t)}{2s\big|x_k(t)-x_i(t)\big|^{1+2s}},$$

which ends the justification of the system (4.12). □

We recall that, till now, in Theorem 4.2 we considered the initial data as a superposition of transitions all occurring with the same orientation (see (4.11)), i.e. the initial dislocation is a monotone function (all the atoms are initially moved to the right).

Of course, for concrete applications, it is interesting to consider also the case in which the atoms may dislocate in both directions, i.e. the transitions can occur with different orientations (the atoms may be initially displaced to the left or to the right of their equilibrium position).

To model the different orientations of the dislocations, we introduce a parameter $\xi_i \in \{-1, 1\}$ (roughly speaking $\xi_i = 1$ corresponds to a dislocation to the right and $\xi_i = -1$ to a dislocation to the left).

The main result in this case is the following (see [76]):

Theorem 4.3. *There exists a viscosity solution of*

$$\begin{cases} \partial_t v_\epsilon = \dfrac{1}{\epsilon}\Big(-(-\Delta)^s v_\epsilon - \dfrac{1}{\epsilon^{2s}} W'(v_\epsilon) + \sigma_\epsilon \Big) & \text{in } (0,+\infty)\times\mathbb{R}, \\ v_\epsilon(0,x) = \dfrac{\epsilon^{2s}}{W''(0)}\sigma(0,x) + \displaystyle\sum_{i=1}^N u\Big(\xi_i \frac{x-x_i^0}{\epsilon}\Big) & \text{for } x\in\mathbb{R} \end{cases}$$

such that

$$\lim_{\epsilon\to 0} v_\epsilon(t,x) = \sum_{i=1}^N H\Big(\xi_i\big(x - x_i(t)\big)\Big),$$

where $\big(x_i(t)\big)_{i=1,\dots,N}$ *is solution to*

$$\begin{cases} \dot{x_i} = \gamma\Big(-\xi_i\sigma(t,x_i) + \displaystyle\sum_{j\neq i} \xi_i\xi_j \frac{x_i - x_j}{2s|x_i-x_j|^{2s+1}} \Big) & \text{in } (0,+\infty), \\ x_i(0) = x_i^0. \end{cases} \tag{4.21}$$

We observe that Theorem 4.3 reduces to Theorem 4.2 when $\xi_1 = \cdots = \xi_n = 1$. In fact, the case discussed in Theorem 4.3 is richer than the one in Theorem 4.2, since, in the case of different initial orientations, collisions can occur, i.e. it may happen that $x_i(T_c) = x_{i+1}(T_c)$ for some $i \in \{1,\dots,N-1\}$ at a collision time T_c.

For instance, in the case $N = 2$, for $\xi_1 = 1$ and $\xi_2 = -1$ (two initial dislocations with different orientations) we have that

$$\text{if } \sigma \leqslant 0 \text{ then } T_c \leqslant \frac{s\theta_0^{1+2s}}{(2s+1)\gamma},$$

$$\text{if } \theta_0 < (2s\|\sigma\|_\infty)^{-\frac{1}{2s}} \text{ then } T_c \leqslant \frac{s\theta_0^{1+2s}}{\gamma(1-2s\theta_0\|\sigma\|_\infty)},$$

where $\theta_0 := x_2^0 - x_1^0$ is the initial distance between the dislocated atoms. That is, if either the external force has the right sign, or the initial distance is suitably small with respect to the external force, then the dislocation time is finite, and collisions occur in a finite time (on the other hand, when these conditions are violated, there are examples in which collisions do not occur).

This and more general cases of collisions, with precise estimates on the collision times, are discussed in detail in [76].

An interesting feature of the system is that the dislocation function v_ϵ does not annihilate at the collision time. More precisely, in the appropriate scale, we have

that v_ϵ at the collision time vanishes outside the collision points, but it still preserves a non-negligible asymptotic contribution exactly at the collision points. A formal statement is the following (see [76]):

Theorem 4.4. *Let $N = 2$ and assume that a collision occurs. Let x_c be the collision point, namely $x_c = x_1(T_c) = x_2(T_c)$. Then*

$$\lim_{t\to T_c}\lim_{\varepsilon\to 0} v_\varepsilon(t,x) = 0 \quad \text{for any} \quad x \neq x_c, \tag{4.22}$$

but

$$\limsup_{\substack{t\to T_c\\ \varepsilon\to 0}} v_\varepsilon(t,x_c) \geqslant 1. \tag{4.23}$$

Formulas (4.22) and (4.23) describe what happens in the crystal at the collision time. On the one hand, formula (4.22) states that at any point that is not the collision point and at a large scale, the system relaxes at the collision time. On the other hand, formula (4.23) states that the behavior at the collision points at the collision time is quite "singular". Namely, the system does not relax immediately (in the appropriate scale).

What happens is that a slightly larger time is needed before the system relaxes exponentially fast: a detailed description of this relaxation phenomenon is presented in [77]. For instance, in the case $N = 2$, the dislocation function decays to zero exponentially fast, slightly after collision, as given by the following result:

Theorem 4.5. *Let $N = 2$, $\xi_1 = 1$, $\xi_2 = -1$, and let v_ϵ be the solution given by Theorem 4.3, with $\sigma \equiv 0$. Then there exist $\epsilon_0 > 0$, $c > 0$, $T_\epsilon > T_c$ and $\rho_\epsilon > 0$ satisfying*

$$\lim_{\epsilon\to 0} T_\epsilon = T_c$$
$$\text{and} \qquad \lim_{\epsilon\to 0} \varrho_\epsilon = 0$$

such that for any $\epsilon < \epsilon_0$ we have

$$|v_\epsilon(t,x)| \leqslant \varrho_\epsilon e^{c\frac{T_\epsilon - t}{\epsilon^{2s+1}}}, \quad \text{for any } x \in \mathbb{R} \text{ and } t \geqslant T_\epsilon. \tag{4.24}$$

The estimate in (4.24) states, roughly speaking, that at a suitable time T_ϵ (only slightly bigger than the collision time T_c) the dislocation function gets below a small threshold ρ_ϵ, and later it decays exponentially fast (the constant of this exponential becomes large when ϵ is small).

The reader may compare Theorem 4.4 and 4.5 and notice that different asymptotics are considered by the two results. A result similar to Theorem 4.5 holds for a larger number of dislocated atoms. For instance, in the case of three atoms with alternate dislocations, one has that, slightly after collision, the dislocation function decays exponentially fast to the basic layer solution. More precisely (see again [77]), we have that:

Theorem 4.6. *Let $N = 3$, $\xi_1 = \xi_3 = 1$, $\xi_2 = -1$, and let v_ϵ be the solution given by Theorem 4.3, with $\sigma \equiv 0$. Then there exist $\epsilon_0 > 0$, $c > 0$, $T^1_\epsilon, T^2_\epsilon > T_c$ and $\rho_\epsilon > 0$ satisfying*

$$\lim_{\epsilon\to 0} T^1_\epsilon = \lim_{\epsilon\to 0} T^2_\epsilon = T_c,$$
$$\text{and} \qquad \lim_{\epsilon\to 0} \varrho_\epsilon = 0$$

and points $\bar y_\epsilon$ *and* $\bar z_\epsilon$ *satisfying*

$$\lim_{\epsilon\to 0}|\bar z_\epsilon-\bar y_\epsilon|=0$$

such that for any $\epsilon<\epsilon_0$ *we have*

$$v_\epsilon(t,x)\leqslant u\left(\frac{x-\bar y_\epsilon}{\epsilon}\right)+\varrho_\epsilon e^{-\frac{c(t-T^1_\epsilon)}{\epsilon^{2s+1}}},\qquad \text{for any } x\in\mathbb{R} \text{ and } t\geqslant T^1_\epsilon, \tag{4.25}$$

and

$$v_\epsilon(t,x)\geqslant u\left(\frac{x-\bar z_\epsilon}{\epsilon}\right)-\varrho_\epsilon e^{-\frac{c(t-T^2_\epsilon)}{\epsilon^{2s+1}}},\qquad \text{for any } x\in\mathbb{R} \text{ and } t\geqslant T^2_\epsilon, \tag{4.26}$$

where u *is the basic layer solution introduced in* (4.9).

Roughly speaking, formulas (4.25) and (4.26) say that for times T^1_ϵ, T^2_ϵ just slightly bigger than the collision time T_c, the dislocation function v_ϵ gets trapped between two basic layer solutions (centered at points $\bar y_\epsilon$ and $\bar z_\epsilon$), up to a small error. The error gets eventually to zero, exponentially fast in time, and the two basic layer solutions which trap v_ϵ get closer and closer to each other as ϵ goes to zero (that is, the distance between $\bar y_\epsilon$ and $\bar z_\epsilon$ goes to zero with ϵ).

We refer once more to [77] for a series of figures describing in details the results of Theorems 4.5 and 4.6.

4.3. **An approach to the extension problem via the Fourier transform.** We will discuss here the extension operator of the fractional Laplacian via the Fourier transform approach (see [19] and [90] for other approaches and further results).

Some readers may find the details of this part rather technical: if so, she or he can jump directly to Section 5 on page 42, without affecting the subsequent reading.

We fix at first a few pieces of notation. We denote points in $\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times(0,+\infty)$ as $X=(x,y)$, with $x\in\mathbb{R}^n$ and $y>0$. When taking gradients in $\mathbb{R}^{n+1}_+$, we write $\nabla_X=(\nabla_x,\partial_y)$, where ∇_x is the gradient in $\mathbb{R}^n$. Also, in $\mathbb{R}^{n+1}_+$, we will often take the Fourier transform in the variable x only, for fixed $y>0$. We also set

$$a:=1-2s\in(-1,1).$$

We will consider the fractional Sobolev space $\widehat{H}^s(\mathbb{R}^n)$ defined as the set of functions u that satisfy

$$\|u\|_{L^2(\mathbb{R}^n)}+[\widehat{u}]_G<+\infty,$$

where

$$[v]_G:=\sqrt{\int_{\mathbb{R}^n}|\xi|^{2s}\,|v(\xi)|^2\,d\xi}.$$

For any $u\in W^{1,1}_{\rm loc}((0,+\infty))$, we consider the functional

$$G(u):=\int_0^{+\infty}t^a\Big(\big|u(t)\big|^2+\big|u'(t)\big|^2\Big)\,dt. \tag{4.27}$$

By Theorem 4 of [85], we know that the functional G attains its minimum among all the functions $u\in W^{1,1}_{\rm loc}((0,+\infty))\cap C^0([0,+\infty))$ with $u(0)=1$. We call g such minimizer and

$$C_\sharp:=G(g)=\min_{\substack{u\in W^{1,1}_{\rm loc}((0,+\infty))\cap C^0([0,+\infty))\\ u(0)=1}}G(u). \tag{4.28}$$

The main result of this section is the following.

Theorem 4.7. *Let $u \in \mathcal{S}(\mathbb{R}^n)$ and let*

$$U(x,y) := \mathcal{F}^{-1}\Big(\widehat{u}(\xi)\, g(|\xi|y)\Big). \tag{4.29}$$

Then

$$\operatorname{div}(y^a \nabla U) = 0 \tag{4.30}$$

for any $X = (x,y) \in \mathbb{R}^{n+1}_+$. In addition,

$$-\, y^a \partial_y U\Big|_{\{y=0\}} = C_\sharp (-\Delta)^s u \tag{4.31}$$

in $\mathbb{R}^n$, both in the sense of distributions and as a pointwise limit.

In order to prove Theorem 4.7, we need to make some preliminary computations. At first, let us recall a few useful properties of the minimizer function g of the operator G introduced in (4.27).

We know from formula (4.5) in [85] that

$$0 \leqslant g \leqslant 1, \tag{4.32}$$

and from formula (2.6) in [85] that

$$g' \leqslant 0. \tag{4.33}$$

We also cite formula (4.3) in [85], according to which g is a solution of

$$g''(t) + at^{-1} g'(t) = g(t) \tag{4.34}$$

for any $t > 0$, and formula (4.4) in [85], according to which

$$\lim_{t \to 0^+} t^a g'(t) \;=\; -C_\sharp. \tag{4.35}$$

Now, for any $V \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ we set

$$[V]_a := \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |\nabla_X V(X)|^2 \, dX}.$$

Notice that $[V]_a$ is well defined (possibly infinite) on such space. Also, one can compute $[V]_a$ explicitly in the following interesting case:

Lemma 4.8. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and*

$$U(x,y) := \mathcal{F}^{-1}\Big(\psi(\xi)\, g(|\xi|y)\Big). \tag{4.36}$$

Then

$$[U]_a^2 = C_\sharp\, [\psi]_G^2. \tag{4.37}$$

Proof. By (4.32), for any fixed $y > 0$, the function $\xi \mapsto \psi(\xi)\, g(|\xi|y)$ belongs to $L^2(\mathbb{R}^n)$, and so we may consider its (inverse) Fourier transform. This says that the definition of U is well posed.

By the inverse Fourier transform definition (3.2), we have that

$$\begin{aligned}
\nabla_x U(x,y) &= \nabla_x \int_{\mathbb{R}^n} \psi(\xi)\, g(|\xi|y)\, e^{ix\cdot\xi}\, d\xi \\
&= \int_{\mathbb{R}^n} i\xi\psi(\xi)\, g(|\xi|y)\, e^{ix\cdot\xi}\, d\xi \\
&= \mathcal{F}^{-1}\Big(i\xi\psi(\xi) g(|\xi|y)\Big)(x).
\end{aligned}$$

Thus, by Plancherel Theorem,

$$\int_{\mathbb{R}^n} |\nabla_x U(x,y)|^2\,dx = \int_{\mathbb{R}^n} \left|\xi\psi(\xi)g(|\xi|y)\right|^2 d\xi.$$

Integrating over $y>0$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla_x U(X)|^2\,dX &= \int_{\mathbb{R}^n} |\xi|^2|\psi(\xi)|^2\left[\int_0^{+\infty} y^a\left|g(|\xi|y)\right|^2 dy\right] d\xi\\ &= \int_{\mathbb{R}^n} |\xi|^{1-a}|\psi(\xi)|^2\left[\int_0^{+\infty} t^a\left|g(t)\right|^2 dt\right] d\xi\\ &= \int_0^{+\infty} t^a\left|g(t)\right|^2 dt\cdot\int_{\mathbb{R}^n}|\xi|^{2s}|\psi(\xi)|^2\,d\xi\\ &= [\psi]_G^2\int_0^{+\infty} t^a\left|g(t)\right|^2 dt.\end{aligned} \tag{4.38}$$

Let us now prove that the following identity is well posed

$$\partial_y U(x,y) = \mathcal{F}^{-1}\Big(|\xi|\,\psi(\xi)\,g'(|\xi|y)\Big). \tag{4.39}$$

For this, we observe that

$$|g'(t)| \leqslant C_\sharp t^{-a}. \tag{4.40}$$

To check this, we define $\gamma(t) := t^a\,|g'(t)|$. From (4.33) and (4.34), we obtain that

$$\gamma'(t) = -\frac{d}{dt}\big(t^a g'(t)\big) = -t^a\left(g''(t) + at^{-1}g'(t)\right) = -t^a g(t) \leqslant 0.$$

Hence

$$\gamma(t) \leqslant \lim_{\tau\to 0^+}\gamma(\tau) = \lim_{\tau\to 0^+}\tau^a|g'(\tau)| = C_\sharp,$$

where formula (4.35) was used in the last identity, and this establishes (4.40).

From (4.40) we have that $|\xi|\,|\psi(\xi)|\,|g'(|\xi|y)| \leqslant C_\sharp y^{-a}\,|\xi|^{1-a}\,|\psi(\xi)| \in L^2(\mathbb{R}^n)$, and so (4.39) follows.

Therefore, by (4.39) and the Plancherel Theorem,

$$\int_{\mathbb{R}^n}|\partial_y U(x,y)|^2\,dx = \int_{\mathbb{R}^n}|\xi|^2\,|\psi(\xi)|^2\,|g'(|\xi|y)|^2\,d\xi.$$

Integrating over $y>0$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} y^a|\partial_y U(x,y)|^2\,dx &= \int_{\mathbb{R}^n}|\xi|^2\,|\psi(\xi)|^2\left[\int_0^{+\infty} y^a|g'(|\xi|y)|^2\,dy\right] d\xi\\ &= \int_{\mathbb{R}^n}|\xi|^{1-a}\,|\psi(\xi)|^2\left[\int_0^{+\infty} t^a|g'(t)|^2 dt\right] d\xi\\ &= \int_0^{+\infty} t^a|g'(t)|^2 dt\cdot\int_{\mathbb{R}^n}|\xi|^{2s}\,|\psi(\xi)|^2\,d\xi\\ &= [\psi]_G^2\int_0^{+\infty} t^a|g'(t)|^2 dt.\end{aligned}$$

By summing this with (4.38), and recalling (4.28), we obtain the desired result $[U]_a^2 = C_\sharp\,[\psi]_G^2$. This concludes the proof of the Lemma. □

Now, given $u \in L^1_{\rm loc}(\mathbb{R}^n)$, we consider the space X_u of all the functions $V \in W^{1,1}_{\rm loc}(\mathbb{R}^{n+1}_+)$ such that, for any $x \in \mathbb{R}^n$, the map $y \mapsto V(x,y)$ is in $C^0\big([0,+\infty)\big)$, with $V(x,0) = u(x)$ for any $x \in \mathbb{R}^n$. Then the problem of minimizing $[\,\cdot\,]_a$ over X_u has a somehow explicit solution.

Lemma 4.9. *Assume that $u \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\min_{V\in X_u} [V]_a^2 = [U]_a^2 = C_\sharp\,[\widehat{u}]_G^2, \tag{4.41}$$

$$U(x,y) := \mathcal{F}^{-1}\Big(\widehat{u}(\xi)\,g(|\xi|y)\Big). \tag{4.42}$$

Proof. We remark that (4.42) is simply (4.36) with $\psi := \widehat{u}$, and by Lemma 4.8 we have that

$$[U]_a^2 = C_\sharp [\widehat{u}]_G^2.$$

Furthermore, we claim that

$$U \in X_u. \tag{4.43}$$

In order to prove this, we first observe that

$$|g(T) - g(t)| \leqslant \frac{C_\sharp\,|T^{2s} - t^{2s}|}{2s}. \tag{4.44}$$

To check this, without loss of generality, we may suppose that $T \geqslant t \geqslant 0$. Hence, by (4.33) and (4.40),

$$\begin{aligned} |g(T) - g(t)| &\leqslant \int_t^T |g'(r)|\,dr \\ &\leqslant C_\sharp \int_t^T r^{-a}\,dr \\ &= \frac{C_\sharp\,(T^{1-a} - t^{1-a})}{1-a}, \end{aligned}$$

that is (4.44).

Then, by (4.44), for any y, $\tilde{y} \in (0,+\infty)$, we see that

$$\Big|g(|\xi|\,y) - g(|\xi|\,\tilde{y})\Big| \leqslant \frac{C_\sharp\,|\xi|^{2s}|y^{2s} - \tilde{y}^{2s}|}{2s}.$$

Accordingly,

$$\begin{aligned} |U(x,y) - U(x,\tilde{y})| &= \left|\mathcal{F}^{-1}\Big(\widehat{u}(\xi)\,\Big(g(|\xi|\,y) - g(|\xi|\,\tilde{y})\Big)\Big)\right| \\ &\leqslant \int_{\mathbb{R}^n} \Big|\widehat{u}(\xi)\,\Big(g(|\xi|\,y) - g(|\xi|\,\tilde{y})\Big)\Big|\,d\xi \\ &\leqslant \frac{C_\sharp\,|y^{2s} - \tilde{y}^{2s}|}{2s} \int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{u}(\xi)|\,d\xi, \end{aligned}$$

and this implies (4.43).

Thanks to (4.43) and (4.37), in order to complete the proof of (4.41), it suffices to show that, for any $V \in X_u$, we have that

$$[V]_a^2 \geqslant [U]_a^2. \tag{4.45}$$

To prove this, let us take $V \in X_u$. Without loss of generality, since $[U]_a < +\infty$ thanks to (4.37), we may suppose that $[V]_a < +\infty$. Hence, fixed a.e. $y > 0$, we have that

$$y^a \int_{\mathbb{R}^n} |\nabla_x V(x,y)|^2 \, dx \leqslant y^a \int_{\mathbb{R}^n} |\nabla_X V(x,y)|^2 \, dx < +\infty,$$

hence the map $x \in |\nabla_x V(x,y)|$ belongs to $L^2(\mathbb{R}^n)$. Therefore, by Plancherel Theorem,

$$\int_{\mathbb{R}^n} |\nabla_x V(x,y)|^2 \, dx = \int_{\mathbb{R}^n} \Big|\mathcal{F}\big(\nabla_x V(x,y)\big)(\xi)\Big|^2 \, d\xi. \tag{4.46}$$

Now by the Fourier transform definition (3.1)

$$\begin{aligned} \mathcal{F}\big(\nabla_x V(x,y)\big)(\xi) &= \int_{\mathbb{R}^n} \nabla_x V(x,y)\, e^{-ix\cdot\xi} \, dx \\ &= \int_{\mathbb{R}^n} i\xi\, V(x,y)\, e^{-ix\cdot\xi} \, dx \\ &= i\xi\, \mathcal{F}\big(V(x,y)\big)(\xi), \end{aligned}$$

hence (4.46) becomes

$$\int_{\mathbb{R}^n} |\nabla_x V(x,y)|^2 \, dx = \int_{\mathbb{R}^n} |\xi|^2 \, |\mathcal{F}\big(V(x,y)\big)(\xi)|^2 \, d\xi. \tag{4.47}$$

On the other hand

$$\mathcal{F}\big(\partial_y V(x,y)\big)(\xi) = \partial_y \mathcal{F}\big(V(x,y)\big)(\xi)$$

and thus, by Plancherel Theorem,

$$\int_{\mathbb{R}^n} |\partial_y V(x,y)|^2 \, dx = \int_{\mathbb{R}^n} \Big|\mathcal{F}\big(\partial_y V(x,y)\big)(\xi)\Big|^2 \, d\xi = \int_{\mathbb{R}^n} |\partial_y \mathcal{F}\big(V(x,y)\big)(\xi)|^2 \, d\xi.$$

We sum up this latter result with identity (4.47) and we use the notation $\phi(\xi,y) := \mathcal{F}\big(V(x,y)\big)(\xi)$ to conclude that

$$\int_{\mathbb{R}^n} |\nabla_X V(x,y)|^2 \, dx = \int_{\mathbb{R}^n} |\xi|^2 \, |\phi(\xi,y)|^2 + |\partial_y \phi(\xi,y)|^2 \, d\xi. \tag{4.48}$$

Accordingly, integrating over $y > 0$, we deduce that

$$[V]_a^2 = \int_{\mathbb{R}^{n+1}_+} y^a \Big(|\xi|^2 \, |\phi(\xi,y)|^2 + |\partial_y \phi(\xi,y)|^2\Big) \, d\xi \, dy. \tag{4.49}$$

Let us first consider the integration over y, for any fixed $\xi \in \mathbb{R}^n \backslash \{0\}$, that we now omit from the notation when this does not generate any confusion. We set $h(y) := \phi(\xi, |\xi|^{-1} y)$. We have that $h'(y) = |\xi|^{-1} \partial_y \phi(\xi, |\xi|^{-1} y)$ and therefore, using the substitution $t = |\xi|\, y$, we obtain

$$\begin{aligned} &\int_0^{+\infty} y^a \Big(|\xi|^2 \, |\phi(\xi,y)|^2 + \big|\partial_y \phi(\xi,y)\big|^2\Big) \, dy \\ &= |\xi|^{1-a} \int_0^{+\infty} t^a \Big(|\phi(\xi, |\xi|^{-1} t)|^2 + |\xi|^{-2} \big|\partial_y \phi(\xi, |\xi|^{-1} t)\big|^2\Big) \, dt \\ &= |\xi|^{1-a} \int_0^{+\infty} t^a \Big(|h(t)|^2 + |h'(t)|^2\Big) \, dt \\ &= |\xi|^{2s} \, G(h). \end{aligned} \tag{4.50}$$

Now, for any $\lambda \in \mathbb{R}$, we show that

$$\min_{w \in W^{1,1}_{\rm loc}((0,+\infty)) \cap C^0([0,+\infty))} w(0) = \lambda G(w) = \lambda^2 C_\sharp. \tag{4.51}$$

Indeed, when $\lambda = 0$, the trivial function is an allowed competitor and $G(0) = 0$, which gives (4.51) in this case. If, on the other hand, $\lambda \neq 0$, given w as above with $w(0) = \lambda$ we set $w_\lambda(x) := \lambda^{-1} w(x)$. Hence we see that $w_\lambda(0) = 1$ and thus $G(w) = \lambda^2 G(w_\lambda) \leqslant \lambda^2 G(g) = \lambda^2 C_\sharp$, due to the minimality of g. This proves (4.51). From (4.51) and the fact that

$$h(0) = \phi(\xi, 0) = \mathcal{F}\big(V(x,0)\big)(\xi) = \widehat{u}(\xi),$$

we obtain that

$$G(h) \geqslant C_\sharp \left|\widehat{u}(\xi)\right|^2.$$

As a consequence, we get from (4.50) that

$$\int_0^{+\infty} y^a \Big(|\xi|^2 \left|\phi(\xi,y)\right|^2 + \left|\partial_y \phi(\xi,y)\right|^2 \Big)\, dy \geqslant C_\sharp \left|\xi\right|^{2s} \left|\widehat{u}(\xi)\right|^2.$$

Integrating over $\xi \in \mathbb{R}^n \backslash \{0\}$ we obtain that

$$\int_{\mathbb{R}^{n+1}_+} y^a \Big(|\xi|^2 \left|\phi(\xi,y)\right|^2 + \left|\partial_y \phi(\xi,y)\right|^2 \Big)\, d\xi\, dy \geqslant C_\sharp \left[\widehat{u}\right]^2_G.$$

Hence, by (4.49),

$$[V]^2_a \geqslant C_\sharp \left[\widehat{u}\right]^2_G,$$

which proves (4.45), and so (4.41). □

We can now prove the main result of this section.

Proof of Theorem 4.7. Formula (4.30) follows from the minimality property in (4.41), by writing that $[U]^2_a \leqslant [U + \epsilon\varphi]^2_a$ for any φ smooth and compactly supported inside $\mathbb{R}^{n+1}_+$ and any $\epsilon \in \mathbb{R}$.

Now we take $\varphi \in C^\infty_0(\mathbb{R}^n)$ (notice that its support may now hit $\{y = 0\}$). We define $u_\epsilon := u + \epsilon\varphi$, and U_ϵ as in (4.29), with $\widehat{u}$ replaced by $\widehat{u}_\epsilon$ (notice that (4.29) is nothing but (4.42)), hence we will be able to exploit Lemma 4.9.

We also set

$$\varphi_*(x,y) := \mathcal{F}^{-1}\Big(\widehat{\varphi}(\xi)\, g(|\xi| y)\Big).$$

We observe that

$$\varphi_*(x,0) = \mathcal{F}^{-1}\Big(\widehat{\varphi}(\xi)\, g(0)\Big) = \mathcal{F}^{-1}\Big(\widehat{\varphi}(\xi)\Big) = \varphi(x) \tag{4.52}$$

and that

$$U_\epsilon = U + \epsilon \mathcal{F}^{-1}\Big(\widehat{\varphi}(\xi)\, g(|\xi| y)\Big) = U + \epsilon \varphi_*.$$

As a consequence

$$[U_\epsilon]^2_a = [U_\epsilon]^2_a + 2\epsilon \int_{\mathbb{R}^{n+1}_+} y^a \nabla_X U \cdot \nabla_X \varphi_*\, dX + o(\epsilon).$$

Hence, using (4.30), (4.52) and the Divergence Theorem,

$$\begin{aligned} [U_\epsilon]^2_a &= [U]^2_a + 2\epsilon \int_{\mathbb{R}^{n+1}_+} \operatorname{div}\Big(\varphi_*\, y^a \nabla_X U\Big)\, dX + o(\epsilon) \\ &= [U]^2_a - 2\epsilon \int_{\mathbb{R}^n \times \{0\}} \varphi\, y^a \partial_y U\, dx + o(\epsilon). \end{aligned} \tag{4.53}$$

Moreover, from Plancherel Theorem, and the fact that the image of φ is in the reals,

$$\begin{aligned}[\widehat{u}_\epsilon]_G^2 &= [\widehat{u}]_G + 2\epsilon \int_{\mathbb{R}^n} |\xi|^{2s} \widehat{u}(\xi)\, \overline{\widehat{\varphi}(\xi)}\, d\xi + o(\epsilon)\\ &= [\widehat{u}]_G + 2\epsilon \int_{\mathbb{R}^n} \mathcal{F}^{-1}\Big(|\xi|^{2s}\widehat{u}(\xi)\Big)(x)\, \overline{\varphi(x)}\, dx + o(\epsilon)\\ &= [\widehat{u}]_G + 2\epsilon \int_{\mathbb{R}^n} (-\Delta)^s u(x)\, \varphi(x)\, dx + o(\epsilon).\end{aligned}$$

By comparing this with (4.53) and recalling (4.41) we obtain that

$$\begin{aligned}&[U]_a^2 - 2\epsilon \int_{\mathbb{R}^n\times\{0\}} \varphi\, y^a \partial_y U\, dx + o(\epsilon)\\ &= [U_\epsilon]_a^2\\ &= C_\sharp [u_\epsilon]_G^2\\ &= C_\sharp [\widehat{u}]_G + 2C_\sharp\epsilon \int_{\mathbb{R}^n} (-\Delta)^s u(x)\, \varphi(x)\, dx + o(\epsilon)\\ &= [U]_a^2 + 2C_\sharp\epsilon \int_{\mathbb{R}^n} (-\Delta)^s u\, \varphi\, dx + o(\epsilon)\end{aligned}$$

and so

$$-\int_{\mathbb{R}^n\times\{0\}} \varphi\, y^a \partial_y U\, dx = C_\sharp \int_{\mathbb{R}^n} (-\Delta)^s u\, \varphi\, dx,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, that is the distributional formulation of (4.31).

Furthermore, by (4.29), we have that

$$y^a \partial_y U(x,y) = \mathcal{F}^{-1}\Big(|\xi|\, \widehat{u}(\xi)\, y^a\, g(|\xi|y)\Big) = \mathcal{F}^{-1}\Big(|\xi|^{1-a}\, \widehat{u}(\xi)\, (|\xi|y)^a\, g(|\xi|y)\Big).$$

Hence, by (4.35), we obtain

$$\begin{aligned}\lim_{y\to 0^+} y^a \partial_y U(x,y) =&\ -C_\sharp \mathcal{F}^{-1}\Big(|\xi|^{1-a}\, \widehat{u}(\xi)\Big)\\ =&\ -C_\sharp \mathcal{F}^{-1}\Big(|\xi|^{2s}\, \widehat{u}(\xi)\Big)\\ =&\ -(-\Delta)^s u(x),\end{aligned}$$

that is the pointwise limit formulation of (4.31). This concludes the proof of Theorem 4.7. □

5. Nonlocal phase transitions

Now, we consider a nonlocal phase transition model, in particular described by the Allen-Cahn equation. A fractional analogue of a conjecture of De Giorgi, that deals with possible one-dimensional symmetry of entire solutions, naturally arises from treating this model, and will be consequently presented. There is a very interesting connection with nonlocal minimal surfaces, that will be studied in Section 6.

We introduce briefly the classical case[8]. The Allen-Cahn equation has various applications, for instance, in the study of interfaces (both in gases and solids), in the theory of superconductors and superfluids or in cosmology. We deal here with a two-phase transition model, in which a fluid can reach two pure phases (say 1 and

[8]We would like to thank Alberto Farina who, during a summer-school in Cortona (2014), gave a beautiful introduction on phase transitions in the classical case.

-1) forming an interface of separation. The aim is to describe the pattern and the separation of the two phases.

The formation of the interface is driven by a variational principle. Let $u(x)$ be the function describing the state of the fluid at position x in a bounded region Ω. As a first guess, the phase separation can be modeled via the minimization of the energy

$$\mathcal{E}_0(u) = \int_\Omega W\big(u(x)\big)\,dx,$$

where W is a double-well potential (which is smooth in $[-1,1]$, positive in $(-1,1)$ and vanishing at ±1). The classical example is

$$W(u) := \frac{(u^2-1)^2}{4}. \tag{5.1}$$

On the other hand, the functional in $\mathcal{E}_0$ produces an ambiguous outcome, since any function u that attains only the values ±1 is a minimizer for the energy. That is, the energy functional in $\mathcal{E}_0$ alone cannot detect any geometric feature of the interface.

To avoid this, one is led to consider an additional energy term that penalizes the formation of unnecessary interfaces. The typical energy functional provided by this procedure has the form

$$\mathcal{E}(u) := \int_\Omega W\big(u(x)\big)\,dx + \frac{\varepsilon^2}{2}\int_\Omega |\nabla u(x)|^2\,dx. \tag{5.2}$$

In this way, the potential energy that forces the pure phases is compensated by a small term, that is due to the elastic effect of the reaction of the particles. As a curiosity, we point out that in the classical mechanics framework, the analogue of (5.2) is a Lagrangian action of a particle, with $n = 1$, x representing a time coordinate and $u(x)$ the position of the particle at time x. In this framework the term involving the square of the derivative of u has the physical meaning of a kinetic energy. With a slight abuse of notation, we will keep referring to the gradient term in (5.2) as a kinetic energy. Perhaps a more appropriate term would be elastic energy, but in concrete applications also the potential may arise from elastic reactions, therefore the only purpose of these names in our framework is to underline the fact that (5.2) occurs as a superposition of two terms, a potential one, which only depends on u, and one, which will be called kinetic, which only depends on the variation of u (and which, in principle, possesses no real "kinetic" feature).

The energy minimizers will be smooth functions, taking values between -1 and 1, forming layers of interfaces of ε-width. If we send $\varepsilon \to 0$, the transition layer will tend to a minimal surface. To better explain this, consider the energy

$$J(u) = \int \frac{1}{2}|\nabla u|^2 + W(u)\,dx, \tag{5.3}$$

whose minimizers solve the Allen-Cahn equation

$$-\Delta u + W'(u) = 0. \tag{5.4}$$

In particular, for the explicit potential in (5.1), equation (5.4) reduces (up to normalizations constants) to

$$-\Delta u = u - u^3. \tag{5.5}$$

In this setting, the behavior of u in large domains reflects into the behavior of the rescaled function $u_\varepsilon(x) = u\big(\frac{x}{\varepsilon}\big)$ in B_1. Namely, the minimizers of J in $B_{1/\varepsilon}$ are the

minimizers of J_ε in B_1, where J_ε is the rescaled energy functional

$$J_\varepsilon(u) = \int_{B_1} \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,dx. \tag{5.6}$$

We notice then that

$$J_\varepsilon(u) \geqslant \int_{B_1} \sqrt{2W(u)}\,|\nabla u|\,dx$$

which, using the Co-area Formula, gives

$$J_\varepsilon(u) \geqslant \int_{-1}^{1} \sqrt{2W(t)}\,\mathcal{H}^{n-1}\left(\{u=t\}\right)\,dt.$$

The above formula may suggest that the minimizers of J_ε have the tendency to minimize the $(n-1)$-dimensional measure of their level sets. It turns out that indeed the level sets of the minimizers of J_ε converge to a minimal surface as $\varepsilon \to 0$: for more details see, for instance, [79] and the references therein.

In this setting, a famous De Giorgi conjecture comes into place. In the late 70's, De Giorgi conjectured that entire, smooth, monotone (in one direction), bounded solutions of (5.5) in the whole of $\mathbb{R}^n$ are necessarily one-dimensional, i.e., there exist $\omega \in S^{n-1}$ and $u_0 : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = u_0(\omega \cdot x) \quad \text{for any} \quad x \in \mathbb{R}^n.$$

In other words, the conjecture above asks if the level sets of the entire, smooth, monotone (in one direction), bounded solutions are necessarily hyperplanes, at least in dimension $n \leqslant 8$.

One may wonder why the number eight has a relevance in the problem above. A possible explanation for this is given by the Bernstein Theorem, as we now try to describe.

The Bernstein problem asks on whether or not all minimal graphs (i.e. surfaces that locally minimize the perimeter and that are graph in a given direction) in $\mathbb{R}^n$ must be necessarily affine. This is indeed true in dimensions n at most eight. On the other hand, in dimension $n \geqslant 9$ there are global minimal graphs that are not hyperplanes (see e.g. [59]).

The link between the problem of Bernstein and the conjecture of De Giorgi could be suggested by the fact that minimizers approach minimal surfaces in the limit. In a sense, if one is able to prove that the limit interface is a hyperplane and that this rigidity property gets inherited by the level sets of the minimizers u_ε (which lie nearby such limit hyperplane), then, by scaling back, one obtains that the level sets of u are also hyperplanes. Of course, this link between the two problems, as stated here, is only heuristic, and much work is needed to deeply understand the connections between the problem of Bernstein and the conjecture of De Giorgi. We refer to [51] for a more detailed introduction to this topic.

We recall that this conjecture by De Giorgi was proved for $n \leqslant 3$, see [58, 8, 4]. Also, the case $4 \leqslant n \leqslant 8$ with the additional assumption that

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \quad \text{for any} \quad x' \in \mathbb{R}^{n-1} \tag{5.7}$$

was proved in [80].

For $n \geqslant 9$ a counterexample can be found in [37]. Notice that, if the above limit is uniform (and De Giorgi conjecture with this additional assumption is known as

the Gibbons conjecture), the result extends to all possible n (see for instance [50, 51] for further details).

The goal of the next part of this paper is then to discuss an analogue of these questions for the nonlocal case and present related results.

5.1. **The fractional Allen-Cahn equation.** The extension of the Allen-Cahn equation in (5.4) from a local to a nonlocal setting has theoretical interest and concrete applications. Indeed, the study of long range interactions naturally leads to the analysis of phase transitions and interfaces of nonlocal type.

Given an open domain $\Omega \subset \mathbb{R}^n$ and the double well potential W (as in (5.1)), our goal here is to study the fractional Allen-Cahn equation

$$(-\Delta)^s u + W'(u) = 0 \quad \text{in} \quad \Omega,$$

for $s \in (0,1)$ (when $s = 1$, this equation reduces to (5.4)). The solutions are the critical points of the nonlocal energy

$$\mathcal{E}(u,\Omega) := \int_\Omega W\big(u(x)\big)\,dx + \frac{1}{2} \iint\limits_{\mathbb{R}^{2n}\setminus(\Omega^{\mathcal{C}})^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \tag{5.8}$$

up to normalization constants that we omitted for simplicity. The reader can compare (5.8) with (5.2). Namely, in (5.8) the kinetic energy is modified, in order to take into account long range interactions. That is, the new kinetic energy still depends on the variation of the phase parameter. But, in this case, far away changes in phase may influence each other (though the influence is weaker and weaker towards infinity).

Notice that in the nonlocal framework, we prescribe the function on $\Omega^{\mathcal{C}} \times \Omega^{\mathcal{C}}$ and consider the kinetic energy on the remaining regions (see Figure 8). The prescription of values in $\Omega^{\mathcal{C}} \times \Omega^{\mathcal{C}}$ reflects into the fact that the domain of integration of the kinetic integral in (5.8) is $\mathbb{R}^{2n}\setminus(\Omega^{\mathcal{C}})^2$. Indeed, this is perfectly compatible with the local case in (5.2), where the domain of integration of the kinetic term was simply Ω. To see this compatibility, one may think that the domain of integration of the kinetic energy is simply the complement of the set in which the values of the functions are prescribed. In the local case of (5.2), the values are prescribed on $\partial\Omega$, or, one may say, in $\Omega^{\mathcal{C}}$: then the domain of integration of the kinetic energy is the complement of $\Omega^{\mathcal{C}}$, which is simply Ω. In analogy with that, in the nonlocal case of (5.8), the values are prescribed on $\Omega^{\mathcal{C}} \times \Omega^{\mathcal{C}} = (\Omega^{\mathcal{C}})^2$, i.e. outside Ω for both the variables x and y. Then, the kinetic integral is set on the complement of $(\Omega^{\mathcal{C}})^2$, which is indeed $\mathbb{R}^{2n}\setminus(\Omega^{\mathcal{C}})^2$.

Of course, the potential energy has local features, both in the local and in the nonlocal case, since in our model the nonlocality only occurs in the kinetic interaction, therefore the potential integrals are set over Ω both in (5.2) and in (5.8).

For the sake of shortness, given disjoint sets $A, B \subseteq \mathbb{R}^n$ we introduce the notation

$$u(A,B) := \int_A \int_B \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

and we write the new kinetic energy in (5.8) as

$$\mathcal{K}(u,\Omega) = \frac{1}{2}u(\Omega,\Omega) + u(\Omega,\Omega^{\mathcal{C}}). \tag{5.9}$$

Let us define the energy minimizers and provide a density estimate for the minimizers.

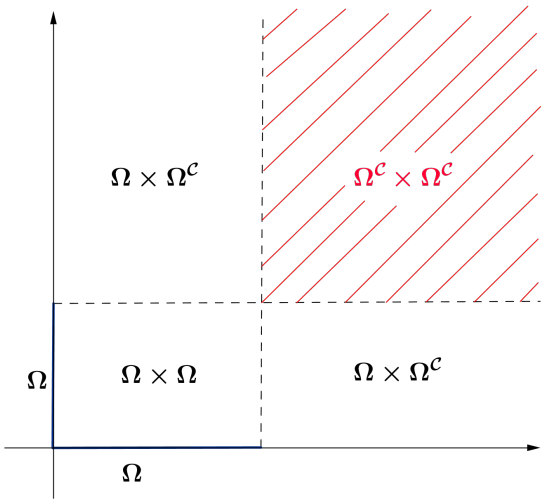

FIGURE 8. The kinetic energy

Definition 5.1. *The function u is a minimizer for the energy $\mathcal{E}$ in B_R if $\mathcal{E}(u, B_R) \leqslant \mathcal{E}(v, B_R)$ for any v such that $u = v$ outside B_R.*

The energy of the minimizers satisfy the following uniform bound property on large balls.

Theorem 5.2. *Let u be a minimizer in B_{R+2} for a large R, say $R \geqslant 1$. Then*

$$\lim_{R\to+\infty} \frac{1}{R^n}\mathcal{E}(u, B_R) = 0. \tag{5.10}$$

More precisely,

$$\mathcal{E}(u, B_R) \leqslant \begin{cases} CR^{n-1} & \text{if} \quad s \in \left(\frac{1}{2}, 1\right), \\ CR^{n-1}\log R & \text{if} \quad s = \frac{1}{2}, \\ CR^{n-2s} & \text{if} \quad s \in \left(0, \frac{1}{2}\right). \end{cases}$$

Here, C is a positive constant depending only on n, s and W.

Notice that for $s \in \left(0, \frac{1}{2}\right)$, $R^{n-2s} > R^{n-1}$. These estimates are optimal (we refer to [83] for further details).

Proof. We introduce at first some auxiliary functions. Let

$$\psi(x) := -1 + 2\min\Big\{(|x| - R - 1)_+, 1\Big\}, \quad v(x) := \min\Big\{u(x), \psi(x)\Big\},$$

$$d(x) := \max\Big\{(R + 1 - |x|), 1\Big\}.$$

Then, for $|x - y| \leqslant d(x)$ we have that

$$|\psi(x) - \psi(y)| \leqslant \frac{2|x - y|}{d(x)}. \tag{5.11}$$

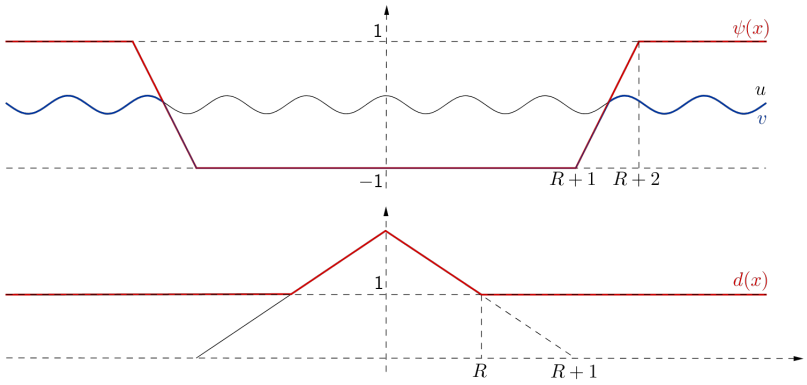

FIGURE 9. The functions ψ, v and d

Indeed, if $|x| \leqslant R$, then $d(x) = R + 1 - |x|$ and

$$|y| \leqslant |x - y| + |x| \leqslant d(x) + |x| \leqslant R + 1,$$

thus $\psi(x) = \psi(y) = 0$ and the inequality is trivial. Else, if $|x| \geqslant R$, then $d(x) = 1$, and so the inequality is assured by the Lipschitz continuity of ψ (with 2 as the Lipschitz constant).

Also, we prove that we have the following estimates for the function d:

$$\int_{B_{R+2}} d(x)^{-2s}\,dx \leqslant \begin{cases} CR^{n-1} & \text{if} \quad s \in \left(\frac{1}{2}, 1\right), \\ CR^{n-1}\log R & \text{if} \quad s = \frac{1}{2}, \\ CR^{n-2s} & \text{if} \quad s \in \left(0, \frac{1}{2}\right). \end{cases} \tag{5.12}$$

To prove this, we observe that in the ring $B_{R+2}\backslash B_R$, we have $d(x) = 1$. Therefore, the contribution to the integral in (5.12) that comes from the ring $B_{R+2}\backslash B_R$ is bounded by the measure of the ring, and so it is of order R^{n-1}, namely

$$\int_{B_{R+2}\backslash B_R} d(x)^{-2s}\,dx = |B_{R+2}\backslash B_R| \leqslant CR^{n-1}, \tag{5.13}$$

for some $C > 0$. We point out that this order is always negligible with respect to the right hand side of (5.12).

Therefore, to complete the proof of (5.12), it only remains to estimate the contribution to the integral coming from B_R.

For this, we use polar coordinates and perform the change of variables $t = \rho/(R+1)$. In this way, we obtain that

$$\begin{aligned} \int_{B_R} d(x)^{-2s}\,dx &= C\int_0^R \frac{\rho^{n-1}}{(R+1-\rho)^{2s}}\,d\rho \\ &= C\,(R+1)^{n-2s}\int_0^{1-\frac{1}{R+1}} t^{n-1}(1-t)^{-2s}\,dt \\ &\leqslant C\,(R+1)^{n-2s}\int_0^{1-\frac{1}{R+1}} (1-t)^{-2s}\,dt, \end{aligned}$$

for some $C > 0$. Now we observe that

$$\int_0^{1-\frac{1}{R+1}} (1-t)^{-2s}\,dt \leqslant \begin{cases} \displaystyle\int_0^1 (1-t)^{-2s}\,dt = C & \text{if} \quad s \in \left(0, \tfrac{1}{2}\right), \\ -\log(1-t)\Big|_0^{1-\frac{1}{R+1}} \leqslant \log R & \text{if} \quad s = \tfrac{1}{2}, \\ -\frac{(1-t)^{1-2s}}{1-2s}\Big|_0^{1-\frac{1}{R+1}} \leqslant CR^{2s-1} & \text{if} \quad s \in \left(\tfrac{1}{2}, 1\right). \end{cases}$$

The latter two formulas and (5.13) imply (5.12).

Now, we define the set

$$A := \{v = \psi\}$$

and notice that $B_{R+1} \subseteq A \subseteq B_{R+2}$. We prove that for any $x \in A$ and any $y \in A^{\mathcal{C}}$

$$|v(x) - v(y)| \leqslant \max\Big\{|u(x) - u(y)|, |\psi(x) - \psi(y)|\Big\}. \tag{5.14}$$

Indeed, for $x \in A$ and $y \in A^{\mathcal{C}}$ we have that

$$v(x) = \psi(x) \leqslant u(x) \quad \text{and} \quad v(y) = u(y) \leqslant \psi(y),$$

therefore

$$v(x) - v(y) \leqslant u(x) - u(y) \quad \text{and} \quad v(y) - v(x) \leqslant \psi(y) - \psi(x),$$

which establishes (5.14). This leads to

$$v(A, A^{\mathcal{C}}) \leqslant u(A, A^{\mathcal{C}}) + \psi(A, A^{\mathcal{C}}). \tag{5.15}$$

Notice now that

$$\mathcal{E}(u, B_{R+2}) \leqslant \mathcal{E}(v, B_{R+2})$$

since u is a minimizer in B_{R+2} and $v = u$ outside B_{R+2}. We have that

$$\begin{aligned} \mathcal{E}(u, B_{R+2}) &= \frac{1}{2}\,u(B_{R+2}, B_{R+2}) + u(B_{R+2}, B_{R+2}^{\mathcal{C}}) + \int_{B_{R+2}} W(u)\,dx \\ &= \frac{1}{2}\,u(A, A) + u(A, A^{\mathcal{C}}) \\ &\quad + \frac{1}{2}\,u(B_{R+2}\backslash A, B_{R+2}\backslash A) + u(B_{R+2}\backslash A, B_{R+2}^{\mathcal{C}}) \\ &\quad + \int_A W(u)\,dx + \int_{B_{R+2}\backslash A} W(u)\,dx. \end{aligned}$$

Since u and v coincide on $A^{\mathcal{C}}$, by using the inequality (5.15) we obtain that

$$\begin{aligned} 0 &\leqslant \mathcal{E}(v, B_{R+2}) - \mathcal{E}(u, B_{R+2}) \\ &= \frac{1}{2}\,v(A, A) - \frac{1}{2}u(A, A) + v(A, A^{\mathcal{C}}) - u(A, A^{\mathcal{C}}) + \int_A \Big(W(v) - W(u)\Big)\,dx \\ &\leqslant \frac{1}{2}\,v(A, A) - \frac{1}{2}u(A, A) + \psi(A, A^{\mathcal{C}}) + \int_A \Big(W(v) - W(u)\Big)\,dx. \end{aligned}$$

Moreover, $v = \psi$ on A and we have that

$$\frac{1}{2}\,u(A, A) + \int_A W(u)\,dx \leqslant \frac{1}{2}\,\psi(A, A) + \psi(A, A^{\mathcal{C}}) + \int_A W(\psi)\,dx) = \mathcal{E}(\psi, A),$$

and therefore, since $B_{R+1} \subseteq A \subseteq B_{R+2}$,

$$\frac{1}{2}\,u(B_{R+1}, B_{R+1}) + \int_{B_{R+1}} W(u)\,dx \leqslant \mathcal{E}(\psi, B_{R+2}). \tag{5.16}$$

We estimate now $\mathcal{E}(\psi, B_{R+2})$. For a fixed $x \in B_{R+2}$ we observe that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy \\
&= \int_{|x-y|\leqslant d(x)} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy + \int_{|x-y|\geqslant d(x)} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy \\
&\leqslant C\left(\frac{1}{d(x)^2}\int_{|x-y|\leqslant d(x)} |x-y|^{-n-2s+2}\,dy + \int_{|x-y|\geqslant d(x)} |x-y|^{-n-2s}\,dy\right),
\end{aligned}$$

where we have used (5.11) and the boundedness of ψ. Passing to polar coordinates, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy &\leqslant C\left(\frac{1}{d(x)^2}\int_0^{d(x)} \rho^{-2s+1}\,d\rho + \int_{d(x)}^{\infty} \rho^{-2s-1}\,d\rho\right) \\
&= Cd(x)^{-2s}.
\end{aligned}$$

Recalling that $\psi(x) = -1$ on B_{R+1} and $W(-1) = 0$, we obtain that

$$\begin{aligned}
\mathcal{E}(\psi, B_{R+2}) &= \int_{B_{R+2}}\int_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy\,dx + \int_{B_{R+2}} W(\psi)\,dx \\
&\leqslant \int_{B_{R+2}} d(x)^{-2s}dx + \int_{B_{R+2}\setminus B_{R+1}} W(\psi)\,dx.
\end{aligned}$$

Therefore, making use of (5.12),

$$\mathcal{E}(\psi, B_{R+2}) \leqslant \begin{cases} CR^{n-1} & \text{if} \quad s \in \left(\frac{1}{2}, 1\right), \\ CR^{n-1}\log R & \text{if} \quad s = \frac{1}{2}, \\ CR^{n-2s} & \text{if} \quad s \in \left(0, \frac{1}{2}\right). \end{cases} \tag{5.17}$$

For what regards the right hand-side of inequality (5.16), we have that

$$\begin{aligned}
\frac{1}{2}\,u(B_{R+1}, B_{R+1}) + \int_{B_{R+1}} W(u)\,dx \geqslant \frac{1}{2}\,u(B_R, B_R) + u(B_R, B_{R+1}\setminus B_R) \\
+ \int_{B_R} W(u)\,dx.
\end{aligned} \tag{5.18}$$

We prove now that

$$u(B_R, B_{R+1}^{\mathcal{C}}) \leqslant \int_{B_{R+2}} d(x)^{-2s}\,dx. \tag{5.19}$$

For this, we observe that if $x \in B_R$, then $d(x) = R + 1 - |x|$. So, if $x \in B_R$ and $y \in B_{R+1}^{\mathcal{C}}$, then

$$|x-y| \geqslant |y| - |x| \geqslant R + 1 - |x| = d(x).$$

Therefore, by changing variables $z = x - y$ and then passing to polar coordinates, we have that

$$\begin{aligned}
u(B_R, B_{R+1}^{\mathcal{C}}) &\leqslant 4\int_{B_R} dx \int_{B_{d(x)}^{\mathcal{C}}} |z|^{-n-2s}\,dz \\
&\leqslant C\int_{B_R} dx \int_{d(x)}^{\infty} \rho^{-2s-1}\,d\rho \\
&= C\int_{B_R} d(x)^{-2s}\,dx.
\end{aligned}$$

This establishes (5.19).

Hence, by (5.12) and (5.19), we have that

$$u(B_R, B_{R+1}^{\mathcal{C}}) \leqslant \int_{B_{R+2}} d(x)^{-2s}\,dx \leqslant \begin{cases} CR^{n-1} & \text{if} \quad s \in \left(\frac{1}{2}, 1\right), \\ CR^{n-1}\log R & \text{if} \quad s = \frac{1}{2}, \\ CR^{n-2s} & \text{if} \quad s \in \left(0, \frac{1}{2}\right). \end{cases} \tag{5.20}$$

We also observe that, by adding $u(B_R, B_{R+1}^{\mathcal{C}})$ to inequality (5.18), we obtain that

$$\begin{aligned} &\frac{1}{2}\,u(B_{R+1}, B_{R+1}) + \int_{B_{R+1}} W(u)\,dx + u(B_R, B_{R+1}^{\mathcal{C}}) \\ &\qquad \geqslant \frac{1}{2}\,u(B_R, B_R) + u(B_R, B_{R+1}\backslash B_R) + \int_{B_R} W(u)\,dx + u(B_R, B_{R+1}^{\mathcal{C}}) \\ &\qquad = \mathcal{E}(u, B_R). \end{aligned}$$

This and (5.16) give that

$$\mathcal{E}(u, B_R) \leqslant \mathcal{E}(\psi, B_{R+2}) + u(B_R, B_{R+1}^{\mathcal{C}}).$$

Combining this with The estimates in (5.17) and (5.20), we obtain the desired result. □

Another type of estimate can be given in terms of the level sets of the minimizers (see Theorem 1.4 in [83]).

Theorem 5.3. *Let u be a minimizer of $\mathcal{E}$ in B_R. Then for any $\theta_1, \theta_2 \in (-1, 1)$ such that*

$$u(0) > \theta_1$$

we have that there exist $\overline{R}$ and $C > 0$ such that

$$\Big|\{u > \theta_2\} \cap B_R\Big| \geqslant CR^n.$$

if $R \geqslant \overline{R}(\theta_1, \theta_2)$. The constant $C > 0$ depends only on n, s and W and $\overline{R}(\theta_1, \theta_2)$ is a large constant that depends also on θ_1 and θ_2.

The statement of Theorem 5.3 says that the level sets of minimizers always occupy a portion of a large ball comparable to the ball itself. In particular, both phases occur in a large ball, and the portion of the ball occupied by each phase is comparable to the one occupied by the other.

Of course, the simplest situation in which two phases split a ball in domains with comparable, and in fact equal, size is when all the level sets are hyperplanes. This question is related to a fractional version of a classical conjecture of De Giorgi and to nonlocal minimal surfaces, that we discuss in the following Sections 5.2 and 6.

5.2. **A nonlocal version of a conjecture by De Giorgi.** In this section we consider the fractional counterpart of the conjecture by De Giorgi that was discussed before in the classical case. Namely, we consider the nonlocal Allen-Cahn equation

$$-(-\Delta)^s u + W(u) = 0 \quad \text{in} \quad \mathbb{R}^n,$$

where W is a double-well potential, and u is smooth, bounded and monotone in one direction, namely $|u| \leqslant 1$ and $\partial_{x_n} u > 0$. We wonder if it is also true, at least in low dimension, that u is one-dimensional. In this case, the conjecture was initially proved for $n = 2$ and $s = \frac{1}{2}$ in [16]. In the case $n = 2$, for any $s \in (0, 1)$, the result is proved using the harmonic extension of the fractional Laplacian in [15] and [88].

For $n=3$, the proof can be found in [13] for $s\in\left[\frac12,1\right]$. The conjecture is still open for $n=3$ and $s\in\left[0,\frac12\right]$ and for $n\geqslant 4$. Also, the Gibbons conjecture (that is the De Giorgi conjecture with the additional condition that limit in (5.7) is uniform) is also true for any $s\in(0,1)$ and in any dimension n, see [52].

To keep the discussion as simple as possible, we focus here on the case $n=2$ and any $s\in(0,1)$, providing an alternative proof that does not make use of the harmonic extension. This part is completely new and not available in the literature. The proof is indeed quite general and it will be further exploited in [28].

We define (as in (5.9)) the total energy of the system to be

$$\mathcal{E}(u,B_R)=\mathcal{K}_R(u)+\int_{B_R}W(u)dx, \tag{5.21}$$

where the kinetic energy is

$$\mathcal{K}_R(u):=\frac12\iint_{Q_R}\frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+2s}}\,dx\,d\bar x, \tag{5.22}$$

and $Q_R:=\mathbb{R}^{2n}\backslash(B_R^{\mathcal{C}})^2=(B_R\times B_R)\cup(B_R\times(\mathbb{R}^n\backslash B_R))\cup((\mathbb{R}^n\backslash B_R)\times B_R)$. We recall that the kinetic energy can also be written as

$$\mathcal{K}_R(u)=\frac12 u(B_R,B_R)+u(B_R,B_R^{\mathcal{C}}), \tag{5.23}$$

where for two sets A,B

$$u(A,B)=\int_A\int_B\frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+2s}}\,dx\,d\bar x. \tag{5.24}$$

The main result of this section is the following.

Theorem 5.4. *Let u be a minimizer of the energy defined in (5.21) in any ball of $\mathbb{R}^2$. Then u is 1-D, i.e. there exist $\omega\in S^1$ and $u_0:\mathbb{R}\to\mathbb{R}$ such that*

$$u(x)=u_0(\omega\cdot x)\quad \textit{for any}\quad x\in\mathbb{R}^2.$$

The proof relies on the following estimate for the kinetic energy, that we prove by employing a domain deformation technique.

Lemma 5.5. *Let $R>1$, $\varphi\in C_0^\infty(B_1)$. Also, for any $y\in\mathbb{R}^n$, let*

$$\Psi_{R,+}(y):=y+\varphi\Big(\frac{y}{R}\Big)e_1\quad\textit{and}\quad\Psi_{R,-}(y):=y-\varphi\Big(\frac{y}{R}\Big)e_1. \tag{5.25}$$

Then, for large R, the maps $\Psi_{R,+}$ and $\Psi_{R,-}$ are diffeomorphisms on $\mathbb{R}^n$. Furthermore, if we define $u_{R,\pm}(x):=u(\Psi_{R,\pm}^{-1}(x))$, we have that

$$\mathcal{K}_R(u_{R,+})+\mathcal{K}_R(u_{R,-})-2\mathcal{K}_R(u)\leqslant\frac{C}{R^2}\mathcal{K}_R(u), \tag{5.26}$$

for some $C>0$.

Proof. First of all, we compute the Jacobian of $\Psi_{R,\pm}$. For this, we write $\Psi_{R,+,i}$ to denote the i^{th} component of the vector $\Psi_{R,+}=(\Psi_{R,+,1},\cdots,\Psi_{R,+,n})$ and we observe that

$$\frac{\partial\Psi_{R,+,i}(y)}{\partial y_j}=\frac{\partial}{\partial y_j}\Big(y_i\pm\varphi\Big(\frac{y}{R}\Big)\delta_{i1}\Big)=\delta_{ij}\pm\frac1R\partial_j\varphi\Big(\frac{y}{R}\Big)\delta_{i1}. \tag{5.27}$$

The latter term is bounded by $\mathcal{O}(R^{-1})$, and this proves that $\Psi_{R,\pm}$ is a diffeomorphism if R is large enough.

For further reference, we point out that if $J_{R,\pm}$ is the Jacobian determinant of $\Psi_{R,\pm}$, then the change of variable

$$x := \Psi_{R,\pm}(y), \qquad \bar{x} := \Psi_{R,\pm}(\bar{y}) \tag{5.28}$$

gives that

$$\begin{aligned} dx\, d\bar{x} &= J_{R,\pm}(y)\, J_{R,\pm}(\bar{y})\, dy\, d\bar{y} \\ &= \Big(1 \pm \Big(\frac{1}{R}\Big)\partial_1\varphi\Big(\frac{y}{R}\Big) + \mathcal{O}\Big(\frac{1}{R^2}\Big)\Big)\Big(1 \pm \frac{1}{R}\partial_1\varphi\Big(\frac{\bar{y}}{R}\Big) + \mathcal{O}\Big(\frac{1}{R^2}\Big)\Big) dy d\bar{y} \\ &= 1 \pm \frac{1}{R}\partial_1\varphi\Big(\frac{y}{R}\Big) \pm \frac{1}{R}\partial_1\varphi\Big(\frac{\bar{y}}{R}\Big) + \mathcal{O}\Big(\frac{1}{R^2}\Big)\, dy\, d\bar{y}, \end{aligned}$$

thanks to (5.27). Therefore

$$\begin{aligned} &\frac{|u_{R,\pm}(x) - u_{R,\pm}(\bar{x})|^2}{|x-\bar{x}|^{n+2s}}\, dx\, d\bar{x} \\ &= \frac{\big|u(\Psi^{-1}_{R,\pm}(x)) - u(\Psi^{-1}_{R,\pm}(\bar{x}))\big|^2}{|\Psi^{-1}_{R,\pm}(x) - \Psi^{-1}_{R,\pm}(\bar{x})|^{n+2s}} \cdot \left(\frac{|x-\bar{x}|^2}{|\Psi^{-1}_{R,\pm}(x) - \Psi^{-1}_{R,\pm}(\bar{x})|^2}\right)^{-\frac{n+2s}{2}} dx\, d\bar{x} \\ &= \frac{|u(y) - u(\bar{y})|^2}{|y-\bar{y}|^{n+2s}} \cdot \left(\frac{\big|\Psi_{R,\pm}(y) - \Psi_{R,\pm}(\bar{y})\big|^2}{|y-\bar{y}|^2}\right)^{-\frac{n+2s}{2}} \\ &\quad \cdot \left(1 \pm \frac{1}{R}\partial_1\varphi\Big(\frac{y}{R}\Big) \pm \frac{1}{R}\partial_1\varphi\Big(\frac{\bar{y}}{R}\Big) + \mathcal{O}\Big(\frac{1}{R^2}\Big)\right) dy\, d\bar{y}. \end{aligned} \tag{5.29}$$

Now, for any y, $\bar{y} \in \mathbb{R}^n$ we calculate

$$\begin{aligned} &\big|\Psi_{R,\pm}(y) - \Psi_{R,\pm}(\bar{y})\big|^2 \\ &\quad = \Big|(y-\bar{y}) \pm \Big(\varphi\Big(\frac{y}{R}\Big) - \varphi\Big(\frac{\bar{y}}{R}\Big)\Big)\, e_1\Big|^2 \\ &\quad = |y-\bar{y}|^2 + \Big|\varphi\Big(\frac{y}{R}\Big) - \varphi\Big(\frac{\bar{y}}{R}\Big)\Big|^2 \pm 2\Big(\varphi\Big(\frac{y}{R}\Big) - \varphi\Big(\frac{\bar{y}}{R}\Big)\Big)\,(y_1 - \bar{y}_1). \end{aligned} \tag{5.30}$$

Notice also that

$$\Big|\varphi\Big(\frac{y}{R}\Big) - \varphi\Big(\frac{\bar{y}}{R}\Big)\Big| \leqslant \frac{1}{R}\|\varphi\|_{C^1(\mathbb{R}^n)}|y-\bar{y}|, \tag{5.31}$$

hence (5.30) becomes

$$\frac{\big|\Psi_{R,\pm}(y) - \Psi_{R,\pm}(\bar{y})\big|^2}{|y-\bar{y}|^2} = 1 + \eta_\pm$$

where

$$\eta_\pm := \frac{\Big|\varphi\Big(\frac{y}{R}\Big) - \varphi\Big(\frac{\bar{y}}{R}\Big)\Big|^2}{|y-\bar{y}|^2} \pm 2\frac{\Big(\varphi\Big(\frac{y}{R}\Big) - \varphi\Big(\frac{\bar{y}}{R}\Big)\Big)\,(y_1 - \bar{y}_1)}{|y-\bar{y}|^2} = \mathcal{O}\Big(\frac{1}{R}\Big). \tag{5.32}$$

As a consequence

$$\left(\frac{\left|\Psi_{R,\pm}(y)-\Psi_{R,\pm}(\bar y)\right|^2}{|y-\bar y|^2}\right)^{-\frac{n+2s}{2}} = (1+\eta_\pm)^{-\frac{n+2s}{2}} = 1-\frac{n+2s}{2}\eta_\pm+\mathcal{O}(\eta_\pm^2).$$

We plug this information into (5.29) and use (5.32) to obtain

$$\begin{aligned}
&\frac{|u_{R,\pm}(x)-u_{R,\pm}(\bar x)|^2}{|x-\bar x|^{n+2s}}\,dx\,d\bar x\\
=\;&\frac{|u(y)-u(\bar y)|^2}{|y-\bar y|^{n+2s}}\cdot\left(1-\frac{n+2s}{2}\eta_\pm+\mathcal{O}\Big(\frac{1}{R^2}\Big)\right)\\
&\cdot\left(1\pm\frac{1}{R}\partial_1\varphi\Big(\frac{y}{R}\Big)\pm\frac{1}{R}\partial_1\varphi\Big(\frac{\bar y}{R}\Big)+\mathcal{O}\Big(\frac{1}{R^2}\Big)\right)dy\,d\bar y\\
=\;&\frac{|u(y)-u(\bar y)|^2}{|y-\bar y|^{n+2s}}\cdot\left[1-\frac{n+2s}{2}\eta_\pm+\left(\pm\frac{1}{R}\partial_1\varphi\Big(\frac{y}{R}\Big)\pm\frac{1}{R}\partial_1\varphi\Big(\frac{\bar y}{R}\Big)\right)\right.\\
&\left.+\mathcal{O}\Big(\frac{1}{R^2}\Big)\right]dy\,d\bar y.
\end{aligned}$$

Using this and the fact that

$$\eta_+ + \eta_- = 2\,\frac{\left|\varphi\Big(\frac{y}{R}\Big)-\varphi\Big(\frac{\bar y}{R}\Big)\right|^2}{|y-\bar y|^2} = \mathcal{O}\Big(\frac{1}{R^2}\Big),$$

thanks to (5.31), we obtain

$$\begin{aligned}
&\frac{|u_{R,+}(x)-u_{R,+}(\bar x)|^2}{|x-\bar x|^{n+2s}}+\frac{|u_{R,-}(x)-u_{R,-}(\bar x)|^2}{|x-\bar x|^{n+2s}}\,dx\,d\bar x\\
=\;&\frac{|u(y)-u(\bar y)|^2}{|y-\bar y|^{n+2s}}\cdot\left(2+\mathcal{O}\Big(\frac{1}{R^2}\Big)\right)\,dy\,d\bar y.
\end{aligned}$$

Thus, if we integrate over Q_R we find that

$$\mathcal{K}_R(u_{R,+})+\mathcal{K}_R(u_{R,+}) = 2\mathcal{K}_R(u)+\iint\limits_{Q_R}\mathcal{O}\Big(\frac{1}{R^2}\Big)\,\frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+2s}}\,dx\,d\bar x.$$

This establishes (5.26). □

Proof of Theorem 5.4. We organize this proof into four steps.

Step 1. A geometrical consideration

In order to prove that the level sets are flat, it suffices to prove that u is monotone in any direction. Indeed, if u is monotone in any direction, the level set $\{u=0\}$ is both convex and concave, thus it is flat.

Step 2. Energy estimates

Let $\varphi\in C_0^\infty(B_1)$ such that $\varphi=1$ in $B_{1/2}$, and let $e=(1,0)$. We define as in Lemma 5.5

$$\Psi_{R,+}(y) := y+\varphi\Big(\frac{y}{R}\Big)e \quad\text{and}\quad \Psi_{R,-}(y) := y-\varphi\Big(\frac{y}{R}\Big)e,$$

which are diffeomorphisms for large R, and the functions $u_{R,\pm}(x) := u(\Psi^{-1}_{R,+}(x))$. Notice that

$$u_{R,+}(y) = u(y) \qquad \text{for } y \in B_R^{\mathcal{C}} \tag{5.33}$$

$$u_{R,+}(y) = u(y-e) \qquad \text{for } y \in B_{R/2}. \tag{5.34}$$

By computing the potential energy, it is easy to see that

$$\int_{B_R} W(u_{R,+}(x))\,dx + \int_{B_R} W(u_{R,-}(x))\,dx - 2\int_{B_R} W(u(x))\,dx \leqslant \frac{C}{R^2}\int_{B_R} W(u(x))\,dx.$$

Using this and (5.26), we obtain the following estimate for the total energy

$$\mathcal{E}(u_{R,+}, B_R) + \mathcal{E}(u_{R,-}, B_R) - 2\mathcal{E}(u, B_R) \leqslant \frac{C}{R^2}\mathcal{E}(u, B_R). \tag{5.35}$$

Also, since $u_{R,\pm} = u$ in $B_R^{\mathcal{C}}$, we have that

$$\mathcal{E}(u, B_R) \leqslant \mathcal{E}(u_{R,-}, B_R).$$

This and (5.35) imply that

$$\mathcal{E}(u_{R,+}, B_R) - \mathcal{E}(u, B_R) \leqslant \frac{C}{R^2}\mathcal{E}(u, B_R). \tag{5.36}$$

As a consequence of this estimate and (5.10), it follows that

$$\lim_{R\to+\infty} \Big(\mathcal{E}(u_{R,+}, B_R) - \mathcal{E}(u, B_R)\Big) = 0. \tag{5.37}$$

Step 3. Monotonicity

We claim that u is monotone. Suppose by contradiction that u is not monotone. That is, up to translation and dilation, we suppose that the value of u at the origin stays above the values of e and $-e$, with $e := (1, 0)$, i.e.

$$u(0) > u(e) \quad \text{and} \quad u(0) > u(-e).$$

Take R to be large enough, say $R > 8$. Let now

$$v_R(x) := \min\big\{u(x), u_{R,+}(x)\big\} \quad \text{and} \quad w_R(x) := \max\big\{u(x), u_{R,+}(x)\big\}. \tag{5.38}$$

By (5.33) we have that $v_R = w_R = u$ outside B_R. Then, since u is a minimizer in B_R and $w_R = u$ outside B_R, we have that

$$\mathcal{E}(w_R, B_R) \geqslant \mathcal{E}(u, B_R). \tag{5.39}$$

Moreover, the sum of the energies of the minimum and the maximum is less than or equal to the sum of the original energies: this is obvious in the local case, since equality holds, and in the nonlocal case the proof is based on the inspection of the different integral contributions, see e.g. formula (38) in [75]. So we have that

$$\mathcal{E}(v_R, B_R) + \mathcal{E}(w_R, B_R) \leqslant \mathcal{E}(u, B_R) + \mathcal{E}(u_{R,+}, B_R)$$

hence, recalling (5.39),

$$\mathcal{E}(v_R, B_R) \leqslant \mathcal{E}(u_{R,+}, B_R). \tag{5.40}$$

We claim that v_R is not identically neither u, nor $u_{R,+}$. Indeed, since $u(0) = u_{R,+}(e)$ and $u(-e) = u_{R,+}(0)$ we have that

$$\begin{aligned} v_R(0) &= \min\big\{u(0), u_{R,+}(0)\big\} = \min\big\{u(0), u(-e)\big\} \\ &= u(-e) = u_{R,+}(0) < u(0) \quad \text{and} \\ v_R(e) &= \min\big\{u(e), u_{R,+}(e)\big\} = \min\big\{u(e), u(0)\big\} \\ &= u(e) < u(0) = u_{R,+}(e). \end{aligned}$$

By continuity of u and $u_{R,+}$, we have that

$$\begin{aligned} v_R &= u_{R,+} < u \text{ in a neighborhood of } 0 \quad \text{and} \\ v_R &= u < u_{R,+} \text{ in a neighborhood of } e. \end{aligned} \tag{5.41}$$

We focus our attention on the energy in the smaller ball B_2. We claim that v_R is not minimal for $\mathcal{E}(\cdot, B_2)$. Indeed, if v_R were minimal in B_2, then on B_2 both v_R and u would satisfy the same equation. However, $v_R \leqslant u$ in $\mathbb{R}^2$ by definition and $v_R = u$ in a neighborhood of e by the second statement in (5.41). The Strong Maximum Principle implies that they coincide everywhere, which contradicts the first line in (5.41).

Hence v_R is not a minimizer in B_2. Let then v_R^* be a minimizer of $\mathcal{E}(\cdot, B_2)$, that agrees with v_R outside the ball B_2, and we define the positive quantity

$$\delta_R := \mathcal{E}(v_R, B_2) - \mathcal{E}(v_R^*, B_2). \tag{5.42}$$

We claim that

$$\text{as } R \text{ goes to infinity, } \delta_R \text{ remains bounded away from zero.} \tag{5.43}$$

To prove this, we assume by contradiction that

$$\lim_{R\to+\infty} \delta_R = 0. \tag{5.44}$$

Consider $\tilde{u}$ to be the translation of u, that is $\tilde{u}(x) := u(x-e)$. Let also

$$m(x) := \min\big\{u(x), \tilde{u}(x)\big\}.$$

We notice that in $B_{R/2}$ we have that $\tilde{u}(x) = u_{R,+}(x)$. This and (5.38) give that

$$m = v_R \text{ in } B_{R/2}. \tag{5.45}$$

Also, from (5.41) and (5.45), it follows that m cannot be identically neither u nor $\tilde{u}$, and

$$\begin{aligned} m &< u \text{ in a neighborhood of } 0 \quad \text{and} \\ m &= u \text{ in a neighborhood of } e. \end{aligned} \tag{5.46}$$

Let z be a competitor for m in the ball B_2, that agrees with m outside B_2. We take a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi = 1$ in $B_{R/4}$, $\psi = 0$ in $B_{R/2}^{\mathcal{C}}$. Let

$$z_R(x) := \psi(x) z(x) + \big(1 - \psi(x)\big) v_R(x).$$

Then we have that $z_R = z$ on $B_{R/4}$ and

$$z_R = v_R \text{ on } B_{R/2}^{\mathcal{C}}. \tag{5.47}$$

In addition, by (5.45), we have that $z = m = v_R$ in $B_{R/2}\backslash B_2$. So, it follows that

$$z_R(x) = \psi(x) v_R(x) + (1 - \psi(x)) v_R(x) = v_R(x) = z(x) \quad \text{on} \quad B_{R/2}\backslash B_2.$$

This and (5.47) imply that $z_R = v_R$ on $B_2^{\mathcal{C}}$.

We summarize in the next lines these useful identities (see also Figure 10).

$$\begin{aligned}
&\text{in } B_2 && u_{R,+} = \tilde{u}, \quad m = v_R, \quad z = z_R \\
&\text{in } B_{R/2}\backslash B_2 && u_{R,+} = \tilde{u}, \quad v_R^* = v_R = m = z = z_R \\
&\text{in } B_R\backslash B_{R/2} && v_R^* = v_R = z_R, \quad m = z \\
&\text{in } B_R^{\mathcal{C}} && u_{R,+} = u = v_R = v_R^* = z_R, \quad m = z.
\end{aligned}$$

We compute now

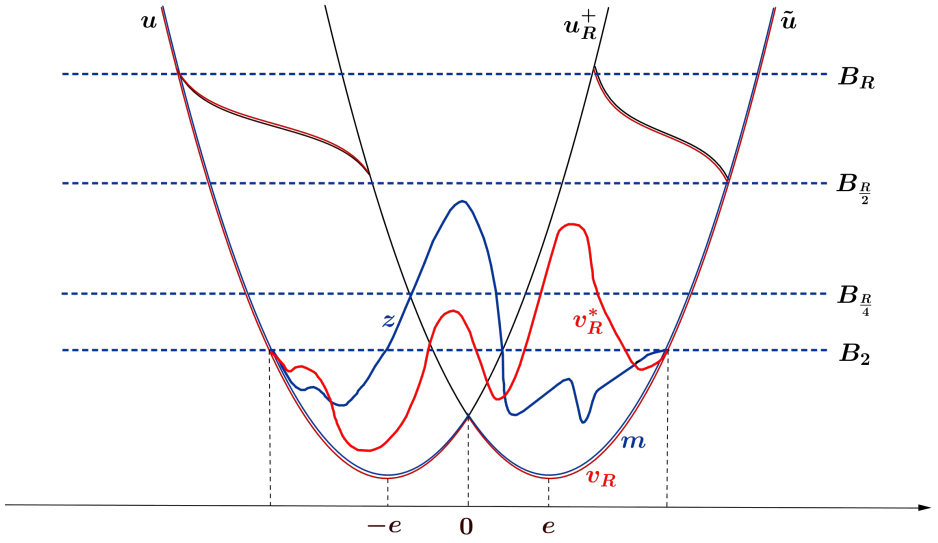

FIGURE 10. Energy estimates

$$\begin{aligned}
&\mathcal{E}(m\ , B_2) - \mathcal{E}(z, B_2) \\
&\quad = \mathcal{E}(m, B_2) - \mathcal{E}(v_R, B_2) + \mathcal{E}(v_R, B_2) - \mathcal{E}(z_R, B_2) + \mathcal{E}(z_R, B_2) - \mathcal{E}(z, B_2).
\end{aligned}$$

By the definition of δ_R in (5.42), we have that

$$\begin{aligned}
&\mathcal{E}(m\ , B_2) - \mathcal{E}(z, B_2) \\
&\quad = \mathcal{E}(m, B_2) - \mathcal{E}(v_R, B_2) + \delta_R + \mathcal{E}(v_R^*, B_2) - \mathcal{E}(z_R, B_2) + \mathcal{E}(z_R, B_2) - \mathcal{E}(z, B_2).
\end{aligned} \tag{5.48}$$

Using the formula for the kinetic energy given in (5.23) together with (5.24) we have that

$$\begin{aligned}
&\mathcal{E}(m\ , B_2) - \mathcal{E}(v_R, B_2) \\
&\quad = \frac{1}{2} m(B_2, B_2) + m(B_2, B_2^{\mathcal{C}}) + \int_{B_2} W\big(m(x)\big)\, dx \\
&\qquad - \frac{1}{2} v_R(B_2, B_2) - v_R(B_2, B_2^{\mathcal{C}}) - \int_{B_2} W\big(v_R(x)\big)\, dx.
\end{aligned}$$

Since $m = v_R$ on $B_{R/2}$ (recall (5.45)), we obtain

$$\begin{aligned}&\mathcal{E}(m\,, B_2) - \mathcal{E}(v_R, B_2)\\ &\quad= \int_{B_2} dx \int_{B^{\mathcal{C}}_{R/2}} dy \frac{|m(x)-m(y)|^2 - |m(x)-v_R(y)|^2}{|x-y|^{n+2s}}.\end{aligned}$$

Notice now that m and v_R are bounded on $\mathbb{R}^n$ (since so is u). Also, if $x \in B_2$ and $y \in B^{\mathcal{C}}_{R/2}$ we have that $|x-y| \geqslant |y| - |x| \geqslant |y|/2$ if R is large. Accordingly,

$$\mathcal{E}(m, B_2) - \mathcal{E}(v_R, B_2) \leqslant C \int_{B_2} dx \int_{B^{\mathcal{C}}_{R/2}} \frac{1}{|y|^{n+2s}}\, dy \leqslant CR^{-2s}, \tag{5.49}$$

up to renaming constants. Similarly, $z_R = z$ on $B_{R/2}$ and we have the same bound

$$\mathcal{E}(z_R, B_2) - \mathcal{E}(z, B_2) \leqslant CR^{-2s}. \tag{5.50}$$

Furthermore, since v_R^* is a minimizer for $\mathcal{E}(\cdot, B_2)$ and $v_R^* = z_R$ outside of B_2, we have that

$$\mathcal{E}(v_R^*, B_2) - \mathcal{E}(z_R, B_2) \leqslant 0.$$

Using this, (5.49) and (5.50) in (5.48), it follows that

$$\mathcal{E}(m, B_2) - \mathcal{E}(z, B_2) \leqslant CR^{-2s} + \delta_R.$$

Therefore, by sending $R \to +\infty$ and using again (5.44), we obtain that

$$\mathcal{E}(m, B_2) \leqslant \mathcal{E}(z, B_2). \tag{5.51}$$

We recall that z can be any competitor for m, that coincides with m outside of B_2. Hence, formula (5.51) means that m is a minimizer for $\mathcal{E}(\cdot, B_2)$. On the other hand, u is a minimizer of the energy in any ball. Then, both u and m satisfy the same equation in B_2. Moreover, they coincide in a neighborhood of e, as stated in the second line of (5.46). By the Strong Maximum Principle, they have to coincide on B_2, but this contradicts the first statement of (5.46). The proof of (5.43) is thus complete.

Now, since $v_R^* = v_R$ on $B_2^{\mathcal{C}}$, from definition (5.42) we have that

$$\delta_R = \mathcal{E}(v_R, B_R) - \mathcal{E}(v_R^*, B_R).$$

Also, $\mathcal{E}(v_R^*, B_R) \geqslant \mathcal{E}(u, B_R)$, thanks to the minimizing property of u. Using these pieces of information and inequality (5.40), it follows that

$$\delta_R \leqslant \mathcal{E}(u_{R,+}, B_R) - \mathcal{E}(u, B_R).$$

Now, by sending $R \to +\infty$ and using (5.43), we have that

$$\lim_{R\to+\infty} \mathcal{E}(u_{R,+}, B_R) - \mathcal{E}(u, B_R) > 0,$$

which contradicts (5.37). This implies that indeed u is monotone, and this concludes the proof of this Step.

Step 4. Conclusions
In Step 3, we have proved that u is monotone, in any given direction e. Then, Step 1 gives the desired result. This concludes the proof of Theorem 5.4. □

We remark that the exponent two in the energy estimate (5.26) is related to the expansions of order two and not to the dimension of the space. Indeed, the energy estimates hold for any n. However, the two power in the estimate (5.26) allows us to prove the fractional version of De Giorgi conjecture only in dimension two. In other words, the proof of Theorem 5.4 is not applicable for $n > 2$. One can verify this by checking the limit in (5.37)

$$\lim_{R\to+\infty}\Big(\mathcal{E}(u_{R,+},B_R)-\mathcal{E}(u,B_R)\Big)=0,$$

which was necessary for the Proof of Theorem 5.4 in the case $n = 2$. We know from Theorem 5.2 that

$$\lim_{R\to+\infty}\frac{C}{R^n}\mathcal{E}(u,B_R)=0.$$

Confronting this result with inequality (5.36)

$$\mathcal{E}(u_{R,+},B_R)-\mathcal{E}(u,B_R)\leqslant\frac{C}{R^2}\mathcal{E}(u,B_R),$$

we see that we need to have $n = 2$ in order for the the limit in (5.37) to be zero.

6. Nonlocal minimal surfaces

In this section we introduce nonlocal minimal surfaces and focus on two main results, a Bernstein type result in any dimension and the non-existence of nontrivial s-minimal cones in dimension 2. Moreover, some boundary regularity results will be discussed at the end of this chapter.

Let $\Omega\subset\mathbb{R}^n$ be an open bounded domain, and $E\subset\mathbb{R}^n$ be a measurable set, fixed outside Ω. We will consider for $s\in(0,1/2)$ minimizers of the H^s norm

$$\begin{aligned}||\chi_E||^2_{H^s}=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ =&2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\chi_E(x)\chi_{E^c}(y)}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

Notice that only interactions between E and $E^{\mathcal{C}}$ contribute to the norm.

In order to define the fractional perimeter of E in Ω, we need to clarify the contribution of Ω to the H^s norm here introduced. Namely, as E is fixed outside Ω, we aim at minimizing the "Ω-contribution" to the norm among all measurable sets that "vary" inside Ω. We consider thus interactions between $E\cap\Omega$ and $E^{\mathcal{C}}$ and between $E\backslash\Omega$ and $\Omega\backslash E$, neglecting the data that is fixed outside Ω and that does not contribute to the minimization of the norm (see Figure 11).

We define the interaction $I(A,B)$ of two disjoint subsets of $\mathbb{R}^n$ as

$$\begin{aligned}I(A,B):=&\int_A\int_B\frac{dx\,dy}{|x-y|^{n+2s}}\\ =&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\chi_A(x)\chi_B(x)}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}\tag{6.1}$$

Then, one defines the nonlocal s-perimeter functional of E in Ω as

$$\mathrm{Per}_s(E,\Omega):=I(E\cap\Omega,E^{\mathcal{C}})+I(E\backslash\Omega,\Omega\backslash E).\tag{6.2}$$

Equivalently, one may write

$$\mathrm{Per}_s(E,\Omega)=I(E\cap\Omega,\Omega\backslash E)+I(E\backslash\Omega,\Omega^{\mathcal{C}}\backslash E)+I(E\backslash\Omega,\Omega\backslash E).$$

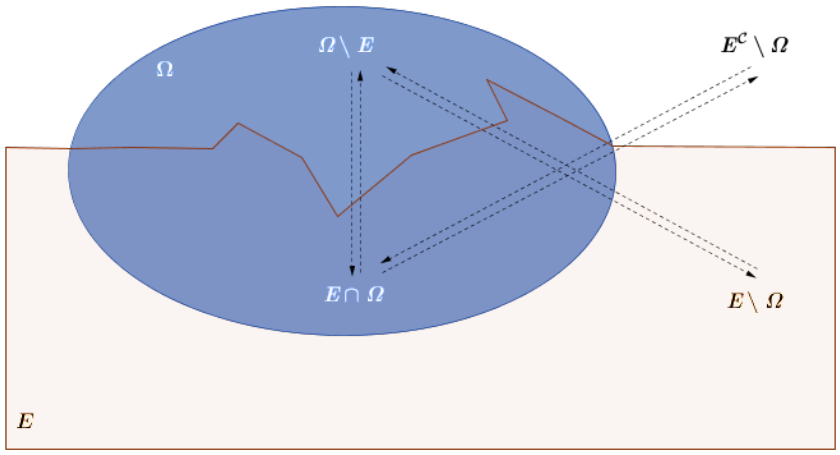

FIGURE 11. Fractional Perimeter

Definition 6.1. *Let Ω be an open domain of $\mathbb{R}^n$. A measurable set $E \subset \mathbb{R}^n$ is s-minimal in Ω if $Per_s(E,\Omega)$ is finite and if, for any measurable set F such that $E\backslash\Omega = F\backslash\Omega$, we have that*

$$Per_s(E,\Omega) \leqslant Per_s(F,\Omega).$$

A measurable set is s-minimal in $\mathbb{R}^n$ if it is s-minimal in any ball B_r, where $r > 0$.

When $s \to \frac{1}{2}$, the fractional perimeter Per_s approaches the classical perimeter, as proved in [23]. A simple, formal statement, up to renormalizing constants, is the following:

Theorem 6.2. *Let $\alpha \in (0,1)$, $R > 0$ and E be a set with $C^{1,\alpha}$-boundary in B_R. Then*

$$\lim_{s\to\frac{1}{2}} \Big(\frac{1}{2} - s\Big) Per_s(E, B_r) = Per\,(E, B_r)$$

for almost any $r \in (0,R)$.

The boundaries of s-minimal sets are referred to as *nonlocal minimal surfaces*.

In [17] it is proved that s-minimizers satisfy a suitable integral equation (see in particular Theorem 5.1 in [17]), that is the Euler-Lagrange equation corresponding to the s-perimeter functional Per_s. If E is s-minimal in Ω and ∂E is smooth enough, this Euler-Lagrange equation can be written as

$$\int_{\mathbb{R}^n} \frac{\chi_E(x_0+y) - \chi_{\mathbb{R}^n\setminus E}(x_0+y)}{|y|^{n+2s}}\, dy = 0, \tag{6.3}$$

for any $x_0 \in \Omega \cap \partial E$.

Therefore, in analogy with the case of the classical minimal surfaces, which have zero mean curvature, one defines the *nonlocal mean curvature* of E at $x_0 \in \partial E$ as

$$H_E^s(x_0) := \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y-x_0|^{n+2s}}\, dy. \tag{6.4}$$

In this way, equation (6.3) can be written as H_E^s along ∂E.

It is also suggestive to think that the function $\tilde{\chi}_E := \chi_E - \chi_{E^c}$ averages out to zero at the points of ∂E, if ∂E is smooth enough, since at these points the local

contribution of E compensates the one of $E^{\mathcal{C}}$. Using this notation, one may take the liberty of writing

$$\begin{aligned} H_E^s(x_0) &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x_0+y)+\tilde{\chi}_E(x_0-y)}{|y|^{n+2s}}\,dy \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x_0+y)+\tilde{\chi}_E(x_0-y)-2\tilde{\chi}_E(x_0)}{|y|^{n+2s}}\,dy \\ &= \frac{-(-\Delta)^s\tilde{\chi}_E(x_0)}{C(n,s)}, \end{aligned}$$

using the notation of (2.1). Using this suggestive representation, the Euler-Lagrange equation in (6.3) becomes

$$(-\Delta)^s\tilde{\chi}_E = 0 \text{ along } \partial E.$$

We refer to [2] for further details on this argument.

It is also worth recalling that the nonlocal perimeter functionals find applications in motions of fronts by nonlocal mean curvature (see e.g. [21] and [27]), problems in which aggregating and disaggregating terms compete towards an equilibrium (see e.g. [53] and [39]) and nonlocal free boundary problems (see e.g. [18] and [44]).

In the classical case of the local perimeter functional, it is known that minimal surfaces are smooth in dimension $n \leqslant 7$. Moreover, if $n \geqslant 8$ minimal surfaces are smooth except on a small singular set of Hausdorff dimension $n-8$. Furthermore, minimal surfaces that are graphs are called minimal graphs, and they reduce to hyperplanes if $n \leqslant 8$ (this is called the Bernstein property, which was also discussed at the beginning of this section). If $n \geqslant 9$, there exist global minimal graphs that are not affine (see e.g. [59]).

Differently from the classical case, the regularity theory for s-minimizers is still quite open. We present here some of the partial results obtained in this direction:

Theorem 6.3. *In the plane, s-minimal sets are smooth. More precisely:*
a) If E is an s-minimal set in $\Omega \subset \mathbb{R}^2$, then $\partial E \cap \Omega$ is a C^∞-curve.
b) Let E be s-minimal in $\Omega \subset \mathbb{R}^n$ and let $\Sigma_E \subset \partial E \cap \Omega$ denote its singular set. Then $\mathcal{H}^d(\Sigma_E) = 0$ for any $d > n-3$.

See [82] for the proof of this results (as a matter of fact, in [82] only $C^{1,\alpha}$ regularity is proved, but then [5] proved that s-minimal sets with $C^{1,\alpha}$-boundary are automatically C^∞). Further regularity results of the s minimal surfaces can be found in [24]. There, a regularity theory when s is near $\frac{1}{2}$ is stated, as we see in the following Theorem:

Theorem 6.4. *There exists $\epsilon_0 \in \left(0, \frac{1}{2}\right)$ such that if $s \geqslant \frac{1}{2} - \epsilon_0$, then*
a) if $n \leqslant 7$, any s-minimal set is of class C^∞,
b) if $n = 8$ any s-minimal surface is of class C^∞ except, at most, at countably many isolated points,
c) any s-minimal surface is of class C^∞ outside a closed set Σ of Hausdorff dimension $n-8$.

We will focus the following material on the Bernstein type results, in particular on the following theorem:

Theorem 6.5. *Let $E = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } t < u(x)\}$ be an s-minimal graph, and assume there are no singular cones in dimensions n (that is, if $\mathcal{K} \subset \mathbb{R}^n$ is an s-minimal cone, then $\mathcal{K}$ is a half-space). Then u is an affine function (thus E is a half-space).*

To be able to prove Theorem 6.5, we recall some useful auxiliary results. In the following lemma we state a dimensional reduction result (see Theorem 10.1 in [17]).

Lemma 6.6. *Let $E = F \times \mathbb{R}$. Then if E is s-minimal if and only if F is s-minimal.*

We define then the blow-up and blow-down of the set E are, respectively

$$E_0 := \lim_{r \to 0} E_r \quad \text{and} \quad E_\infty := \lim_{r \to +\infty} E_r, \quad \text{where} \quad E_r = \frac{E}{r}.$$

A first property of the blow-up of E is the following (see Lemma 3.1 in [54]).

Lemma 6.7. *If E_∞ is affine, then so is E.*

We recall also a regularity result for the s-minimal surfaces (see [54] and [5] for details and proof).

Lemma 6.8. *Let E be s-minimal. Then:*
a) If E is Lipschitz, then $E \in C^{1,\alpha}$.
b) If $E \in C^{1,\alpha}$, then $E \in C^\infty$.

We give here a sketch of the proof of Theorem 6.5 (see [54] for all the details).

Sketch of proof of Theorem 6.5. If $E \subset \mathbb{R}^{n+1}$ is an s-minimal graph, then the blow-down E_∞ is an s-minimal cone (see Theorem 9.2 in [17] for the proof of this statement). By applying the dimensional reduction argument in Lemma 6.6 we obtain an s-minimal cone in dimension n. According to the assumption that no singular s-minimal cones exist in dimension n, it follows that necessarily E_∞ can be singular only at the origin.

We consider a bump function $w_0 \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\begin{aligned} &w_0(t) = 0 \text{ in } \left(-\infty, \frac{1}{4}\right) \cup \left(\frac{3}{4}, +\infty\right) \\ &w_0(t) = 1 \text{ in } \left(\frac{2}{5}, \frac{3}{5}\right) \\ &w(t) = w_0(|t|). \end{aligned}$$

The blow-down of E is

$$E_\infty = \big\{(x', x_{n+1}) \text{ s.t. } x_{n+1} \leqslant u_\infty(x')\big\}.$$

For a fixed $\sigma \in \partial B_1$, let

$$F_t := \big\{(x', x_{n+1}) \text{ s.t. } x_{n+1} \leqslant u_\infty\big(x' + t\theta w(x')\sigma\big) - t\big\}$$

be a family of sets, where $t \in (0,1)$ and $\theta > 0$. Then for θ small, we have that

$$F_1 \text{ is below } E_\infty. \tag{6.5}$$

Indeed, suppose by contradiction that this is not true. Then, there exists $\theta_k \to 0$ such that

$$u_\infty\big(x'_k + \theta_k w(x'_k)\sigma\big) - 1 \geqslant u_\infty(x'_k). \tag{6.6}$$

But $x'_k \in \mathrm{supp} w$, which is compact, therefore $x'_\infty := \lim_{k\to+\infty} x'_k$ belongs to the support of w, and $w(x'_\infty)$ is defined. Then, by sending $k \to +\infty$ in (6.6) we have that

$$u_\infty(x'_\infty) - 1 \geqslant u_\infty(x'_\infty),$$

which is a contradiction. This establishes (6.5).

Now consider the smallest $t_0 \in (0,1)$ for which F_t is below E_∞. Since E_∞ is a graph, then F_{t_0} touches E_∞ from below in one point $X_0 = (x'_0, x^0_{n+1})$, where $x'_0 \in \mathrm{supp} w$. Now, since E_∞ is s-minimal, we have that the nonlocal mean curvature (defined in (6.4)) of the boundary is null. Also, since F_{t_0} is a C^2 diffeomorphism of E_∞ we have that

$$H^s_{F_{t_0}}(p) \simeq \theta t_0, \tag{6.7}$$

and there is a region where E_∞ and F_{t_0} are well separated by t_0, thus

$$\big|\big(E_\infty \backslash F_{t_0}\big) \cap \big(B_3 \backslash B_2\big)\big| \geqslant c t_0,$$

for some $c > 0$. Therefore, we see that

$$H^s_{F_{t_0}}(p) = H^s_{F_{t_0}}(p) - H^s_E(p) \geqslant c t_0.$$

This and (6.7) give that $\theta t_0 \geqslant c t_0$, for some $c > 0$ (up to renaming it). If θ is small enough, this implies that $t_0 = 0$.

In particular, we have proved that there exists $\theta > 0$ small enough such that, for any $t \in (0,1)$ and any $\sigma \in \partial B_1$, we have that

$$u_\infty\big(x' + t\theta w(x')\sigma\big) - t \leqslant u_\infty(x').$$

This implies that

$$\frac{u_\infty\big(x' + t\theta w(x')\sigma\big) - u_\infty(x')}{t\theta} \leqslant \frac{1}{\theta},$$

hence, letting $t \to 0$, we have that

$$\nabla u_\infty(x') w(x') \sigma \leqslant \frac{1}{\theta}, \text{ for any } x \in \mathbb{R}^n \backslash \{0\}, \text{ and } \sigma \in B_1.$$

We recall now that $w = 1$ in $B_{3/5} \backslash B_{2/5}$ and σ is arbitrary in ∂B_1. Hence, it follows that

$$|\nabla u_\infty(x)| \leqslant \frac{1}{\theta}, \text{ for any } x \in B_{3/5} \backslash B_{2/5}.$$

Therefore u_∞ is globally Lipschitz. By the regularity statement in Lemma 6.8, we have that u_∞ is C^∞. This says that u is smooth also at the origin, hence (being a cone) it follows that E_∞ is necessarily a half-space. Then by Lemma 6.7, we conclude that E is a half-space as well. □

We now prove the non-existence of singular s-minimal cones in dimension 2, as stated in the next result (from this, the more general statement in Theorem 6.3 follows after a blow-up procedure):

Theorem 6.9. *If E is an s-minimal cone in $\mathbb{R}^2$, then E is a half-plane.*

We remark that, as a combination of Theorems 6.5 and 6.9, we obtain the following result of Bernstein type:

Corolary 6.10. *Let $E = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } t < u(x)\}$ be an s-minimal graph, and assume that $n \in \{1,2\}$. Then u is an affine function.*

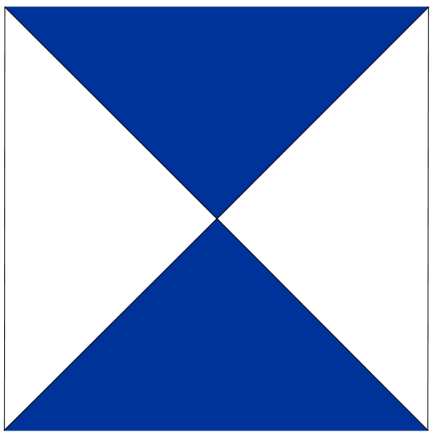

FIGURE 12. The cone $\mathcal{K}$

Let us first consider a simple example, given by the cone in the plane

$$\mathcal{K} := \Big\{(x,y) \in \mathbb{R}^2 \text{ s.t. } y^2 > x^2\Big\},$$

see Figure 12.

Proposition 6.11. *The cone $\mathcal{K}$ depicted in Figure 12 is not s-minimal in $\mathbb{R}^2$.*

Notice that, by symmetry, one can prove that $\mathcal{K}$ satisfies (6.3) (possibly in the viscosity sense). On the other hand, Proposition 6.11 gives that $\mathcal{K}$ is not s-minimal. This, in particular, provides an example of a set that satisfies the Euler-Lagrange equation in (6.3), but is not s-minimal (i.e., the Euler-Lagrange equation in (6.3) is implied by, but not necessarily equivalent to, the s-minimality property).

Proof of Proposition 6.11. The proof of the non-minimality of $\mathcal{K}$ is due to an original idea by Luis Caffarelli.

Suppose by contradiction that the cone $\mathcal{K}$ is minimal in $\mathbb{R}^2$. We add to $\mathcal{K}$ a small square adjacent to the origin (see Figure 13), and call $\mathcal{K}'$ the set obtained. Then $\mathcal{K}$ and $\mathcal{K}'$ have the same s-perimeter. This is due to the interactions considered in the s-perimeter functional and the unboundedness of the regions. We remark that in Figure 13 we represent bounded regions, of course, sets A, B, C, D, A', B', C' and D' are actually unbounded.

Indeed, we notice that in the first image, the white square M interacts with the dark regions A, B, C, D, while in the second the now dark square M interacts with the regions A', B', C', D', and all the other interactions are unmodified. Therefore, the difference between the s-perimeter of $\mathcal{K}$ and that of $\mathcal{K}'$ consists only of the interactions $I(A,M) + I(B,M) + I(C,M) + I(D,M) - I(A',M) - I(B',M) - I(C',M) - I(D',M)$. But $A \cup B = A' \cup B'$ and $C \cup D = C' \cup D'$ (since these sets are all unbounded), therefore the difference is null, and the s-perimeter of $\mathcal{K}$ is equal to that of $\mathcal{K}'$. Consequently, $\mathcal{K}'$ is also s-minimal, and therefore it satisfies the Euler-Lagrange equation in (6.3) at the origin. But this leads to a contradiction, since the the dark region now contributes more than the white one, namely

$$\int_{\mathbb{R}^2} \frac{\chi_{\mathcal{K}'}(y) - \chi_{\mathbb{R}^2\setminus\mathcal{K}'}(y)}{|y|^{2+s}}\,dy > 0.$$

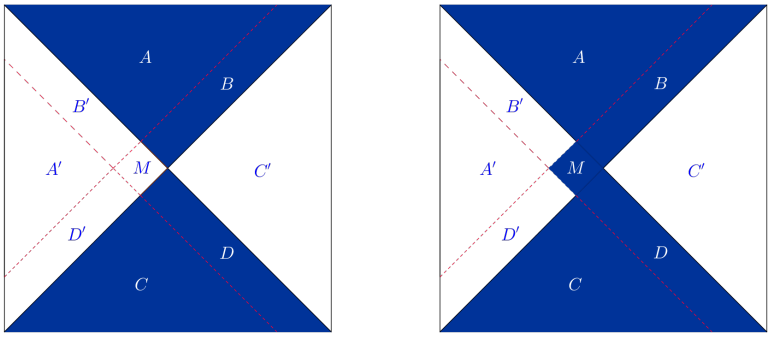

FIGURE 13. Interaction of M with $A, B, C, D, A', B', C', D'$

Thus $\mathcal{K}$ cannot be s-minimal, and this concludes our proof. □

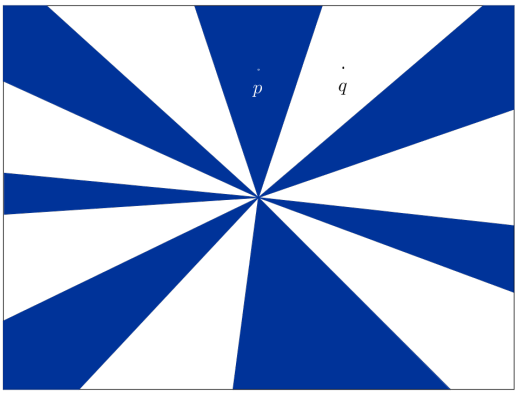

FIGURE 14. Cone in $\mathbb{R}^2$

This geometric argument cannot be extended to a more general case (even, for instance, to a cone in $\mathbb{R}^2$ made of many sectors, see Figure 14). As a matter of fact, the proof of Theorem 6.9 will be completely different than the one of Proposition 6.11 and it will rely on an appropriate domain perturbation argument.

The proof of Theorem 6.9 that we present here is actually different than the original one in [82]. Indeed, in [82], the result was proved by using the harmonic extension for the fractional Laplacian. Here, the extension will not be used; furthermore, the proof follows the steps of Theorem 5.4 and we will recall here just the main ingredients.

Proof of Theorem 6.9. The idea of the proof is the following: if $E \subset \mathbb{R}^2$ is an s-minimal cone, then let $\tilde{E}$ be a translation of E in $B_{R/2}$ which coincides with E outside B_R. Then the difference between the energies of $\tilde{E}$ and E tends to 0 as $R \to +\infty$. This implies that also the energy of $E \cap \tilde{E}$ is arbitrarily close to the energy of E. On the other hand if E is not a half-plane, the set $\tilde{E} \cap E$ can be modified locally to decrease its energy by a fixed small amount and we reach a contradiction.

The details of the proof go as follows. Let

$$u := \chi_E - \chi_{\mathbb{R}^2\setminus E}.$$

From definition (5.24) we have that

$$u(B_R, B_R) = 2I(E \cap B_R, B_R\setminus E)$$

and

$$u(B_R, B_R^{\mathcal{C}}) = I(B_R \cap E, E^{\mathcal{C}}\setminus B_R) + I(B_R\setminus E, E\setminus B_R),$$

thus

$$\mathrm{Per}_s(E, B_R) = \mathcal{K}_R(u), \tag{6.8}$$

where $\mathcal{K}_R(u)$ is given in (5.22) and $\mathrm{Per}_s(E, B_R)$ is the s-perimeter functional defined in (6.2). Then E is s-minimal if u is a minimizer of the energy $\mathcal{K}_R$ in any ball B_R, with $R > 0$. We recall the estimate obtained in (5.26), that, combined with the minimality of u, gives

$$\mathcal{K}_R(u_{R,+}) - \mathcal{K}_R(u) \leqslant \frac{C}{R^2}\mathcal{K}_R(u).$$

But u is a minimizer in any ball, and by the energy estimate in Theorem 5.2 we have that

$$\mathcal{K}_R(u_{R,+}) - \mathcal{K}_R(u) \leqslant CR^{-2s}.$$

This implies that

$$\lim_{R\to+\infty} \mathcal{K}_R(u_{R,+}) - \mathcal{K}_R(u) = 0. \tag{6.9}$$

Now, we argue by contradiction, and suppose that E is an s-minimal cone different from the half-space. Up to rotations, we may suppose that a sector of E has an angle smaller than π and is bisected by e_2. Thus there exists $M \geqslant 1$ and $p \in E \cap B_M$ on the e_2-axis such that $p \pm e_1 \in \mathbb{R}^2\setminus E$ (see Figure 14).

Now we take $\varphi \in C_0^\infty(B_1)$, such that $\varphi(x) = 1$ in $B_{1/2}$. For R large (say $R > 8M$), we define

$$\Psi_{R,+}(y) := y + \varphi\Big(\frac{y}{R}\Big)\, e_1.$$

We point out that, for R large, $\Psi_{R,+}$ is a diffeomorphism on $\mathbb{R}^2$.

Furthermore, we define $u_R^+(x) := u(\Psi_{R,+}^{-1}(x))$. Then

$$\begin{aligned} & u_R^+(y) = u(y - e_1) && \text{for } p \in B_{2M} \\ \text{and} \quad & u_R^+(y) = u(y) && \text{for } p \in \mathbb{R}^2\setminus B_R. \end{aligned}$$

Let now

$$v_R(x) := \min\{u(x), u_R^+(x)\} \qquad \text{and} \qquad w_R(x) := \max\{u(x), u_R^+(x)\}.$$

We claim that v_R is not identically u nor u_R^+. Indeed

$$\begin{aligned} & u_R^+(p) = u(p - e_1) = (\chi_E - \chi_{\mathbb{R}^2\setminus E})(p - e_1) = -1 \quad \text{and} \\ & u(p) = (\chi_E - \chi_{\mathbb{R}^2\setminus E})(p) = 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} & u_R^+(p + e_1) = u(p) = 1 \quad \text{and} \\ & u(p + e_1) = (\chi_E - \chi_{\mathbb{R}^2\setminus E})(p + e_1) = -1. \end{aligned}$$

By the continuity of u and u_R^+, we obtain that

$$v_R = u_R^+ < u \text{ in a neighborhood of } p \tag{6.10}$$

and

$$v_R = u < u_R^+ \text{ in a neighborhood of } p + e_1. \tag{6.11}$$

Now, by the minimality property of u,

$$\mathcal{K}_R(u) \leqslant \mathcal{K}_R(v_R).$$

Moreover (see e.g. formula (38) in [75]),

$$\mathcal{K}_R(v_R) + \mathcal{K}_R(w_R) \leqslant \mathcal{K}_R(u) + \mathcal{K}_R(u_R^+).$$

The latter two formulas give that

$$\mathcal{K}_R(v_R) \leqslant \mathcal{K}_R(u_R^+). \tag{6.12}$$

We claim that

$$v_R \text{ is not minimal for } \mathcal{K}_{2M} \tag{6.13}$$

with respect to compact perturbations in B_{2M}. Indeed, assume by contradiction that v_R is minimal, then in B_{2M} both v_R and u would satisfy the same equation. Recalling (6.11) and applying the Strong Maximum Principle, it follows that $u = v_R$ in B_{2M}, which contradicts (6.10). This establishes (6.13).

Now, we consider a minimizer u_R^* of $\mathcal{K}_{2M}$ among the competitors that agree with v_R outside B_{2M}. Therefore, we can define

$$\delta_R := \mathcal{K}_{2M}(v_R) - \mathcal{K}_{2M}(u_R^*).$$

In light of (6.13), we have that $\delta_R > 0$.

The reader can now compare Step 3 in the proof of Theorem 5.4. There we proved that

$$\delta_R \text{ remains bounded away from zero as } R \to +\infty. \tag{6.14}$$

Furthermore, since u_R^* and v_R agree outside B_{2M} we obtain that

$$\mathcal{K}_R(u_R^*) + \delta_R = \mathcal{K}_R(v_R).$$

Using this, (6.12) and the minimality of u, we obtain that

$$\delta_R = \mathcal{K}_R(v_R) - \mathcal{K}_R(u_R^*) \leqslant \mathcal{K}_R(u_{R,+}) - \mathcal{K}_R(u).$$

Now we send R to infinity, recall (6.9) and (6.14), and we reach a contradiction. Thus, E is a half-space, and this concludes the proof of Theorem 6.9. □

As already mentioned, the regularity theory for s-minimal sets is still widely open. Little is known beyond Theorems 6.3 and 6.4, so it would be very interesting to further investigate the regularity of s-minimal surfaces in higher dimension and for small s.

Regarding this problem, let us mention the recent papers [31] and [32]. Among other very interesting results, it is proved there that suitable singular cones of symmetric type are unstable up to dimension 6 but become stable in dimension 7 for small s (these cones can be seen as the nonlocal analogue of the Lawson cones in the classical minimal surface theory, and the stability property is in principle weaker than minimality, since it deals with the positivity of the second order derivative of the functional).

This phenomenon may suggest the conjecture that the s-minimal surfaces are always smooth up to dimension 6 but may develop singularity in dimension 7 and higher.

In [32], interesting examples of surfaces with vanishing nonlocal mean curvature are provided. Remarkably, the surfaces in [32] are the nonlocal analogue of the

catenoids, but, differently from the classical case (in which catenoids grow logarithmically), they approach a singular cone at infinity.

See also [14] for the construction of surfaces with constant mean curvature with the structure of onduloids, and [14] and [29] for the proof that bounded surfaces with constant mean curvature are necessarily spheres (this is the analogue of a celebrated result by Alexandrov for surfaces of constant classical mean curvature).

The boundary regularity of the nonlocal minimal surfaces is also a very interesting, and surprising, topic. Indeed, differently from the classical case, nonlocal minimal surfaces do not always attain boundary data in a continuous way (not even in low dimension). A possible boundary behavior is, on the contrary, a combination of stickiness to the boundary and smooth separation from the adjacent portions. Namely, the nonlocal minimal surfaces may have a portion that sticks at the boundary and that separates from it in a $C^{\frac{1}{2}+s}$-way.

As an example, we can consider, for any $\delta > 0$, the spherical cap

$$K_\delta := \big(B_{1+\delta} \setminus B_1\big) \cap \{x_n < 0\},$$

and obtain the following stickiness result:

Theorem 6.12. *There exists $\delta_0 > 0$, depending on n and s, such that for any $\delta \in (0, \delta_0]$, we have that the s-minimal set in B_1 that coincides with K_δ outside B_1 is K_δ itself.*

That is, the s-minimal set with datum K_δ outside B_1 is empty inside B_1.

Other stickiness examples occur at the sides of slabs in the plane. For instance, given $M > 1$, one can consider the s-minimal set E_M in $(-1,1) \times \mathbb{R}$ with datum outside $(-1,1) \times \mathbb{R}$ given by the "jump" set $J_M := J_M^- \cup J_M^+$, where

$$\begin{aligned} & J_M^- := (-\infty, -1] \times (-\infty, -M) \\ \text{and} \qquad & J_M^+ := [1, +\infty) \times (-\infty, M). \end{aligned}$$

Then, if M is large enough, the minimal set E_M sticks at the boundary of the slab:

Theorem 6.13. *There exist $M_o > 0$, $c_o \in (0,1)$, depending on s, such that if $M \geqslant M_o$ then*

$$[-1,1) \times [c_o M, M] \subseteq E_M^c \tag{6.15}$$

$$\textit{and} \qquad (-1,1] \times [-M, -c_o M] \subseteq E_M. \tag{6.16}$$

For the proof of Theorems 6.12 and 6.13, and other results on the boundary behavior of nonlocal minimal surfaces, see [45].

To conclude this section, we make a remark on the connection between solutions of the fractional Allen-Cahn equation and s-minimal surfaces. Namely, a suitably scaled version of the functional in (5.8) Γ-converges to either the classical perimeter or the nonlocal perimeter functional, depending on the fractional parameter s. The Γ-convergence is a type of convergence of functionals that is compatible with the minimization of the energy, and turns out to be very useful when dealing with variational problems indexed by a parameter. This notion was introduced by De Giorgi, see e.g. [33] for details.

In the nonlocal case, some care is needed to introduce the "right" scaling of the functional, which comes from the dilation invariance of the space coordinates and

possesses a nontrivial energy in the limit. For this, one takes first the rescaled energy functional

$$J_\varepsilon(u,\Omega) := \varepsilon^{2s}\mathcal{K}(u,\Omega) + \int_\Omega W(u)\,dx.$$

Then, one considers the functional

$$F_\varepsilon(u,\Omega) := \begin{cases} \varepsilon^{-2s} J_\varepsilon(u,\Omega) & \text{if } s \in (0,\,1/2), \\ |\varepsilon \log \varepsilon|^{-1} J_\varepsilon(u,\Omega) & \text{if } s = 1/2, \\ \varepsilon^{-1} J_\varepsilon(u,\Omega) & \text{if } s \in (1/2,\,1). \end{cases}$$

The limit functional of F_ε as $\varepsilon \to 0$ depends on s. Namely, when $s \in (0,1/2)$, the limit functional is (up to dimensional constants that we neglect) the fractional perimeter, i.e.

$$F(u,\Omega) := \begin{cases} \mathrm{Per}_s(E,\Omega) & \text{if } u|_\Omega = \chi_E - \chi_{E^c}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{6.17}$$

On the other hand, when $s \in [1/2,1)$, the limit functional of F_ε is (again, up to normalizing constants) the classical perimeter, namely

$$F(u,\Omega) := \begin{cases} \mathrm{Per}(E,\Omega) & \text{if } u|_\Omega = \chi_E - \chi_{E^c}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{6.18}$$

That is, the following limit statement holds true:

Theorem 6.14. *Let $s \in (0,1)$. Then, F_ε Γ-converges to F, as defined in either (6.17) or (6.18), depending on whether $s \in (0,1/2)$ or $s \in [1/2,1)$.*

For precise statements and further details, see [81]. Additionally, we remark that the level sets of the minimizers of the functional in (5.8), after a homogeneous scaling in the space variables, converge locally uniformly to minimizers either of the fractional perimeter (if $s \in (0,1/2)$) or of the classical perimeter (if $s \in [1/2,1)$): that is, the "functional" convergence stated in Theorem 6.14 has also a "geometric" counterpart: for this, see Corollary 1.7 in [83].

One can also interpret Theorem 6.14 by saying that a nonlocal phase transition possesses two parameters, ε and s. When $\varepsilon \to 0$, the limit interface approaches a minimal surface either in the fractional case (when $s \in (0,1/2)$) or in the classical case (when $s \in [1/2,1)$). This bifurcation at $s = 1/2$ somehow states that for lower values of s the nonlocal phase transition possesses a nonlocal interface in the limit, but for larger values of s the limit interface is characterized only by local features (in a sense, when $s \in (0,1/2)$ the "surface tension effect" is nonlocal, but for $s \in [1/2,1)$ this effect localizes).

It is also interesting to compare Theorems 6.2 and 6.14, since the bifurcation at $s = 1/2$ detected by Theorem 6.14 is perfectly compatible with the limit behavior of the fractional perimeter, which reduces to the classical perimeter exactly for this value of s, as stated in Theorem 6.2.

7. Nonlocal nonlinear Schrödinger type equation

The type of problems introduced in this section are connected to solitary solutions of nonlinear dispersive wave equations (such as the Benjamin-Ono equation, the Benjamin-Bona-Mahony equation and the fractional Schrödinger equation).

Let $n \geqslant 2$ be the dimension of the reference space, $s \in (0,1)$ be the fractional parameter, and $\epsilon > 0$ be a small parameter. We consider the so-called fractional Sobolev exponent

$$2_s^* := \begin{cases} \dfrac{2n}{n-2s} & \text{for } n \geqslant 3, \text{ or } n = 2 \text{ and } s \in (0,1/2) \\ +\infty & \text{for } n = 1 \text{ and } s \in (0,1/2] \end{cases}$$

and introduce the following nonlocal nonlinear Schrödinger equation

$$\begin{cases} \epsilon^{2s}(-\Delta)^s u + u = u^p & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{7.1}$$

in the subcritical case $p \in (1, 2_s^* - 1)$, namely when $p \in \left(1, \dfrac{n+2s}{n-2s}\right)$.

This equation arises in the study of the fractional Schrödinger equation when looking for standing waves. Namely, the fractional Schrödinger equation considers solutions $\Psi = \Psi(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ of

$$i\hbar \partial_t \Psi = \big(\hbar^{2s}(-\Delta)^s + V\big)\Psi, \tag{7.2}$$

where $s \in (0,1)$, $\hbar$ is the reduced Planck constant and $V = V(x,t,|\Psi|)$ is a potential. This equation is of interest in quantum mechanics (see e.g. [67] and the appendix in [30] for details and physical motivations). Roughly speaking, the quantity $|\Psi(x,t)|^2\, dx\, dt$ represents the probability density of finding a quantum particle in the space region dx and in the time interval dt.

The simplest solutions of (7.2) are the ones for which this probability density is independent of time, i.e. $|\Psi(x,t)| = u(x)$ for some $u : \mathbb{R}^n \to [0, +\infty)$. In this way, one can write Ψ as u times a phase that oscillates (very rapidly) in time: that is one may look for solutions of (7.2) of the form

$$\Psi(x,t) := u(x)\, e^{i\omega t/\hbar},$$

for some frequency $\omega \in \mathbb{R}$. Choosing $V = V(|\Psi|) = -|\Psi|^{p-1} = -u^{p-1}$, a substitution into (7.2) gives that

$$\left(\hbar^{2s}(-\Delta)^s u + \omega u - u^p\right) e^{i\omega t/\hbar} = \hbar^{2s}(-\Delta)^s \Psi - i\hbar\partial_t \Psi + V\Psi = 0,$$

which is (7.1) (with the normalization convention $\omega := 1$ and $\epsilon := \hbar$).

The goal of this section is to construct solutions of problem (7.1) that concentrate at interior points of the domain Ω for sufficiently small values of ϵ. We perform a blow-up of the domain, defined as

$$\Omega_\epsilon := \frac{1}{\epsilon}\Omega = \left\{ \frac{x}{\epsilon}, x \in \Omega \right\}.$$

We can also rescale the solution of (7.1) on Ω_ϵ,

$$u_\epsilon(x) = u(\epsilon x).$$

The problem (7.1) for u_ϵ reads

$$\begin{cases} (-\Delta)^s u + u = u^p & \text{in } \Omega_\epsilon \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega_\epsilon. \end{cases} \tag{7.3}$$

When $\epsilon \to 0$, the domain Ω_ϵ invades the whole of the space. Therefore, it is also natural to consider (as a first approximation) the equation on the entire space

$$(-\Delta)^s u + u = u^p \text{ in } \mathbb{R}^n. \tag{7.4}$$

The first result that we need is that there exists an entire positive radial least energy solution $w \in H^s(\mathbb{R}^n)$ of (7.4), called the *ground state solution*. Here follow some relevant results on this. The interested reader can find their proofs in [56].

(1) The ground state solution $w \in H^s(\mathbb{R}^n)$ is unique up to translations.
(2) The ground state solution $w \in H^s(\mathbb{R}^n)$ is nondegenerate, i.e., the derivatives $D_i w$ are solutions to the linearized equation
$$(-\Delta)^s Z + Z = pZ^{p-1}. \tag{7.5}$$
(3) The ground state solution $w \in H^s(\mathbb{R}^n)$ decays polynomially at infinity, namely there exist two constants $\alpha, \beta > 0$ such that
$$\alpha|x|^{-(n+2s)} \leqslant u(x) \leqslant \beta|x|^{-(n+2s)}.$$

Unlike the fractional case, we remark that for the (classical) Laplacian, at infinity the ground state solution decays exponentially fast.

The main theorem of this section establishes the existence of a solution that concentrates at interior points of the domain for sufficiently small values of ϵ. This concentration phenomena is written in terms of the ground state solution w. Namely, the first approximation for the solution is exactly the ground state w, scaled and concentrated at an appropriate point ξ of the domain. More precisely, we have:

Theorem 7.1. *If ϵ is sufficiently small, there exist a point $\xi \in \Omega$ and a solution U_ϵ of the problem (7.1) such that*
$$\left| U_\epsilon(x) - w\Big(\frac{x-\xi}{\epsilon}\Big) \right| \leqslant C\epsilon^{n+2s},$$
and $\mathrm{dist}(\xi, \partial\Omega) \geqslant \delta > 0$. Here, C and δ are constants independent of ϵ or Ω, and the function w is the ground state solution of problem (7.4).

The concentration point ξ in Theorem 7.1 is influenced by the global geometry of the domain. On the one hand, when $s = 1$, the point ξ is the one that maximizes the distance from the boundary. On the other hand, when $s \in (0, 1)$, such simple characterization of ξ does not hold anymore: in this case, ξ turns out to be asymptotically the maximum of a (complicated, but rather explicit) nonlocal functional: see [30] for more details.

We state here the basic idea of the proof of Theorem 7.1 (we refer again to [30] for more details).

Sketch of the proof of theorem 7.1. The proof makes use of a Lyapunov-Schmidt procedure. Namely, rather than looking for the solution in an infinite-dimensional functional space, one decomposes the problem into two orthogonal subproblems. One of these problem is still infinite-dimensional, but it has the advantage to bifurcate from a known object (in this case, a translation of the ground state). Solving this auxiliary subproblem does not provide a true solution of the original problem, since a leftover in the orthogonal direction may remain. To kill this remainder term, one solves a second subproblem, which turns out to be finite-dimensional (in our case, this subproblem is set in $\mathbb{R}^n$, which corresponds to the action of the translations on the ground state).

A structural advantage of the problem considered lies in its variational structure. Indeed, equation (7.3) is the Euler-Lagrange equation of the energy functional
$$I_\epsilon(u) = \frac{1}{2}\int_{\Omega_\epsilon} \Big((-\Delta)^s u(x) + u(x)\Big)u(x)\,dx - \frac{1}{p+1}\int_{\Omega_\epsilon} u^{p+1}(x)\,dx \tag{7.6}$$

for any $u \in H_0^s(\Omega_\epsilon) := \{u \in H^s(\mathbb{R}^n) \text{ s.t. } u = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega_\epsilon\}$. Therefore, the problem reduces to finding critical points of I_ϵ.

To this goal, we consider the *ground state* solution w and for any $\xi \in \mathbb{R}^n$ we let $w_\xi := w(x-\xi)$. For a given $\xi \in \Omega_\epsilon$ a first approximation $\overline{u}_\xi$ for the solution of problem (7.3) can be taken as the solution of the linear problem

$$\begin{cases} (-\Delta)^s \overline{u}_\xi + \overline{u}_\xi = w_\xi^p & \text{in } \Omega_\epsilon, \\ \overline{u}_\xi = 0 & \text{in } \mathbb{R}^n \backslash \Omega_\epsilon. \end{cases} \tag{7.7}$$

The actual solution will be obtained as a small perturbation of $\overline{u}_\xi$ for a suitable point $\xi = \xi(\epsilon)$. We define the operator $\mathcal{L} := (-\Delta)^s + I$, where I is the identity and we notice that $\mathcal{L}$ has a unique fundamental solution that solves

$$\mathcal{L}\Gamma = \delta_0 \quad \text{in } \mathbb{R}^n.$$

The Green function G_ϵ of the operator $\mathcal{L}$ in Ω_ϵ satisfies

$$\begin{cases} \mathcal{L} G_\epsilon(x,y) = \delta_y(x) & \text{if } x \in \Omega_\epsilon, \\ G_\epsilon(x,y) = 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega_\epsilon. \end{cases} \tag{7.8}$$

It is convenient to introduce the regular part of G_ϵ, which is often called the Robin function. This function is defined by

$$H_\epsilon(x,y) := \Gamma(x-y) - G_\epsilon(x,y) \tag{7.9}$$

and it satisfies, for a fixed $y \in \mathbb{R}^n$,

$$\begin{cases} \mathcal{L} H_\epsilon(x,y) = 0 & \text{if } x \in \Omega_\epsilon, \\ H_\epsilon(x,y) = \Gamma(x-y) & \text{if } x \in \mathbb{R}^n \backslash \Omega_\epsilon. \end{cases} \tag{7.10}$$

Then

$$\overline{u}_\xi(x) = \int_{\Omega_\epsilon} \overline{u}_\xi(y) \delta_0(x-y)\, dy,$$

and by (7.8)

$$\overline{u}_\xi(x) = \int_{\Omega_\epsilon} \overline{u}_\xi(y) \mathcal{L} G_\epsilon(x,y)\, dy.$$

The operator $\mathcal{L}$ is self-adjoint and thanks to the above identity and to equation (7.7) it follows that

$$\begin{aligned} \overline{u}_\xi(x) &= \int_{\Omega_\epsilon} \mathcal{L}\overline{u}_\xi(y) G_\epsilon(x,y)\, dy \\ &= \int_{\Omega_\epsilon} w_\xi^p(y) G_\epsilon(x,y)\, dy. \end{aligned}$$

So, we use (7.9) and we obtain that

$$\overline{u}_\xi(x) = \int_{\Omega_\epsilon} w_\xi^p(y) \Gamma(x-y)\, dy - \int_{\Omega_\epsilon} w_\xi^p(y) H_\epsilon(x,y)\, dy.$$

Now we notice that, since w_ξ is solution of (7.4) and Γ is the fundamental solution of $\mathcal{L}$, we have that

$$\begin{aligned} \int_{\mathbb{R}^n} w_\xi^p(y) \Gamma(x-y)\, dy &= \int_{\mathbb{R}^n} \mathcal{L} w_\xi(y) \Gamma(x-y)\, dy \\ &= \int_{\mathbb{R}^n} w_\xi(y) \mathcal{L}\Gamma(x-y)\, dy \\ &= w_\xi(x). \end{aligned}$$

Therefore we have obtained that

$$\overline{u}_\xi(x) = w_\xi(x) - \int_{\mathbb{R}^n\setminus\Omega_\epsilon} w_\xi^p(y)\Gamma(x-y)\,dy - \int_{\Omega_\epsilon} w_\xi^p(y)H_\epsilon(x,y)\,dy. \tag{7.11}$$

Now we can insert (7.11) into the energy functional (7.6) and expand the errors in powers of ϵ. For $\operatorname{dist}(\xi,\partial\Omega_\epsilon) \geqslant \dfrac{\delta}{\epsilon}$ with δ fixed and small, the energy of $\overline{u}_\xi$ is a perturbation of the energy of the ground state w and one finds (see Theorem 4.1 in [30]) that

$$I_\epsilon(\overline{u}_\xi) = I(w) + \frac{1}{2}\mathcal{H}_\epsilon(\xi) + \mathcal{O}(\epsilon^{n+4s}), \tag{7.12}$$

where

$$\mathcal{H}_\epsilon(\xi) := \int_{\Omega_\epsilon}\int_{\Omega_\epsilon} H_\epsilon(x,y)w_\xi^p(x)w_\xi^p(y)\,dx\,dy$$

and I is the energy computed on the whole space $\mathbb{R}^n$, namely

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n}\Big((-\Delta)^s u(x) + u(x)\Big)u(x)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}(x)\,dx. \tag{7.13}$$

In particular, $I_\epsilon(\overline{u}_\xi)$ agrees with a constant (the term $I(w)$), plus a functional over a finite-dimensional space (the term $\mathcal{H}_\epsilon(\xi)$, which only depends on $\xi\in\mathbb{R}^n$), plus a small error.

We remark that the solution $\overline{u}_\xi$ of equation (7.7) which can be obtained from (7.11) does not provide a solution for the original problem (7.3) (indeed, it only solves (7.7)): for this, we look for solutions u_ξ of (7.3) as perturbations of $\overline{u}_\xi$, in the form

$$u_\xi := \overline{u}_\xi + \psi. \tag{7.14}$$

The perturbation functions ψ are considered among those vanishing outside Ω_ϵ and orthogonal to the space $\mathcal{Z} = \operatorname{Span}(Z_1,\dots,Z_n)$, where $Z_i = \dfrac{\partial w_\xi}{\partial x_i}$ are solutions of the linearized equation (7.5). This procedure somehow "removes the degeneracy", namely we look for the corrector ψ in a set where the linearized operator is invertible. This makes it possible, fixed any $\xi\in\mathbb{R}^n$, to find $\psi=\psi_\xi$ such that the function u_ξ, as defined in (7.14) solves the equation

$$(-\Delta)^s u_\xi + u_\xi = u_\xi^p + \sum_{i=1}^n c_i Z_i \text{ in } \Omega_\epsilon. \tag{7.15}$$

That is, u_ξ is solution of the original equation (7.3), up to an error that lies in the tangent space of the translations (this error is exactly the price that we pay in order to solve the corrector equation for ψ on the orthogonal of the kernel, where the operator is nondegenerate). As a matter of fact (see Theorem 7.6 in [30] for details) one can see that the corrector $\psi=\psi_\xi$ is of order ϵ^{n+2s}. Therefore, one can compute $I_\epsilon(u_\xi) = I_\epsilon(\overline{u}_\xi+\psi_\xi)$ as a higher order perturbation of $I_\epsilon(\overline{u}_\xi)$. From (7.12), one obtains that

$$I_\epsilon(u_\xi) = I(w) + \frac{1}{2}\mathcal{H}_\epsilon(\xi) + \mathcal{O}(\epsilon^{n+4s}), \tag{7.16}$$

see Theorem 7.17 in [30] for details.

Since this energy expansion now depends only on $\xi\in\mathbb{R}^n$, it is convenient to define the operator $J_\epsilon\colon \Omega_\epsilon\to\mathbb{R}$ as

$$J_\epsilon(\xi) := I_\epsilon(u_\xi).$$

This functional is often called the reduced energy functional. From (7.16), we conclude that

$$J_\epsilon(\xi) = I(w) + \frac{1}{2}\mathcal{H}_\epsilon(\xi) + \mathcal{O}(\epsilon^{n+4s}). \tag{7.17}$$

The reduced energy J plays an important role in this framework since critical points of J correspond to true solutions of the original equation (7.3). More precisely (see Lemma 7.16 in [30]) one has that $c_i = 0$ for all $i = 1, \dots, n$ in (7.15) if and only if

$$\frac{\partial J_\epsilon}{\partial \xi}(\xi) = 0. \tag{7.18}$$

In other words, when ϵ approaches 0, to find concentration points, it is enough to find critical points of J, which is a finite-dimensional problem. Also, critical points for J come from critical points of $\mathcal{H}_\epsilon$, up to higher orders, thanks to (7.17). The issue is thus to prove that $\mathcal{H}_\epsilon$ does possess critical points and that these critical points survive after the small error of size ϵ^{n+4s}: in fact, we show that $\mathcal{H}_\epsilon$ possesses a minimum, which is stable for perturbations. For this, one needs a bound for the Robin function H_ϵ from above and below. To this goal, one builds a barrier function β_ξ defined for $\xi \in \Omega_\epsilon$ and $x \in \mathbb{R}^n$ as

$$\beta_\xi(x) := \int_{\mathbb{R}^n \setminus \Omega_\epsilon} \Gamma(z-\xi)\Gamma(x-z)\, dz.$$

Using this function in combination with suitable maximum principles, one obtains the existence of a constant $c \in (0,1)$ such that

$$cH_\epsilon(x,\xi) \leqslant \beta_\xi(x) \leqslant c^{-1}H_\epsilon(x,\xi),$$

for any $x \in \mathbb{R}^n$ and any $\xi \in \Omega_\epsilon$ with $\operatorname{dist}(\xi, \partial\Omega_\epsilon) > 1$, see Lemma 2.1 in [30]. From this it follows that

$$\mathcal{H}_\epsilon(\xi) \simeq d^{-(n+4s)}, \tag{7.19}$$

for all points $\xi \in \Omega_\epsilon$ such that $d \in [5, \delta/\epsilon]$. So, one considers the domain $\Omega_{\epsilon,\delta}$ of the points of Ω_ϵ that lie at distance more than δ/ϵ from the boundary of Ω_ϵ. By (7.19), we have that

$$\mathcal{H}_\epsilon(\xi) \simeq \frac{\epsilon^{n+4s}}{\delta^{n+4s}} \quad \text{for any} \quad \xi \in \partial\Omega_{\epsilon,\delta}. \tag{7.20}$$

Also, up to a translation, we may suppose that $0 \in \Omega$. Thus, $0 \in \Omega_\epsilon$ and its distance from $\partial\Omega_\epsilon$ is of order $1/\epsilon$ (independently of δ). In particular, if δ is small enough, we have that 0 lies in the interior of $\Omega_{\epsilon,\delta}$, and (7.19) gives that

$$\mathcal{H}_\epsilon(0) \simeq \epsilon^{n+4s}.$$

By comparing this with (7.20), we see that $\mathcal{H}_\epsilon$ has an interior minimum in $\Omega_{\epsilon,\delta}$. The value attained at this minimum is of order ϵ^{n+4s}, and the values attained at the boundary of $\Omega_{\epsilon,\delta}$ are of order $\delta^{-n-4s}\epsilon^{n+4s}$, which is much larger than ϵ^{n+4s}, if δ is small enough. This says that the interior minimum of $\mathcal{H}_\epsilon$ in $\Omega_{\epsilon,\delta}$ is nondegenerate and it survives to any perturbation of order ϵ^{n+4s}, if δ is small enough.

This and (7.17) imply that J has also an interior minimum at some point ξ in $\Omega_{\epsilon,\delta}$. By construction, this point ξ satisfies (7.18), and so this completes the proof of Theorem 7.1. □

The variational argument in the proof above (see in particular (7.18)) has a classical and neat geometric interpretation. Namely, the "unperturbed" functional (i.e. the one with $\epsilon = 0$) has a very degenerate geometry, since it has a whole manifold of minimizers with the same energy: this manifold corresponds to the

translation of the ground state w, namely it is of the form $M_0 := \{w_\xi,\ \xi \in \mathbb{R}^n\}$ and, therefore, it can be identified with $\mathbb{R}^n$.

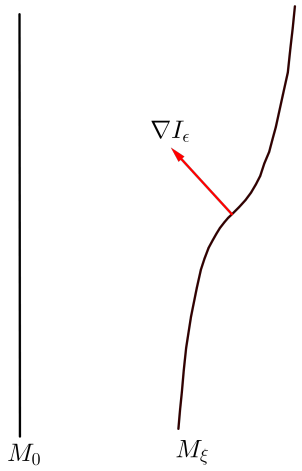

FIGURE 15. Geometric interpretation

For topological arguments, this degenerate picture may constitute a serious obstacle to the existence of critical points for the "perturbed" functional (i.e. the one with $\epsilon \neq 0$). As an obvious example, the reader may think of the function of two variables $f_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ given by $f_\epsilon(x, y) := x^2 + \epsilon y$. When $\epsilon = 0$, this function attains its minimum along the manifold $\{x = 0\}$, but all the critical points on this manifold are "destroyed" by the perturbation when $\epsilon \neq 0$ (indeed $\nabla f_\epsilon(x, y) = (2x, \epsilon)$ never vanishes).

In the situation described in the proof of Theorem 7.1, this pathology does not occur, thanks to the nondegeneracy provided in [56]. Indeed, by the nondegeneracy of the unperturbed critical manifold, when $\epsilon \neq 0$ one can construct a manifold, diffeomorphic to the original one (in our case of the form $M_\epsilon := \{\overline{u}_\xi + \psi(\xi),\ \xi \in \mathbb{R}^n\}$), that enjoys the special feature of "almost annihilating" the gradient of the functional, up to vectors parallel to the original manifold M_0 (this is the meaning of formula (7.15)).

Then, if one finds a minimum of the functional constrained to M_ϵ, the theory of Lagrange multipliers (at least in finite dimension) would suggest that the gradient is normal to M_ϵ. That is, the gradient of the functional is, simultaneously, parallel to M_0 and orthogonal to M_ϵ. But since M_ϵ is diffeomorphically close to M_0, the only vector with this property is the null vector, hence this argument provides the desired critical point.

APPENDIX A. ANOTHER PROOF OF LEMMA 3.3

For completeness, we provide here an alternative proof of Lemma 3.3 that does not use the theory of the fractional Laplacian.

Alternative proof of Lemma 3.3. We first recall some basic properties of the modified Bessel functions (see e.g. [3]). First of all (see formula 9.1.10 on page 360 of [3]),

we have that

$$J_s(z) := \frac{z^s}{2^s}\sum_{k=0}^{+\infty}\frac{(-1)^k z^{2k}}{2^{2k}\,k!\,\Gamma(s+k+1)} = \frac{z^s}{2^s\,\Gamma(1+s)} + \mathcal{O}(|z|^{2+s})$$

as $|z|\to 0$. Therefore (see formula 9.6.3 on page 375 of [3]),

$$\begin{aligned} I_s(z) &:= e^{-\frac{i\pi s}{2}} J_s(e^{\frac{i\pi}{2}} z) \\ &= e^{-\frac{i\pi s}{2}}\left(\frac{e^{\frac{i\pi s}{2}} z^s}{2^s\,\Gamma(1+s)} + \mathcal{O}(|z|^{2+s})\right) \\ &= \frac{z^s}{2^s\,\Gamma(1+s)} + \mathcal{O}(|z|^{2+s}), \end{aligned}$$

as $|z|\to 0$. Using this and formula 9.6.2 on page 375 of [3],

$$\begin{aligned} K_s(z) &:= \frac{\pi}{2\sin(\pi s)}\Big(I_{-s}(z) - I_s(z)\Big) \\ &= \frac{\pi}{2\sin(\pi s)}\left(\frac{z^{-s}}{2^{-s}\,\Gamma(1-s)} - \frac{z^s}{2^s\,\Gamma(1+s)} + \mathcal{O}(|z|^{2-s})\right). \end{aligned}$$

Thus, recalling Euler's reflection formula

$$\Gamma(1-s)\,\Gamma(s) = \frac{\pi}{\sin(\pi s)},$$

and the relation $\Gamma(1+s) = s\Gamma(s)$, we obtain

$$\begin{aligned} K_s(z) &= \frac{\Gamma(1-s)\,\Gamma(s)}{2}\left(\frac{z^{-s}}{2^{-s}\,\Gamma(1-s)} - \frac{z^s}{2^s\,\Gamma(1+s)} + \mathcal{O}(|z|^{2-s})\right) \\ &= \frac{\Gamma(s)\,z^{-s}}{2^{1-s}} - \frac{\Gamma(1-s)\,z^s}{2^{1+s}s} + \mathcal{O}(|z|^{2-s}), \end{aligned}$$

as $|z|\to 0$. We use this and formula (3.100) in [70] (or page 6 in [74]) and get that, for any small $a>0$,

$$\begin{aligned} \int_0^{+\infty}\frac{\cos(2\pi t)}{(t^2+a^2)^{s+\frac12}}\,dt &= \frac{\pi^{s+\frac12}}{a^s\,\Gamma\left(s+\frac12\right)}K_s(2\pi a) \\ &= \frac{\pi^{s+\frac12}}{a^s\,\Gamma\left(s+\frac12\right)}\left[\frac{\Gamma(s)}{2\pi^s a^s} - \frac{\Gamma(1-s)\,\pi^s a^s}{2s} + \mathcal{O}(a^{2-s})\right] \\ &= \frac{\pi^{\frac12}\Gamma(s)}{2a^{2s}\,\Gamma\left(s+\frac12\right)} - \frac{\pi^{2s+\frac12}\,\Gamma(1-s)}{2s\Gamma\left(s+\frac12\right)} + \mathcal{O}(a^{2-2s}). \end{aligned} \tag{A.1}$$

Now we recall the generalized hypergeometric functions ${}_mF_n$ (see e.g. page 211 in [74]): as a matter of fact, we just need that for any b, c, $d>0$,

$${}_1F_2(b;c,d;0) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(b)}\cdot\frac{\Gamma(b)}{\Gamma(c)\Gamma(d)} = 1.$$

We also recall the Beta function relation

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

see e.g. formula 6.2.2 in [3]. Therefore using formula (3.101) in [70] (here with $y := 0$, $\nu := 0$ and $\mu := s - \frac{1}{2}$, or see page 10 in [74]),

$$\begin{aligned}\int_0^{+\infty} \frac{dt}{(a^2+t^2)^{s+\frac{1}{2}}} &= \frac{a^{-2s}}{2} B\left(\frac{1}{2}, s\right) {}_1F_2\left(\frac{1}{2}; 1-s, \frac{1}{2}; 0\right)\\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(s)}{2a^{2s}\Gamma\left(\frac{1}{2}+s\right)}.\end{aligned}$$

Then, we recall that $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$, so, making use of (A.1), for any small $a > 0$,

$$\int_0^{+\infty} \frac{1-\cos(2\pi t)}{(t^2+a^2)^{s+\frac{1}{2}}}\, dt = \frac{\pi^{2s+\frac{1}{2}}\,\Gamma(1-s)}{2s\Gamma\left(s+\frac{1}{2}\right)} + \mathcal{O}(a^{2-2s}).$$

Therefore, sending $a \to 0$ by the Dominated Convergence Theorem we obtain

$$\int_0^{+\infty} \frac{1-\cos(2\pi t)}{t^{1+2s}}\, dt = \lim_{a\to 0} \int_0^{+\infty} \frac{1-\cos(2\pi t)}{(t^2+a^2)^{s+\frac{1}{2}}}\, dt = \frac{\pi^{2s+\frac{1}{2}}\,\Gamma(1-s)}{2s\Gamma\left(s+\frac{1}{2}\right)}. \tag{A.2}$$

Now we recall the integral representation of the Beta function (see e.g. formulas 6.2.1 and 6.2.2 in [3]), namely

$$\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}+s\right)}{\Gamma\left(\frac{n}{2}+s\right)} = B\left(\frac{n-1}{2}, \frac{1}{2}+s\right) = \int_0^{+\infty} \frac{\tau^{\frac{n-3}{2}}}{(1+\tau)^{\frac{n}{2}+s}}\, d\tau. \tag{A.3}$$

We also observe that in any dimension N the $(N-1)$-dimensional measure of the unit sphere is $\frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}$, (see e.g. [61]). Therefore

$$\int_{\mathbb{R}^N} \frac{dY}{(1+|Y|^2)^{\frac{N+1+2s}{2}}} = \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)} \int_0^{+\infty} \frac{\rho^{N-1}}{(1+\rho^2)^{\frac{N+1+2s}{2}}}\, d\rho.$$

In particular, taking $N := n-1$ and using the change of variable $\rho^2 =: \tau$,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \frac{d\eta}{(1+|\eta|^2)^{\frac{n+2s}{2}}} &= \frac{(n-1)\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}+1\right)} \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2s}{2}}}\, d\rho\\ &= \frac{(n-1)\,\pi^{\frac{n-1}{2}}}{2\,\Gamma\left(\frac{n-1}{2}+1\right)} \int_0^{+\infty} \frac{\tau^{\frac{n-3}{2}}}{(1+\tau)^{\frac{n+2s}{2}}}\, d\tau.\end{aligned}$$

By comparing this with (A.3), we conclude that

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \frac{d\eta}{(1+|\eta|^2)^{\frac{n+2s}{2}}} &= \frac{(n-1)\,\pi^{\frac{n-1}{2}}}{2\,\Gamma\left(\frac{n-1}{2}+1\right)} \cdot \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}+s\right)}{\Gamma\left(\frac{n}{2}+s\right)}\\ &= \frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{1}{2}+s\right)}{\Gamma\left(\frac{n}{2}+s\right)}.\end{aligned}$$

Accordingly, with the change of variable $\eta := |\omega_1|^{-1}(\omega_2, \dots, \omega_n)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1-\cos(2\pi\omega_1)}{|\omega|^{n+2s}}\,d\omega \\
&\quad= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \frac{1-\cos(2\pi\omega_1)}{(\omega_1^2+\omega_2^2+\cdots+\omega_n^2)^{\frac{n+2s}{2}}}\,d\omega_2\,\dots\,d\omega_n \right) d\omega_1 \\
&\quad= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \frac{1-\cos(2\pi\omega_1)}{|\omega_1|^{1+2s}(1+|\eta|^2)^{\frac{n+2s}{2}}}\,d\eta \right) d\omega_1 \\
&\quad= \frac{\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac12+s\right)}{\Gamma\left(\frac{n}{2}+s\right)} \int_{\mathbb{R}} \frac{1-\cos(2\pi\omega_1)}{|\omega_1|^{1+2s}}\,d\omega_1 \\
&\quad= \frac{2\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac12+s\right)}{\Gamma\left(\frac{n}{2}+s\right)} \int_0^{+\infty} \frac{1-\cos(2\pi t)}{t^{1+2s}}\,dt.
\end{aligned}$$

Hence, recalling (A.2),

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{1-\cos(2\pi\omega_1)}{|\omega|^{n+2s}}\,d\omega &= \frac{2\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac12+s\right)}{\Gamma\left(\frac{n}{2}+s\right)} \cdot \frac{\pi^{2s+\frac12}\,\Gamma(1-s)}{2s\Gamma\left(s+\frac12\right)} \\
&= \frac{\pi^{\frac{n}{2}+2s}\,\Gamma(1-s)}{s\,\Gamma\left(\frac{n}{2}+s\right)},
\end{aligned}$$

as desired. □

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